Abstract

Solutions of a homogeneous \((r + 1)\)-term linear difference equation are given in two different forms. One involves the elements of a certain matrix, while the other is in terms of certain lower Hessenberg determinants. The results generalize some earlier results of Brown [1] for the solution of a 3-term linear difference equation.

1. Introduction

In a recent paper, Brown [1] has given the solution of the three-term linear difference equation

\[ a_0(n)u_n + a_1(n)u_{n-1} + a_2(n)u_{n-2} = 0, \quad n \geq 2, \quad (1) \]

with \(a_0(n) \neq 0\) for all \(n \geq 2\), in terms of certain tri-diagonal determinants. In this paper, we consider the general \((r + 1)\)-term homogeneous linear difference equation

\[ a_0(n)u_n + a_1(n)u_{n-1} + \ldots + a_r(n)u_{n-r} = 0, \quad n \geq r, \quad (2) \]

with the condition that \(a_0(n) \neq 0\) for all \(n \geq r\). We obtain two forms of the general solution for this difference equation, namely a matrix form, given in Section 2, and a determinantal form, given in Section 3. An interesting determinantal relation is derived in Section 4.

We introduce the following notations:

\[ a_t(n) = 0 \quad \text{if } t \text{ is a negative integer or a positive integer } > r. \]

\[ p_k = \prod_{l=r}^{k} a_0(l), \quad q_k = \prod_{l=r}^{k} a_r(l); \quad \text{empty products will be taken to be } 1. \]
\[ A_k = [a_{i-j}(kr+i-1)], \quad i,j = 1,\ldots,r, \quad k = 1,2,\ldots \]
\[ B_k = [a_{r-\ell}(kr+i-1)], \quad i,j = 1,\ldots,r, \quad k = 1,2,\ldots \]

\[ N = \lceil n/r \rceil, \quad \text{where} \; \lceil x \rceil \text{denotes the integral part of} \; x; \quad N' = n-Nr+1. \]

\[ A_{(n)} = [a_{i-j}(Nr+i-1)], \quad i,j = 1,\ldots,N'. \]
\[ B_{(n)} = [a_{r-\ell}(Nr+i-1)], \quad i = 1,\ldots,N', \quad j = 1,\ldots,r. \]

\[ U_{(r,n)} = [u_r u_{r+1} \ldots u_n]^T; \quad U_{(kr,(k+1)r-1)} = U_k. \]

\[ D_m^n(r,s) = \begin{bmatrix} a_s(m) & a_{s+1}(m+1) & a_1(m+1) & a_0(m+1) & \ldots & 0 \\ a_s(m+r-s) & a_{s-1}(m+r-s) & \ldots & a_0(m+r-s) & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_s(m+r) & a_{s-1}(m+r) & \ldots & \ldots & 0 \\ 0 & a_s(m+r+1) & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \]

for \( n \geq m+1; \; D_m^n(r,s) = a_s(m) \) and \( D^n_\ell(r,s) \equiv D^n_s. \)

2. Solution in terms of matrix elements

We shall first obtain a matrix solution of the linear difference equation (2).

If we put \( n = kr, kr+1,\ldots,(k+1)r-1 \) in (2), we get the matrix reduction formula

\[ A_k U_k = -B_k U_{k-1}, \quad k \geq 1. \quad (3) \]

Therefore,

\[ U_k = (-1)^k \left\{ \prod_{s=1}^{k} (A_s^{-1} B_s) \right\} U_0, \quad k \geq 1, \quad (4) \]

as the matrices \( A_s, s = 1,\ldots,k, \) being lower triangular, are non-singular because of the condition that \( a_0(n) \neq 0 \) for all \( n \geq r. \) Thus, if \( n \equiv t \) (mod \( r \)), then \( n = kr+t, \)
\( 0 \leq t \leq r-1, \) and so, from (4), we obtain the following general solution for the difference equation (2):

\[ u_{kr+t} = \text{the} \; (t+1)\text{th element of the column matrix} \; U_k, \quad 0 \leq t \leq r-1. \quad (5) \]
3. Solution in terms of determinants

In this section, we obtain a solution of the linear difference equation (2) in terms of the determinants $D^n_s$, $s = 1, \ldots, r$.

Let $n = r, r + 1, \ldots, n$ in (2). Then

$$A_{(r,n)} U_{(r,n)} = \begin{bmatrix} B_1 \\ 0_{n-2r+1,r} \end{bmatrix} U_0,$$

where

$$A_{(r,n)} = \begin{bmatrix} A_1 & 0_r & 0_r & \ldots & 0_r & 0_r & 0_{r,N'} \\ B_2 & A_2 & 0_r & \ldots & 0_r & 0_r & 0_{r,N'} \\ 0_r & B_3 & A_3 & \ldots & 0_r & 0_r & 0_{r,N'} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_r & 0_r & 0_r & \ldots & B_{N-1} & A_{N-1} & 0_{r,N'} \\ 0_{N',r} & 0_{N',r} & 0_{N',r} & \ldots & 0_{N',r} & B_{(n)} & A_{(n)} \end{bmatrix},$$

with $0_{r,s}$ denoting the null matrix of dimension $r$ by $s$, and with $0_{r,r} \equiv 0_r$.

Applying Cramer's rule to the linear non-homogeneous system (6), we get, in particular,

$$| A_{(r,n)} | u_n = C_{(r,n)},$$

where

$$C_{(r,n)} = \begin{bmatrix} A_1 & 0_r & 0_r & \ldots & 0_r & 0_{r,N'-1} & -B_1 U_0 \\ B_2 & A_2 & 0_r & \ldots & 0_r & 0_{r,N'-1} & 0_r \\ 0_r & B_3 & A_3 & \ldots & 0_r & 0_{r,N'-1} & 0_r \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_r & 0_r & 0_r & \ldots & B_{N-1} & A_{N-1} & 0_{r,N'-1} \\ 0_{N',r} & 0_{N',r} & 0_{N',r} & \ldots & 0_{N',r} & B_{(n)} & A_{(n)}^* \\ \end{bmatrix},$$

the asterisk in the last row denoting the omission of the last column of the starred matrix. On expanding the determinant $C_{(r,n)}$ along the last column, we find that

$$C_{(r,n)} = (-1)^{n-r+1} \sum_{t=0}^{r-1} (-1)^t \left\{ \sum_{s=1}^{r-t} a_{s+t} (r+t) u_{r-s} \right\} p_{r+t-1} D^n_{r+t+1} (r,1)$$

$$= (-1)^{n-r+1} \sum_{s=1}^{r} u_{r-s} \left\{ \sum_{t=0}^{r-s} (-1)^t a_{s+t} (r+t) p_{r+t-1} D^n_{r+t+1} (r,1) \right\}.$$

The last inner sum is seen to be the expansion, along the first column, of the
determinant $D^r_s$. Hence, by (7), we obtain the following determinantal solution of the linear difference equation (2):

$$u_n = (-1)^{n-r+1} p_{n-r}^{-1} \sum_{s=1}^{r} D^r_s u_{r-s}, \quad n \geq r.$$  

(9)

It is easy to see that, when $r = 2$, this reduces to the solution given by Brown [1], equation (3.10):

$$u_n = (-1)^{n-1} L_{n-1}^{-1} \{u_1 C_1^{n+1} + u_0 F(1) C_2^{n+1}\}, \quad n \geq 3,$$

if one observes that

$$C_m^n = D_{m+1}^{n+1} (2, 1) = D_m^{n+1} (2, 2)/(a_2(m)),$$

so that

$$D_1^n = C_1^{n-1} \quad \text{and} \quad D_2^n = a_2(2) C_2^{n-1}.$$  

(10)

4. A relation between determinants

We can easily obtain a determinantal relation connecting the determinants $D^{n+t}_s$, $t = 0, 1, ..., r-1$, $s = 1, ..., r$.

Let

$$E_r^n = \left| D^{n+i-1}_j \right|, \quad i, j = 1, ..., r,$$

denote the determinant formed from these $r^2$ determinants. On expanding the determinant $D^r_s$ along the last column, we have

$$D^r_s = \sum_{t=1}^{r} (-1)^{t-1} \left\{ \prod_{l=1}^{i-1} a_0(n-l) \right\} a_i(n) D_s^{n-t}, \quad n \geq 2r,$$

(11)

for $s = 1, ..., r$. Making use of relation (11) in the last row of determinant $E_r^n$, we find that

$$E_r^n = \left\{ \prod_{l=n}^{n+r-2} a_0(l) \right\} a_r(n+r-1) E_r^{n-1}, \quad n \geq r+1.$$  

(12)

We now need the following

**Lemma.**

$$E_r^* = \left\{ \prod_{n=r}^{2r-2} p_n \right\} q_{2r-1}.$$  

(13)

**Proof.** Let

$$D^r_r = [D^r_{r-j} \binom{-1}{j}], \quad P_r = [(-1)^{i-1} p_{r+i-1} \delta_{i,j}],$$
where \( i, j = 1, \ldots, r \) and \( \delta^i_j \) is the Kronecker delta. Then the set of solutions (9) for \( n = r, r+1, \ldots, 2r-1 \), can be written as

\[
D_r^* U_0 = -P_r U_1. \tag{14}
\]

Since, by (4),

\[
U_1 = -(A_1^{-1} B_1) U_0,
\]

we have

\[
(P_r^{-1} D_r^*) U_0 = (A_1^{-1} B_1) U_0. \tag{15}
\]

Denoting by \( E_k \) the \( r \)-dimensional unit vector having unity in the \( k \)th place and zeros everywhere else, and taking \( U_0 = E_k, k = 1, 2, \ldots, r \), successively in (15), we get

\[
P_r^{-1} D_r^* = A_1^{-1} B_1,
\]

so that

\[
(-1)^{(r-1)/2} E_r^* = \left\{ \prod_{n=r}^{2r-1} (-1)^{n-r} p_n \right\} p_{2r-1}^{-1} q_{2r-1},
\]

whence (13) follows.

Using the reduction formula (12) and the value (13), after some rearrangement we finally get

\[
E_r^n = \left\{ \prod_{k=n}^{n+r-2} p_k \right\} q_{n+r-1}, \quad n \geq r. \tag{16}
\]

Thus, \( E_r^n \) vanishes if \( q_r(l) = 0 \) for some \( l, r \leq l \leq n+r-1 \).

When \( r = 2 \), (16) reduces to the corresponding relation given by Brown [1], equation (3.14), because of the relations (10).

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Reference