§ 9. If the straight line AB is parallel to A′B′ then for one of the rotations we may substitute a translation, and the change of position is accomplished by a translation and a rotation. Again if AB and A′B′ coincide in situation, without the corresponding points in each coinciding, then the change of position is accomplished by a translation along and a rotation about the same axis. I here assume Chasles' celebrated results, published in 1830 in the *Bulletin Universel des Sciences*, viz :—If any finite displacement be given to a free solid body in space, there exist in the body a set of straight lines, which, after the displacement, are parallel to their initial positions, and in particular there is one straight line, and one only, which after the displacement remains in its original situation. All that need be pointed out at present is that the representation of the displacement of a solid body by a translation and a rotation, and the unique representation by a screw movement, are particular cases of the more general representation by two conjugate rotations.

§ 10. If we consider any two positions of a solid body in space, the displacement of the different points lying in any straight line lie on an hyperbolic paraboloid.

For let AB be the given line, CD its conjugate, then the displacements of every point in AB pass through two straight lines, viz., AB and its displaced position A′B′, and, further, they are all parallel to a plane perpendicular to CD. Hence they lie in an hyperbolic paraboloid. This proposition is given in Routh's *Rigid Dynamics* for infinitesimal motions, and the proof is exactly the same for finite motions.

A Proof of Lagrange's Theorem.

By T. Hugh Miller, M.A.

If $y = z + xf(y)$ where x and z are independent, Lagrange's series gives the expansion of any function of y in terms of x. The coefficients of the powers of x may be found thus. Let $\phi(y)$ be the function to be expanded. Then by Taylor's theorem

$$\phi(y) = \phi\{z + xf(y)\}$$

$$= \phi(z) + xf(y)\phi'(z) + \frac{x^2[f(y)]^2}{\underline{|2}}\phi''(z) + \&c., \qquad (1)$$

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Also

$$f(y)=f(z)+xf(y)f'(z)+\frac{x^2[f(y)]^2}{\underline{|2}}f''(z)+\&c.$$

Now, by substituting the value of $f(y)$, $[f(y)]^2$ from the second series, the co-efficients of x, x^2, &c., can be found. This gives

$$\phi(y)=\phi(z)+xf(z)\phi'(z)+\frac{x^2}{\underline{|2}}[\{f(z)\}^2\phi''(z)+2f(z)f'(z)\phi'(z)]+\&c.,$$

or $$\phi(y)=\phi(z)+xf(z)\phi'(z)+\frac{x^2}{\underline{|2}}\frac{d}{dz}\{[f(z)]^2\phi'(z)\}+\&c.$$

The co-efficients become more complicated, but suppose the above law of formation is true up to the n^{th} term, that is,

Let $$\phi(y)=\phi(z)+xf(z)\phi'(z)+\ldots+\frac{x^{n-1}}{\underline{|n-1}}\frac{d^{n-2}}{dz^{n-2}}[\{f(z)\}^{n-1}\phi'(z)]+\&c.$$

Then $$f(y)=f(z)+xf(z)f'(z)+\ldots+\frac{x^{n-1}}{\underline{|n-1}}\frac{d^{n-2}}{dz^{n-2}}[\{f(z)\}^{n-1}f'(z)]+\&c;$$

and $$\{f(y)\}^r=\{f(z)\}^r+xf(z)r\{f(z)\}^{r-1}f'(z)+\ldots$$
$$+\frac{x^{n-1}}{\underline{|n-1}}\frac{d^{n-2}}{dz^{n-2}}[\{f(z)\}^{n-1}r\{f(z)\}^{r-1}f'(z)]+\&c.$$

Now substitute these values of $f(y)$, $\{f(y)\}^2$ etc. in the series (1), and find the co-efficient of $\frac{x^n}{\underline{|n}}$.

The $(n+1)^{th}$ term of $\phi(y)$ gives $\{f(z)\}^n\phi^n(z)$,

the n^{th} „ $n\phi^{n-1}(z)(n-1)\{f(z)\}^{n-1}f'(z)$

the $(n-1)^{th}$ „ $n(n-1)\phi^{n-2}(z)\frac{1}{\underline{|2}}\frac{d}{dz}\{[f(z)]^2(n-2)[f(z)]^{n-3}f'(z)\}$

&c., „ &c.,

the second „ $\underline{|n}\phi'(z)\cdot\frac{1}{\underline{|n-1}}\frac{d^{n-2}}{dz^{n-2}}\{[f(z)]^{n-1}f'(z)\}$.

Therefore, the co-efficient of $\frac{x^n}{\underline{|n}}$ is

$$\{f(z)\}^n\phi^n(z)+(n-1)\frac{d}{dz}\{f(z)\}^n\phi^{n-1}(z)$$
$$+\frac{(n-1)(n-2)}{\underline{|2}}\frac{d}{dz}\left[\frac{d}{dz}\left\{f(z)\right\}^n\right]\phi^{n-2}(z)+\ldots+\frac{d^{n-2}}{dz^{n-2}}\left[\frac{d}{dz}\left\{f(z)\right\}^n\right]\phi'(z)$$

that is $\frac{d^{n-1}}{dz^{n-1}}[\{f(z)\}^n\phi'(z)]$, by Leibnitz's theorem. Therefore since the law has been proved for the first three terms, it holds universally.

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