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A CHARACTERISATION OF WEAK COMPACTNESS IN BANACH SPACES

WARREN B. MOORS

In this short note we give a new characterisation of weak compactness.

The primary purpose of this paper is to provide a proof of the 'Note added in proof' in [3]. In doing so, we shall also derive a new characterisation of weak compactness in Banach spaces.

For a Banach space X with closed unit ball $B(X)$, we shall denote by $\operatorname{ext} B(X^*)$, the set of all extreme points of the dual ball $B(X^*)$ and we shall denote by σ_e, the weak topology on X generated by $\operatorname{ext} B(X^*)$.

Observe that the σ_e-topology on X is Hausdorff and that the closed unit ball $B(X)$ is also closed in the σ_e-topology. Both of these observations follow from the fact that for each $x \in X$, there exists an element $e \in \operatorname{ext} B(X^*)$ such that $e(x) = \|x\|$. In the case when $X \equiv C(K)$ — (K compact and Hausdorff and $B(X)$ is the supremum norm ball), the σ_e-topology is the pointwise topology on $C(K)$. In the case of a Banach space X, whose dual ball $B(X^*)$ is rotund, the σ_e-topology is the weak topology on X.

THEOREM 0.1. *(Rainwater's theorem, [5]) Let X be a Banach space and let $\{x_n : n \in N\}$ be a bounded sequence in X. Then $\{x_n : n \in N\}$ converges weakly to x if and only if $\{x_n : n \in N\}$ converges to x in the σ_e-topology.*

COROLLARY 0.1. *A bounded sequence in a Banach space X is weakly Cauchy if and only if it is σ_e-Cauchy.*

PROOF: It need only be observed that a sequence $\{x_n : n \in N\}$ is weakly Cauchy if and only if for each pair $\{k_n : n \in N\}$ and $\{j_n : n \in N\}$ of increasing sequences of natural numbers, the sequence $\{x_{k_n} - x_{j_n} : n \in N\}$ converges weakly to 0. ∎

THEOREM 0.2. *Let F be an infinite bounded subset of a Banach space X. Then there exists a countably infinite subset E of F such that $\overline{co}^{\sigma_e} E = \overline{co} E$. In particular,*

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every bounded sequence $\{x_n : n \in N\}$ *in* X *possesses a subsequence* $\{x_{n_k} : k \in N\}$ *such that* $\overline{co}^{\sigma_e}\{x_{n_k} : k \in N\} \subseteq \overline{co}\{x_n : n \in N\}$.

PROOF: Suppose that this is not the case. Then for each countably infinite subset E of F there exists an element $z \in \overline{co}^{\sigma_e}E \backslash \overline{co}E$. Using this, let us show that for any infinite sequence $\{x_n : n \in N\}$ in F, there exists an element

$$z \in \left(\bigcap_{n=1}^{\infty} \overline{co}^{\sigma_e}\{x_k : k \geqslant n\}\right) \Big\backslash \overline{co}\{x_n : n \in N\}$$

We begin by observing that since $B(X)$ is σ_e-closed and convex, we have that $\operatorname{diam} F = \operatorname{diam} \overline{co}^{\sigma_e}F$. So, in particular, we have that $\overline{co}^{\sigma_e}F$ is also bounded. Let $\{x_n : n \in N\}$ be an infinite sequence in F. By our assumption, the Bishop-Phelps theorem and a separation argument, there exists $f \in X^*$ and $z \in \overline{co}^{\sigma_e}\{x_n : n \in N\}$ such that

$$f(z) = \max f(\overline{co}^{\sigma_e}\{x_n : n \in N\}) > \sup f(\overline{co}\{x_n : n \in N\}).$$

We claim that

$$z \in \bigcap_{n=1}^{\infty} \overline{co}^{\sigma_e}\{x_k : k \geqslant n\}.$$

To see this, we note that for each $n \in N$,

$$\overline{co}^{\sigma_e}\{x_k : k \in N\} = co(\overline{co}^{\sigma_e}\{x_k : k > n\} \cup co\{x_1, x_2, \ldots, x_n\}).$$

Therefore, for each $n \in N$, there exists $\lambda_n \in [0,1]$, $z_n \in \overline{co}^{\sigma_e}\{x_k : k > n\}$ and $y_n \in co\{x_1, x_2, \ , \ , x_n\}$ such that $z = \lambda_n z_n + (1 - \lambda_n) y_n$. It now follows that for each $n \in N$, $\lambda_n = 1$ and $z = z_n \in \overline{co}^{\sigma_e}\{x_k : k > n\}$ and so

$$z \in \left(\bigcap_{n=1}^{\infty} \overline{co}^{\sigma_e}\{x_k : k \geqslant n\}\right) \Big\backslash \overline{co}\{x_n : n \in N\}.$$

Let $\{x_n : n \in N\}$ be an arbitrary sequence of distinct elements of F. We shall show that $\{x_n : n \in N\}$ contains a weakly Cauchy subsequence. Let $Y \equiv \overline{sp}\{x_n : n \in N\} \subseteq X$ and denote by $E_Y \equiv \operatorname{ext} B(Y^*)$. Since $(B(Y^*)$, weak$^*)$ is separable and metrisable so is $(E_Y$, weak$^*)$. Let $\{e_Y^n : n \in N\}$ be a dense subset of E_Y. Since for each $m \in N$, the sequence $\{e_Y^m(x_n) : n \in N\}$ is bounded, we may apply a Cantor diagonalisation argument to extract a subsequence $\{x_{n_k} : k \in N\}$ of $\{x_n : n \in N\}$ such that

$$\lim_{k \to \infty} e_Y^m(x_{n_k}) \text{ exists for each } m \in N.$$

We claim that the sequence $\{x_{n_k} : k \in N\}$ is weakly Cauchy. To see this, we consider the following. For each $k \in N$, define $\widehat{x}_{n_k} : E_Y \to R$ by $\widehat{x}_{n_k}(e_Y) \equiv e_Y(x_{n_k})$.

Clearly, each $\widehat{x}_{n_k}$ is continuous on $(E_Y, \text{ weak}^*)$. We shall show that the sequence $\{\widehat{x}_{n_k} : k \in N\}$ is pointwise convergent to some continuous function $\widehat{z}_0$ on $(E_Y, \text{ weak}^*)$; and then apply Rainwater's theorem. Now, for each $x \in \overline{co}^{\sigma_e}\{x_n : n \in N\}$ we may define a continuous 'lift' $\widehat{x}$ of x onto E_Y in the following manner. Define $\widehat{x} : E_Y \to R$, by $\widehat{x}(e_Y) \equiv e(x)$ where e is any member of $B(X^*)$ such that $e|_Y \equiv e_Y$. Our first task is to show that $\widehat{x}$ is well-defined. For each $e_Y \in E_Y$, let $HB(e_Y) \equiv \{e \in B(X^*) : e|_Y = e_Y\}$. It is not too hard to show that the set-valued mapping $e_Y \to HB(e_Y)$ is a weak* cusco on E_Y, that is, for each $e_Y \in E_Y$, $HB(e_Y)$ is non-empty, weak* compact and convex and $e_Y \to HB(e_Y)$ is weak* upper semi-continuous. Furthermore, it is not too hard to see that each $HB(e_Y)$ is an extremal subset of $B(X^*)$ and so $\operatorname{ext} HB(e_Y) \subseteq \operatorname{ext} B(X^*)$. Let us also note that by the Krein-Milman theorem, $HB(e_Y) = \overline{co}^{w^*} \operatorname{ext} HB(e_Y)$, for each $e_Y \in E_Y$. So to show that $\widehat{x}$ is well-defined it suffices to show that if e_1 and $e_2 \in \operatorname{ext} HB(e_Y)$ then $e_1(x) = e_2(x)$ or, equivalently, show that $x \in \ker(e_1 - e_2)$. However, this is obvious since $\{x_n : n \in N\} \subseteq \ker(e_1 - e_2)$ and $\ker(e_1 - e_2)$ is σ_e-closed and convex. The fact that $\widehat{x}$ is continuous on $(E_Y, \text{ weak}^*)$ follows directly from the fact that $e_Y \to HB(e_Y)$ is a weak* cusco on E_Y. Choose

$$z_0 \in \left(\bigcap_{m=1}^{\infty} \overline{co}^{\sigma_e}\{x_{n_k} : k \geqslant m\}\right) \Big\backslash \overline{co}\{x_{n_k} : k \in N\}.$$

We shall show that $\{\widehat{x}_{n_k} : k \in N\}$ converges pointwise to $\widehat{z}_0$ on E_Y. Let us first observe that for each $e_Y \in E_Y$

$$\widehat{z}_0(e_Y) \in \bigcap_{m=1}^{\infty} \overline{co}\{\widehat{x}_{n_k}(e_Y) : k \geqslant m\}.$$

Hence, for each $m \in N$,

$$\widehat{z}_0(e_Y^m) = \lim_{k\to\infty} \widehat{x}_{n_k}(e_Y^m).$$

Next, suppose that for some $e'_Y \in E_Y$,

$$\lim_{k\to\infty} \widehat{x}_{n_k}(e'_Y) \neq \widehat{z}_0(e'_Y).$$

Then there exists an $\varepsilon > 0$ and a subsequence $\{x_{n_{k_l}} : l \in N\}$ of $\{x_{n_k} : k \in N\}$ such that $\left|\widehat{z}_0(e'_Y) - \widehat{x}_{n_{k_l}}(e'_Y)\right| > \varepsilon$ for all $l \in N$. Moreover, by possibly replacing e'_Y by $-e'_Y$ and passing to a subsequence, we may assume that $\widehat{z}_0(e'_Y) < \widehat{x}_{n_{k_l}}(e'_Y) - \varepsilon$ for each $l \in N$. Now choose

$$x_0 \in \left(\bigcap_{m=1}^{\infty} \overline{co}^{\sigma_e}\{x_{n_{k_l}} : l \geqslant m\}\right) \Big\backslash \overline{co}\{x_{n_{k_l}} : l \in N\}.$$

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As before, we have that

$$\widehat{x}_0(e_Y) \in \bigcap_{m=1}^{\infty} \overline{co}\{\widehat{x}_{n_{k_l}}(e_Y) : l \geqslant m\}.$$

Therefore $\widehat{x}_0(e'_Y) \geqslant \widehat{z}_0(e'_Y) + \varepsilon$ while $\widehat{x}_0(e^m_Y) = \widehat{z}_0(e^m_Y)$ for each $m \in N$. But this is not possible, since both $\widehat{x}_0$ and $\widehat{z}_0$ are continuous on $(E_Y$, weak$^*)$. Hence we must have that $\widehat{z}_0$ is the pointwise limit of the sequence $\{\widehat{x}_{n_k} : k \in N\}$. Therefore, by Corollary 0.1, $\{x_{n_k} : k \in N\}$ is weakly Cauchy in Y and so weakly Cauchy in X. The next and penultimate step is to show that the sequence $\{x_{n_k} : k \in N\}$ converges weakly to z_0. We know from Rainwater's theorem that to do this we need only show that

$$\lim_{k\to\infty} e(x_{n_k}) = e(z_0) \quad \text{for each } e \in \operatorname{ext} B(X^*).$$

To this end, let $e \in \operatorname{ext} B(X^*)$. Now, $\{e(x_{n_k}) : k \in N\}$ is a Cauchy sequence in R and

$$e(z_0) \in \bigcap_{m=1}^{\infty} \overline{co}\{e(x_{n_k}) : k \geqslant m\}.$$

Therefore,

$$e(z_0) = \lim_{k\to\infty} e(x_{n_k}).$$

This shows that z_0 is the weak limit of $\{x_{n_k} : k \in N\}$. The final step is to observe that this gives rise to our desired contradiction, since we chose $z_0 \notin \overline{co}\{x_{n_k} : k \in N\}$. ∎

Corollary 0.2. *Every bounded sequence $\{x_n : n \in N\}$ in a Banach space X contains a subsequence $\{x_{n_k} : k \in N\}$ such that*

$$\bigcap_{m=1}^{\infty} \overline{co}^{\sigma_e}\{x_{n_k} : k \geqslant m\} \subseteq \bigcap_{m=1}^{\infty} \overline{co}\{x_n : n \geqslant m\}.$$

Proof: This follows from the previous theorem and a diagonalisation argument. ∎

The next corollary improves the main result in [**1**].

Corollary 0.3. *A bounded subset F of a Banach space X is relatively weakly compact if and only if for each sequence $\{x_n : n \in N\}$ in F*

$$\bigcap_{m=1}^{\infty} \overline{co}^{\sigma_e}\{x_n : n \geqslant m\} \neq \emptyset.$$

In particular, F is relatively weakly compact if and only if F is relatively σ_e-countably compact.

Proof: The proof of this follows from Corollary 0.2 and the fact that F is relatively weakly compact if and only if for each sequence $\{x_n : n \in N\}$ in F

$$\bigcap_{m=1}^{\infty} \overline{co}\{x_n : n \geqslant m\} \neq \emptyset.$$

∎

REMARK. The author benefitted greatly by having an acquaintance with the manuscript [**4**], which contains a simplification of the main result in [**2**].

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Department of Mathematics
The University of Auckland
Private Bag 92019
Auckland
New Zealand