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ALGEBRA HOMOMORPHISMS FROM REAL WEIGHTED L^1 ALGEBRAS

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Abstract We characterize algebra homomorphisms from the Lebesgue algebra $L^1_\omega(\mathbb{R})$ into a Banach algebra $\mathcal{A}$. As a consequence of this result, every bounded algebra homomorphism $\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{A}$ is approached through a uniformly bounded family of fractional homomorphisms, and the Hille–Yosida theorem for C_0-groups is proved.

Keywords: pseudo-resolvents; algebra homomorphisms; Weyl fractional calculus; integrated groups

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1. Introduction

Let $\mathbb{R}$ and $\mathbb{R}^+$ be the sets of real and positive real numbers. Widder's characterization of Laplace transforms of real-valued bounded functions states that, given $r \in C^{(\infty)}(0,\infty)$, there then exists $f \in L^\infty(0,\infty)$ such that

$$r(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} f(t)\,\mathrm{d}t, \quad \lambda > 0,$$

if and only if

$$\sup\left\{\lambda^{n+1}\frac{|r^{(n)}(\lambda)|}{n!} : \lambda > 0,\ n \in \mathbb{N}\right\} < \infty,$$

(see [**17**]). Vector-valued versions of this result have appeared recently (see [**1**, **10**, **12**]).

Let $\mathcal{A}$ be a Banach algebra (with or without identity). For $\Omega \subset \mathbb{R}$, a family $(r_\lambda)_{\lambda\in\Omega}$ of elements of $\mathcal{A}$ is called a *pseudo-resolvent* if the equation $r_\lambda - r_\mu = (\mu - \lambda) r_\lambda r_\mu$ holds for $\lambda, \mu \in \Omega$. Take $\omega \in \mathbb{R}^+ \cup \{0\}$ and let $L^1_\omega(\mathbb{R}^+)$ be the usual Banach algebra with norm given by

$$\|f\|_w := \int_0^\infty |f(t)|\mathrm{e}^{\omega t}\,\mathrm{d}t < \infty,$$

and the convolution product $f * g(t) := \int_0^t f(t-s)g(s)\,\mathrm{d}s$ with $t \geqslant 0$ as its product. Then $(\varepsilon_{-\lambda})_{\lambda\in(\omega,+\infty)}$ with $\varepsilon_{-\lambda}(t) := \mathrm{e}^{-\lambda t}$, $\lambda, t \in \mathbb{R}^+$, is a pseudo-resolvent in $L^1_\omega(\mathbb{R}^+)$ such that

$$\|\underbrace{\varepsilon_{-\lambda} * \cdots * \varepsilon_{-\lambda}}_{n \text{ times}}\|_\omega = \frac{1}{(\lambda-\omega)^n}, \quad \lambda \in (\omega,\infty),\ n \in \mathbb{N}.$$

Moreover, the set $(\varepsilon_{-\lambda})_{\lambda\in(\omega,+\infty)}$ is linearly dense in $L^1_\omega(\mathbb{R}^+)$.

The Kisyński theorem shows the equivalence between algebra homomorphisms $\phi : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ and a class of pseudo-resolvents (see [**5**], and extended versions in [**12**, Theorem 5.1], [**4**, Theorem 3.1] and [**7**, Theorem 1.1]).

Theorem 1.1 (Kisyński). *Let $\mathcal{A}$ be a Banach algebra, $\omega \geqslant 0$, let $(r_\lambda)_{\lambda\in(\omega,+\infty)}$ be a pseudo-resolvent in $\mathcal{A}$, and let*

$$M = \sup\{(\lambda-\omega)^n \|r_\lambda^n\| : n \in \mathbb{N},\ \lambda \in (\omega,\infty)\}.$$

The following conditions are then equivalent:

(i) $M < \infty$;

(ii) *there exists a continuous homomorphism $\phi : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ such that $\phi(\varepsilon_{-\lambda}) = r_\lambda$ for each $\lambda \in (\omega,\infty)$.*

Furthermore, if a continuous homomorphism $\phi : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ satisfying $\phi(\varepsilon_{-\lambda}) = r_\lambda$ for each $\lambda \in (\omega,\infty)$ exists, then it is unique and $\|\phi\| = M$.

This theorem has many interesting applications (see, for example, [**6**,**12**]). It is equivalent to the Hille–Yosida theorem [**5**] and it may be proved by using the Yosida approximation [**3**].

A different vector-valued extension of Widder's characterization is given in [**1**,**10**,**12**]. These references use Lipschitz functions, integration theory and the vector-valued Laplace transform. The result is called an 'integrated version of Widder's theorem'. In fact, both versions are equivalent (see, for example, [**12**, Corollary 7.2]).

In this paper, we consider the Banach algebra $L^1_\omega(\mathbb{R})$ (in Theorem 1.1) and prove a similar result (Theorem 2.2). This point of view is closer to another interesting problem: let D be a convolution algebra of functions or measures on $\mathbb{R}$, let $\mathcal{A}$ be a Banach algebra and let $\Phi : D \to \mathcal{A}$ be a continuous homomorphism of algebras (Φ is called a representation of D in the case when $\mathcal{A}$ is the set of linear and bounded operators on a Banach space X). We investigate the conditions under which a decomposition

$$\Phi(F) = \phi_+(F_+) + \phi_-(F_-), \quad F_+(t) := F(t),\ F_-(t) := F(-t),\ t \geqslant 0,$$

is possible, where $\phi_\pm : D^+ \to \mathcal{A}$ are continuous homomorphisms of a suitable convolution algebra D^+ on $[0,\infty)$. In the cases $D = C_{\mathrm{c}}(\mathbb{R})$ and $D^+ = C_{\mathrm{c}}(\mathbb{R}^+)$ (sets of infinitely differentiable functions with compact support on $\mathbb{R}$ and $\mathbb{R}^+$, respectively), the decomposition of Φ is not always possible [**13**]. We solve the cases $D = L^1_\omega(\mathbb{R})$ and $D^+ = L^1_\omega(\mathbb{R}^+)$ (see Theorem 2.1).

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As in the $L^1_\omega(\mathbb{R}^+)$ case [**15**], we show that a bounded homomorphism $\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{A}$ is equivalent to a uniformly bounded family of algebra homomorphisms $\Phi_\alpha : AC^{(\alpha)}_\omega(\mathbb{R}) \to \mathcal{A}$ for any $\alpha > 0$ (Theorem 3.2). In [**14, 15**], Miana introduced some fractional algebras for functions defined in $\mathbb{R}^+$. In [**9**], fractional Banach algebras for functions defined in $\mathbb{R}$ are considered in the context of quasi-multipliers of a Banach algebra and integrated groups of linear and bounded operators in a Banach space. The algebras $AC^{(\alpha)}_\omega(\mathbb{R})$ are examples of these fractional Banach algebras. Integrated groups in a Banach algebra give the connection between the Kisyński theorem and the integrated version of Widder's theorem (Theorem 3.4). In the last section, we prove the Hille–Yosida theorem for C_0-groups. In fact, both results are equivalent (see similar ideas in [**5**]).

2. The Kisyński theorem in $L^1_\omega(\mathbb{R})$

Let $\omega \in \mathbb{R}^+ \cup \{0\}$ and let $L^1_\omega(\mathbb{R})$ be the usual Banach algebra with the norm given by

$$\|F\|_w := \int_{-\infty}^{\infty} |F(t)| \mathrm{e}^{\omega|t|} \, \mathrm{d}t < +\infty,$$

and the convolution product

$$F * G(t) := \int_{-\infty}^{\infty} F(t-s) G(s) \, \mathrm{d}s,$$

with $t \in \mathbb{R}$. Note that $(\varepsilon_{-\lambda})_{|\lambda|>\omega} \subset L^1_\omega(\mathbb{R})$ (where $\varepsilon_{-\lambda}(t) := \mathrm{e}^{-\lambda t}\chi_{(0,\infty)}(t)$ for $\lambda > \omega$ and $\varepsilon_{-\lambda}(t) := \mathrm{e}^{-\lambda t}\chi_{(-\infty,0)}(t)$ for $\lambda < -\omega$ and $t \in \mathbb{R}$). It is straightforward to check that $(\tilde{\varepsilon}_{-\lambda})_{|\lambda|>\omega}$ (where $\tilde{\varepsilon}_{-\lambda} := \varepsilon_{-\lambda}$ for $\lambda > \omega$ and $\tilde{\varepsilon}_{-\lambda} := -\varepsilon_{-\lambda}$ for $\lambda < -\omega$) is a pseudo-resolvent and is linearly dense in $L^1_\omega(\mathbb{R})$.

Take $F \in L^1_\omega(\mathbb{R})$, and we write $F_+, F_- \in L^1_\omega(\mathbb{R}^+)$ given by $F_+(t) := F(t)$ and $F_-(t) := F(-t)$ for $t \geqslant 0$. It is straightforward to prove that

$$F * G(t) = \begin{cases} (F_+ * G_+ + F_- \circ G_+ + G_- \circ F_+)(t), & t \geqslant 0, \\ (F_- * G_- + F_+ \circ G_- + G_+ \circ F_-)(-t), & t \leqslant 0, \end{cases} \tag{2.1}$$

where $f \circ g \in L^1_\omega(\mathbb{R}^+)$, $f \circ g(t) := \int_t^\infty f(s-t) g(s) \, \mathrm{d}s$, for $t \geqslant 0$ and $f, g \in L^1_\omega(\mathbb{R}^+)$.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra, and let $\phi_1, \phi_2 : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ be two continuous homomorphisms. We define $\Phi := L^1_\omega(\mathbb{R}) \to \mathcal{A}$ by $\Phi(F) := \phi_1(F_+) + \phi_2(F_-)$ with $F \in L^1_\omega(\mathbb{R})$. The following conditions are then equivalent:*

(i) *Φ is a continuous algebra homomorphism;*

(ii) *$\phi_1(f)\phi_2(g) = \phi_1(g \circ f) + \phi_2(f \circ g)$ for $f, g \in L^1_\omega(\mathbb{R}^+)$.*

Proof. (i) $\Rightarrow$ (ii). Since $\Phi(F)\Phi(G) = \Phi(F * G) = \phi_1((F * G)_+) + \phi_2((F * G)_-)$ for $F, G \in L^1_\omega(\mathbb{R})$, we apply (2.1) to obtain

$$\begin{aligned} \phi_2(F_-)\phi_1(G_+) &+ \phi_1(F_+)\phi_2(G_-) \\ &= \phi_1(F_- \circ G_+) + \phi_1(G_- \circ F_+) + \phi_2(F_+ \circ G_-) + \phi_2(G_+ \circ F_-). \end{aligned}$$

Take $f,g \in L^1_\omega(\mathbb{R}^+)$ and we define $F(t) := f(t)\chi_{(0,\infty)}(t)$ and $G(t) = g(t)\chi_{(-\infty,0)}(t)$ with $t \in \mathbb{R}$. Then

$$\phi_1(f)\phi_2(g) = \phi_1(g \circ f) + \phi_2(f \circ g).$$

(ii) $\Rightarrow$ (i). It is clear that Φ is a linear and continuous map. Take $F,G \in L^1_\omega(\mathbb{R})$. Since

$$\begin{aligned}\phi_2(F_-)\phi_1(G_+) &+ \phi_1(F_+)\phi_2(G_-)\\ &= \phi_1(F_- \circ G_+) + \phi_1(G_- \circ F_+) + \phi_2(F_+ \circ G_-) + \phi_2(G_+ \circ F_-),\end{aligned}$$

we obtain $\Phi(F * G) = \Phi(F)\Phi(G)$. □

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra, $\omega \geqslant 0$, let $(r_\lambda)_{|\lambda|>\omega}$ be a pseudo-resolvent in $\mathcal{A}$ and let*

$$M = \sup\{(|\lambda| - \omega)^n \|r_\lambda^n\| : n \in \mathbb{N},\ |\lambda| > \omega\}.$$

Then the following conditions are equivalent:

(i) *$M < +\infty$;*

(ii) *there exists a continuous algebra homomorphism*

$$\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{A}$$

such that $\Phi(\varepsilon_{-\lambda}) = r_\lambda$ for each $\lambda > \omega$ and $\Phi(\varepsilon_{-\lambda}) = -r_\lambda$ for each $\lambda < -\omega$.

If a continuous algebra homomorphism $\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{A}$ satisfying $\Phi(\varepsilon_{-\lambda}) = r_\lambda$ for each $\lambda > \omega$ and $\Phi(\varepsilon_{-\lambda}) = -r_\lambda$ for each $\lambda < -\omega$ exists, then it is unique and $\|\Phi\| = M$.

Proof. We consider two pseudo-resolvents $(r_\lambda)_{\lambda>\omega}$ and $(\tilde{r}_\lambda)_{\lambda>\omega}$, where $\tilde{r}_\lambda = -r_{-\lambda}$ for $\lambda > \omega$.

(i) $\Rightarrow$ (ii). By Theorem 1.1, there exist $\phi_1, \phi_2 : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ such that $\phi_1(\varepsilon_{-\lambda}) = r_\lambda$ and $\phi_2(\varepsilon_{-\lambda}) = \tilde{r}_\lambda$ for $\lambda > \omega$. We define $\Phi(F) := \phi_1(F_+) + \phi_2(F_-)$ with $F \in L^1_\omega(\mathbb{R})$. By Theorem 2.1, it is sufficient to prove that

$$\phi_1(f)\phi_2(g) = \phi_1(g \circ f) + \phi_2(f \circ g), \tag{2.2}$$

for $f,g \in L^1_\omega(\mathbb{R}^+)$. Take $f = \varepsilon_{-\lambda}$ and $g = \varepsilon_{-\mu}$ with $\lambda, \mu > \omega$. Since $(r_\lambda)_{|\lambda|>\omega}$ is a pseudo-resolvent in $\mathcal{A}$, and

$$\varepsilon_{-\lambda} \circ \varepsilon_{-\mu} = \frac{1}{\lambda + \mu}\varepsilon_\mu \quad \text{for } \lambda, \mu > \omega,$$

we have

$$\phi_1(\varepsilon_{-\lambda})\phi_2(\varepsilon_{-\mu}) = -r_\lambda r_{-\mu} = \frac{1}{\lambda+\mu}(r_\lambda - r_{-\mu}) = \phi_1(\varepsilon_{-\mu} \circ \varepsilon_{-\lambda}) + \phi_2(\varepsilon_{-\lambda} \circ \varepsilon_{-\mu}).$$

Since $(\varepsilon_{-\lambda})_{\lambda>\omega}$ is linearly dense in $L^1_\omega(\mathbb{R}^+)$, equality (2.2) holds for any $f,g \in L^1_\omega(\mathbb{R}^+)$. Take $\lambda > \omega$. Then $\Phi(\varepsilon_{-\lambda}) = \phi_1(\varepsilon_{-\lambda}) = r_\lambda$ and if $\lambda < -\omega$, then $\Phi(\varepsilon_{-\lambda}) = \phi_2(\varepsilon_{-\lambda}) = -r_\lambda$.

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(ii) $\Rightarrow$ (i). We consider $\phi_1, \phi_2 : L^1_\omega(\mathbb{R}^+) \to \mathcal{A}$ defined by $\phi_1(f) := \Phi(F_1)$ and $\phi_2(f) := \Phi(F_2)$, where $F_1(t) := f(t)\chi_{(0,\infty)}(t)$ and $F_2(t) := f(-t)\chi_{(-\infty,0)}(t)$ for $t \in \mathbb{R}$. By Theorem 1.1,

$$M_1 = \sup\{(\lambda - \omega)^n \|r^n_\lambda\| : n \in \mathbb{N},\ |\lambda| > \omega\} < \infty,$$
$$M_2 = \sup\{(\lambda - \omega)^n \|\tilde{r}^n_\lambda\| : n \in \mathbb{N},\ |\lambda| > \omega\} < \infty.$$

We define $M := \max(M_1, M_2)$ and obtain (i). Again using $\Phi(F) := \phi_1(F_+) + \phi_2(F_-)$, $\|\Phi\| = \max(\|\phi_1\|, \|\phi_2\|)$ and, by Theorem 1.1, $\max(\|\phi_1\|, \|\phi_2\|) = \max(M_1, M_2) = M$, we finish the proof. □

3. Fractional homomorphisms and integrated groups

In this section we recall some definitions and properties of Weyl fractional calculus. The reader will find a nice introduction and comments in the monograph [**16**].

Let $\mathcal{D}$ be the set of functions of compact support on $\mathbb{R}$ and infinitely differentiable and take $F \in \mathcal{D}$. *Weyl fractional integrals* of F of order $\alpha > 0$, $W^{-\alpha}_+ F$, $W^{-\alpha}_- F$, are defined by

$$W^{-\alpha}_+ F(t) := \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} F(s)\,\mathrm{d}s,$$
$$W^{-\alpha}_- F(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1} F(s)\,\mathrm{d}s,$$

with $t \in \mathbb{R}$ (see, for example, [**16**]). *Weyl fractional derivatives* of F of order $\alpha > 0$, $W^\alpha_+ F$, $W^\alpha_- F$, are given by

$$W^\alpha_+ F := (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} W^{-(n-\alpha)}_+ F, \qquad W^\alpha_- F := \frac{\mathrm{d}^n}{\mathrm{d}t^n} W^{-(n-\alpha)}_- F,$$

with $n \in \mathbb{N}$ and $n > \alpha$ [**16**]. In the case $\alpha = n$, we have

$$(-1)^n W^n_+ F := \frac{\mathrm{d}^n}{\mathrm{d}t^n} F = W^n_- F.$$

Note that $W^\alpha_+(\varepsilon_{-\lambda})(t) = \lambda^\alpha \varepsilon_{-\lambda}(t)$ with $\lambda, t > 0$ and $W^\alpha_-(\varepsilon_{-\lambda})(t) = (-\lambda)^\alpha \varepsilon_{-\lambda}(t)$ for $\lambda, t < 0$.

We define the function $W^\alpha_0(F)$ for $\alpha > 0$ by

$$W^\alpha_0 F(t) := \begin{cases} W^\alpha_- F(t), & t < 0, \\ \mathrm{e}^{\mathrm{i}\pi\alpha} W^\alpha_+ F(t), & t \geqslant 0, \end{cases}$$

for $f \in \mathcal{D}$. Note that, for $\alpha = n$, $\mathrm{e}^{\mathrm{i}\pi\alpha} = (-1)^n$ and the function $W^n_0 F$ is continuous at $t = 0$. Now we consider some fractional Banach algebras that were introduced in [**9**].

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Theorem 3.1. *Suppose $\alpha > 0$ and $\omega \geqslant 0$. If $F \in \mathcal{D}$, then the formula*

$$\|F\|_{(\alpha,\omega)} := \frac{1}{\Gamma(\alpha+1)} \int_{-\infty}^{\infty} |t|^{\alpha} \mathrm{e}^{\omega|t|} |W_0^{\alpha} F(t)| \,\mathrm{d}t$$

defines a norm in $\mathcal{D}$ such that

$$\|F * G\|_{(\alpha,\omega)} \leqslant (2^{\alpha+1}+1)\|F\|_{(\alpha,\omega)}\|G\|_{(\alpha,\omega)},$$

with $F, G \in \mathcal{D}$. We denote by $AC_{\omega}^{(\alpha)}(\mathbb{R})$ the Banach algebra obtained by the completion of $\mathcal{D}$ in this norm; $AC_{\omega}^{(\alpha)}(\mathbb{R}) \hookrightarrow L_{\omega}^1(\mathbb{R})$ and

$$\|F\|_{\omega} \leqslant \|F\|_{(\alpha,\omega)}, \quad F \in AC_{\omega}^{(\alpha)}(\mathbb{R}).$$

Proof. See the proofs of [**9**, Theorem 1.8] and [**15**, Theorem 2.1] to obtain the constant $2^{\alpha+1}+1$. Note that

$$\begin{aligned}\int_0^{\infty} \mathrm{e}^{\omega t}|F(t)\| \,\mathrm{d}t &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \int_0^{s} (s-t)^{\alpha-1} \mathrm{e}^{\omega t} |W_+^{\alpha} F(s)| \,\mathrm{d}t \,\mathrm{d}s \\ &\leqslant \frac{1}{\Gamma(\alpha+1)} \int_0^{\infty} s^{\alpha} \mathrm{e}^{\omega s} |W_+^{\alpha} F(s)| \,\mathrm{d}s,\end{aligned}$$

where we use the Fubini theorem and $\omega \geqslant 0$. Working in the same way on $(-\infty, 0]$ we may obtain the inequality $\|F\|_{\omega} \leqslant \|F\|_{(\alpha,\omega)}$ for $F \in AC_{\omega}^{(\alpha)}(\mathbb{R})$. □

For $\alpha = 0$, $AC_{\omega}^{(0)}(\mathbb{R}) = L_{\omega}^1(\mathbb{R})$ and $AC_{\omega}^{(\beta)}(\mathbb{R}) \hookrightarrow AC_{\omega}^{(\alpha)}(\mathbb{R})$ holds with $0 \leqslant \alpha \leqslant \beta$. Functions $(\varepsilon_{-\lambda})_{|\lambda|>\omega}$ belong to $AC_{\omega}^{(\alpha)}(\mathbb{R})$ with $\alpha \geqslant 0$ (see the similar result [**2**, Lemma II.2]) and

$$\|\varepsilon_{-\lambda}\|_{(\alpha,\omega)} = \frac{|\lambda|^{\alpha}}{(|\lambda|-\omega)^{\alpha+1}}, \quad |\lambda| > \omega.$$

Note that $\|\varepsilon_{-\lambda}\|_{(\alpha,\omega)} \to \|\varepsilon_{-\lambda}\|_{\omega}$ when $\alpha \to 0^+$ and $|\lambda| > \omega$,

$$\underbrace{\varepsilon_{-\lambda} * \cdots * \varepsilon_{-\lambda}}_{n \text{ times}} \in AC_{\omega}^{(\alpha)}(\mathbb{R})$$

and

$$\|\underbrace{\varepsilon_{-\lambda} * \cdots * \varepsilon_{-\lambda}}_{n \text{ times}}\|_{(\alpha,\omega)} \leqslant (2^{\alpha+1}-1)^{n-1}\|\varepsilon_{-\lambda}\|_{(\alpha,\omega)}^n = (2^{\alpha+1}-1)^{n-1} \frac{|\lambda|^{n\alpha}}{(|\lambda|-\omega)^{n(\alpha+1)}},$$

for $|\lambda| > \omega$ (see the constant $(2^{\alpha+1}-1)^{n-1}$ in [**15**, Theorem 2.1]).

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra and let $\omega \geqslant 0$.*

(i) *If $\Phi : L_{\omega}^1(\mathbb{R}) \to \mathcal{A}$ is a continuous homomorphism of Banach algebras, then for every $\alpha > 0$ the restriction $\Phi_{\alpha} := \Phi|_{AC_{\omega}^{(\alpha)}(\mathbb{R})} : AC_{\omega}^{(\alpha)}(\mathbb{R}) \to \mathcal{A}$ is a continuous homomorphism of Banach algebras such that $\|\Phi_{\alpha}\| \leqslant \|\Phi\|$.*

(ii) *Conversely, if for each $\alpha > 0$ there exist continuous homomorphisms $\Phi_\alpha : AC_\omega^{(\alpha)}(\mathbb{R}) \to \mathcal{A}$ of Banach algebras such that $\Phi_\alpha(\varepsilon_{-\lambda})$ does not depend on α for each $|\lambda| > \omega$ and $\limsup_{\alpha\to 0^+} \|\Phi_\alpha\| < \infty$, then there exists a unique continuous homomorphism $\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{A}$ such that $\Phi(\varepsilon_{-\lambda}) = \Phi_\alpha(\varepsilon_{-\lambda})$ for each $|\lambda| > \omega$ and $\|\Phi\| \leqslant \limsup_{\alpha\to 0^+} \|\Phi_\alpha\|$.*

Proof. (i) Take $F \in AC_\omega^{(\alpha)}(\mathbb{R})$. Since $AC_\omega^{(\alpha)}(\mathbb{R}) \hookrightarrow L^1_\omega(\mathbb{R})$ and $\|F\|_\omega \leqslant \|F\|_{(\alpha,\omega)}$ (see Theorem 3.1) we have

$$\|\Phi_\alpha(F)\| = \|\Phi(F)\| \leqslant \|\Phi\|\,\|F\|_\omega \leqslant \|\Phi\|\,\|F\|_{(\alpha,\omega)}.$$

Then $\|\Phi_\alpha\| \leqslant \|\Phi\|$.

(ii) We define $r_\lambda := \Phi_\alpha(\varepsilon_{-\lambda})$ for $\lambda > \omega$ and $r_\lambda := -\Phi_\alpha(\varepsilon_{-\lambda})$ for $\lambda < -\omega$. The family $(r_\lambda)_{|\lambda|>\omega}$ is well defined, a pseudo-resolvent in $\mathcal{A}$ and

$$\|r_\lambda^n\| \leqslant \|\Phi_\alpha\|\,\|\underbrace{\varepsilon_{-\lambda} * \cdots * \varepsilon_{-\lambda}}_{n \text{ times}}\|_{(\alpha,\omega)} \leqslant \|\Phi_\alpha\|(2^{\alpha+1}-1)^{n-1}\frac{|\lambda|^{n\alpha}}{(|\lambda|-\omega)^{n(\alpha+1)}},$$

for $n \in \mathbb{N}$ and $|\lambda| > \omega$. Taking $\alpha \to 0^+$, we obtain

$$\sup\{(|\lambda|-\omega)^n\|r_\lambda^n\| : n \in \mathbb{N},\ |\lambda| > \omega\} \leqslant \limsup_{\alpha\to 0^+}\|\Phi_\alpha\| < \infty.$$

By Theorem 2.2, there exists a unique homomorphism $\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{A}$ such that

$$\Phi(\varepsilon_{-\lambda}) = r_\lambda = \Phi_\alpha(\varepsilon_{-\lambda}) \quad \text{for } |\lambda| > \omega$$

and $\|\Phi\| \leqslant \limsup_{\alpha\to 0^+}\|\Phi_\alpha\|$. □

Remark. Note that $\|\Phi\| = \limsup_{\alpha\to 0^+}\|\Phi_\alpha\| = \sup_{\alpha>0}\|\Phi_\alpha\|$ since $\|\Phi_\beta\| \leqslant \|\Phi_\alpha\|$ with $\beta \geqslant \alpha > 0$.

A different extension of the Widder theorem was taken by Arendt *et al.* [**1**] and Hieber [**11**]. To find a vector-valued version of Widder's theorem on arbitrary Banach space, they consider Lipschitz functions which are integrated from bounded functions in some sense. Then integrated semigroups and groups appear (see [**1**, **11**]). Recall that an α-times integrated semigroup $s_\alpha(\cdot) : [0,\infty) \to \mathcal{A}$ is a continuous mapping with $s_\alpha(0) = 0$ and

$$s_\alpha(t)s_\alpha(s) = \frac{1}{\Gamma(\alpha)}\left(\int_t^{t+s}(t+s-r)^{\alpha-1}s_\alpha(r)\,\mathrm{d}r - \int_0^s (t+s-r)^{\alpha-1}s_\alpha(r)\,\mathrm{d}r\right),$$

with $t, s \geqslant 0$.

Definition 3.3. For any $\alpha > 0$, an α-times integrated group $s_\alpha(\cdot) : \mathbb{R} \to \mathcal{A}$ is a continuous mapping with $s_\alpha(0) = 0$, $s_\alpha(\cdot) : [0,\infty) \to \mathcal{A}$, $\tilde{s}_\alpha(\cdot) : [0,\infty) \to \mathcal{A}$, where $\tilde{s}_\alpha(t) := s_\alpha(-t)$ for $t \geqslant 0$ are α-times integrated semigroups and, if $t < 0 < s$,

$$s_\alpha(t)s_\alpha(s) = \frac{1}{\Gamma(\alpha)}\left(\int_{t+s}^{s}(r-s-t)^{\alpha-1}s_\alpha(r)\,\mathrm{d}r + \int_t^0 (t+s-r)^{\alpha-1}s_\alpha(r)\,\mathrm{d}r\right)$$

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with $t+s \geqslant 0$, and

$$s_\alpha(t)s_\alpha(s) = \frac{1}{\Gamma(\alpha)}\left(\int_t^{t+s}(t+s-r)^{\alpha-1}s_\alpha(r)\,\mathrm{d}r + \int_0^s (r-t-s)^{\alpha-1}s_\alpha(r)\,\mathrm{d}r\right)$$

with $t+s \leqslant 0$.

The set of Bocher–Riesz functions $(R_t^\alpha)_{t\in\mathbb{R}}$, $R_0^\alpha = 0$ and

$$R_t^\alpha(s) := \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\chi_{(0,t)}(s), & t>0,\\[2mm] \dfrac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}\chi_{(t,0)}(s), & t<0,\end{cases}$$

for $\alpha > 0$ is a canonical example of an α-times integrated group in $L^1_\omega(\mathbb{R})$ and

$$\|R_t^\alpha\|_\omega \leqslant \frac{|t|^\alpha \mathrm{e}^{|t|\omega}}{\Gamma(\alpha+1)}, \quad t \in \mathbb{R}.$$

Now we can add more conditions to the Kisyński theorem in $L^1_\omega(\mathbb{R})$.

Theorem 3.4. *Let $\mathcal{A}$ be a Banach algebra, $\omega \geqslant 0$, let $(r_\lambda)_{|\lambda|>\omega}$ be a pseudo-resolvent in $\mathcal{A}$, and let*

$$M = \sup\{(|\lambda|-\omega)^n\|r_\lambda^n\| : n\in\mathbb{N},\ |\lambda|>\omega\}.$$

Then the following conditions are equivalent:

(i) *$M < \infty$;*

(ii) *there exists a continuous algebra homomorphism $\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{A}$ such that $\Phi(\varepsilon_{-\lambda}) = r_\lambda$ for each $\lambda > \omega$ and $\Phi(\varepsilon_{-\lambda}) = -r_\lambda$ for each $\lambda < -\omega$;*

(iii) *for any $\alpha > 0$ there exists an α-times integrated group $(s_\alpha(t))_{t\in\mathbb{R}} \subset \mathcal{A}$ and $C > 0$ such that*

$$\|s_\alpha(t)\| \leqslant \frac{C}{\Gamma(\alpha+1)}|t|^\alpha \mathrm{e}^{\omega|t|} \quad \textit{with } t\in\mathbb{R}$$

and

$$r_\lambda = \lambda^\alpha\int_0^\infty \mathrm{e}^{-\lambda t}s_\alpha(t)\,\mathrm{d}t \qquad \textit{with } \lambda>\omega,$$
$$r_\lambda = -(-\lambda)^\alpha\int_{-\infty}^0 \mathrm{e}^{-\lambda t}s_\alpha(t)\,\mathrm{d}t \quad \textit{with } \lambda<-\omega;$$

(iv) *if for each $\alpha > 0$ there exists a continuous homomorphisms $\Phi_\alpha : AC_\omega^{(\alpha)}(\mathbb{R}) \to \mathcal{A}$ of Banach algebras such that $\Phi_\alpha(\varepsilon_{-\lambda}) = r_\lambda$ for $\lambda > \omega$, $\Phi_\alpha(\varepsilon_{-\lambda}) = -r_\lambda$ for $\lambda < -\omega$ and $\sup_{\alpha>0}\|\Phi_\alpha\| < \infty$.*

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Furthermore, if a continuous algebra homomorphism $\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{A}$ such that $\Phi(\varepsilon_{-\lambda}) = r_\lambda$ for $\lambda > \omega$ and $\Phi(\varepsilon_{-\lambda}) = -r_\lambda$ for $\lambda < -\omega$ exists, then it is unique, $\Phi(F) = \Phi_\alpha(F)$ for $F \in AC^{(\alpha)}_\omega(\mathbb{R})$ and

$$M = \|\Phi\| = \sup_{\alpha>0} \|\Phi_\alpha\| = \inf\left\{C : \|s_\alpha(t)\| \leqslant C\frac{|t|^\alpha \mathrm{e}^{\omega|t|}}{\Gamma(\alpha+1)},\ t \in \mathbb{R}\right\}.$$

Proof. This is similar to the proof in [**15**, Theorem 4.2]. □

4. The Hille–Yosida theorem

Let $\mathcal{A}$ be a Banach algebra and let $\mathrm{Mul}(\mathcal{A})$ be the set of linear and bounded operators on $\mathcal{A}$, $T : \mathcal{A} \to \mathcal{A}$, which verify $T(ab) = aT(b)$ for $a, b \in \mathcal{A}$. These operators are called *multipliers* of $\mathcal{A}$ (for more details see [**6**]).

Take X, a Banach space, and $\mathcal{B}(X)$, the algebra of linear and bounded operators on X. A homomorphism from $\mathcal{A}$ into $\mathcal{B}(X)$ is called a *representation*. Suppose that $\mathcal{A}$ is commutative and has a bounded approximate identity $\{e_n\}_{n\in\mathbb{N}}$. Given $\Phi : \mathcal{A} \to \mathcal{B}(X)$, a continuous representation, the *regularity space* $\mathcal{R}_\Phi$ is the closed linear span of $\{\Phi(a)x \mid a \in \mathcal{A},\ x \in X\}$ and [**6**]

$$\mathcal{R}_\Phi = \left\{x \in X \;\middle|\; \lim_{n\to\infty} \Phi(e_n)x = x\right\}.$$

By the Cohen theorem, $\mathcal{R}_\Phi = \{\Phi(a)x \mid a \in \mathcal{A},\ x \in X\}$, and there exists a unique representation $\hat{\Phi} : \mathrm{Mul}(\mathcal{A}) \to \mathcal{B}(\mathcal{R}_\Phi)$ such that

$$\hat{\Phi}(L_a)x = \Phi(a)x, \quad x \in \mathcal{R}_\Phi,$$

where $L_a(b) := ab$ for $a, b \in \mathcal{A}$ (see [**6**, Theorem 2.4]).

Take the case $\mathcal{A} = L^1_\omega(\mathbb{R})$ and $\mathrm{Mul}(L^1_\omega(\mathbb{R})) = M_\omega(\mathbb{R})$ ($M_\omega(\mathbb{R})$ is the collection of all Borel measures μ on $\mathbb{R}$ for which

$$\int_{-\infty}^{\infty} \mathrm{e}^{\omega|t|}\,\mathrm{d}|\mu|(t) < \infty,$$

where $|\mu|$ denotes the total variation of μ). The Dirac measure is a one-parameter C_0-group $(\delta_t)_{t\in\mathbb{R}}$ of convolution operators on $L^1_\omega(\mathbb{R})$ (i.e. $\delta_t * \delta_s = \delta_{t+s}$ for $t, s \in \mathbb{R}$ and $\delta_t f \to f$ in $L^1_\omega(\mathbb{R})$ when $t \to 0$). The next theorem is well known (see the case $\mathbb{R}^+$ in [**6**, Theorem 3.3]).

Theorem 4.1. *Let X be a Banach space and let $\Phi : L^1_\omega \to \mathcal{B}(X)$ be a continuous representation. For each $t \in \mathbb{R}$, set $S(t) := \hat{\Phi}(\delta_t)$. Then $(S(t))$ is a C_0-group on $\mathcal{R}_\phi$ such that*

$$\|S(t)\| \leqslant \|\hat{\Phi}\|\mathrm{e}^{\omega|t|}, \quad t \in \mathbb{R},$$

and, for $\mu \in M_\omega(\mathbb{R})$,

$$\hat{\Phi}(\mu) = \int_{-\infty}^{\infty} S(t)x\,\mathrm{d}\mu(t), \quad x \in \mathcal{R}_\Phi.$$

Let $R \equiv (R_\lambda)_{|\lambda|>\omega}$ be a pseudo-resolvent on $\mathcal{B}(X)$. It is easy to check that the kernel and the range of R_λ are independent of λ (we denote them by $\ker(R)$ and $\mathrm{Im}(R)$). Note that $(R_\lambda)_{|\lambda|>\omega}$ is the resolvent of a densely defined closed operator $(A, D(A))$ (i.e. $R_\lambda = (\lambda - A)^{-1}$) if and only if $\ker(R) = \{0\}$ and $\overline{\mathrm{Im}(R)} = X$ [**8**, Proposition III.4.6].

The Hille–Yosida theorem for C_0-groups can be found in [**8**, p. 79] (definitions of the infinitesimal generator, C_0-groups and properties can be also be found therein).

Remark (the Hille–Yosida theorem). Let $\omega \geqslant 0$ and $M > 0$. For a linear operator $(A, D(A))$ on a Banach space X, the following properties are equivalent:

(i) $(A, D(A))$ generates a C_0-group $(S(t))_{t\in\mathbb{R}}$ such that $\|S(t)\| \leqslant M\mathrm{e}^{\omega|t|}$ with $t \in \mathbb{R}$;

(ii) $(A, D(A))$ is closed, densely defined and, for every $\lambda \in \mathbb{R}$ with $|\lambda| > \omega$, we have $\lambda \in \rho(A)$ and
$$\|(|\lambda| - \omega)^n R(\lambda, A)^n\| \leqslant M,$$
for all $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). We define $\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{B}(X)$,
$$\Phi(F)x = \int_{-\infty}^{\infty} F(t)S(t)x\,\mathrm{d}t, \quad x \in X.$$
Since $\Phi(\varepsilon_{-\lambda}) = R(\lambda, A)$ with $\lambda > \omega$ and $\Phi(\varepsilon_{-\lambda}) = -R(\lambda, A)$ with $\lambda < -\omega$, we apply Theorem 2.2 to obtain (ii).

(ii) $\Rightarrow$ (i). By Theorem 2.2 there exists $\Phi : L^1_\omega(\mathbb{R}) \to \mathcal{B}(X)$ such that $\Phi(\varepsilon_{-\lambda}) = R(\lambda, A)$ with $\lambda > \omega$ and $\Phi(\varepsilon_{-\lambda}) = -R(\lambda, A)$ with $\lambda < -\omega$. Since $(n\varepsilon_{-n})_{n>\omega}$ is a bounded approximate identity, then $\mathcal{R}_\Phi = \overline{\mathrm{Im}(R_\lambda)} = X$. By Theorem 4.1, there exists a C_0-group on X such that $\|S(t)\| \leqslant M\mathrm{e}^{\omega|t|}$ for $t \in \mathbb{R}$. It is straightforward to check that $(A, D(A))$ is the infinitesimal generator of $(S(t))_{t\in\mathbb{R}}$. □

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