# AN INEQUALITY FOR POLYNOMIALS

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Throughout this note let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n. The following results are immediate.

THEOREM A. *For* $R \geqq 1$

$$(1) \qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta \leqq R^{2n} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta$$

THEOREM B. *If* $p(z)$ *has no zeros in* $|z| < 1$, *then*

$$(2) \qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta \leqq \frac{R^{2n}+1}{2} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta$$

*for* $R \geqq 1$.

Theorem B was proved by Q. I. Rahman [2]. This result is the best possible and equality holds for $p(z) = \alpha+\beta z^n$ where $|\alpha| = |\beta|$.

We prove the following

THEOREM 1. *If* $p(z)$ *has no zeros in* $|z| < k$, $k \geqq 1$, *then*

$$(3) \qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta \leqq \frac{R^{2n}+k^{2n}}{1+k^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta$$

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for $R \geqq k^2$. And

$$(4) \qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta \leqq R^n \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta$$

for $1 \leqq R \leqq k^2$.

If $p(z) = \alpha k^n + \beta z^n$, where $|\alpha| = |\beta|$, then (3) becomes an equality.

Proof of Theorem 1. To start with let us suppose that $p(z)$ has all its zeros on $|z| = k$, so that $p(z) = a_n \prod_{t=1}^{n} (z - z_t)$.

where $|z_1| = |z_2| = \ldots = |z_n| = k$. Hence if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, then $|a_\nu| = k^{n-2\nu} |a_{n-\nu}|$ and

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta = 2\pi \sum_{\nu=0}^{n} |a_\nu|^2 R^{2\nu}$$

$$= \pi \sum_{\nu=0}^{n} \frac{R^{2n-2\nu} + k^{2n-4\nu} R^{2\nu}}{1 + k^{2n-4\nu}}$$

$$\times (|a_{n-\nu}|^2 + |a_\nu|^2) .$$

Now, if $0 \leqq \nu \leqq n$ then

$$\frac{R^{2n} + k^{2n}}{1 + k^{2n}} - \frac{k^{2n-4\nu} R^{2\nu} + R^{2n-2\nu}}{1 + k^{2n-4\nu}}$$

$$= \frac{(R^{2\nu} - k^{4\nu})(R^{2n-2\nu} - 1)k^{2n-4\nu} + (R^{2\nu} - 1)(R^{2n-2\nu} - k^{4n-}}{(1 + k^{2n})(1 + k^{2n-4\nu})}$$

$$\geqq 0$$



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for $R \geqq k^2$. Thus the greatest of the quantities

$$\frac{R^{2n-2\nu} + k^{2n-4\nu} R^{2\nu}}{1 + k^{2n-4\nu}} \quad \text{for } \nu = 0, 1, 2, \ldots, n \text{ is } \frac{R^{2n} + k^{2n}}{1 + k^{2n}}$$

if $R \geqq k^2$, and

$$(5) \qquad \int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta \leqq \frac{R^{2n} + k^{2n}}{1 + k^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta .$$

If $q(z) = (\frac{z}{k})^n \overline{p(\frac{k^2}{\bar{z}})}$, then $|q(z)| = |p(z)|$ for $|z| = k$. If $p(z) \neq 0$ in $|z| < k$ it follows that $|q(z)| \leqq |p(z)|$ for $|z| < k$. On replacing $z$ by $\frac{k^2}{\bar{z}}$ we deduce that for $|z| > k$, $|p(z)| \leqq |q(z)|$. Then it follows from a known result [1: p. 88. Problem 26] that if $p(z) \neq 0$ in $|z| < k$, then for $0 \leqq \eta \leqq 2\pi$ all the zeros of $p(z) + e^{i\eta} q(z)$ lie on the circle $|z| = k$. Applying (5) to the polynomial $p(z) + e^{i\eta} q(z)$ we get

$$\int_0^{2\pi} |p(Re^{i\theta}) + e^{i\eta} q(Re^{i\theta})|^2 d\theta$$

$$\leqq \frac{R^{2n} + k^{2n}}{1 + k^{2n}} \int_0^{2\pi} |p(e^{i\theta}) + e^{i\eta} q(e^{i\theta})|^2 d\theta .$$

We now integrate both sides with respect to $\eta$ from 0 to $2\pi$. From the above it is clear that for $0 \leqq \theta \leqq 2\pi$, $|p(Re^{i\theta})| \leqq |q(Re^{i\theta})|$ and $|p(e^{i\theta})| > |q(e^{i\theta})|$. On inverting the order of integration we obtain

$$\int_0^{2\pi} |1 + e^{i\eta}|^2 d\eta \times \int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta$$

$$\leqq \frac{R^{2n} + k^{2n}}{1 + k^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta \times \int_0^{2\pi} |1 + e^{i\eta}|^2 d\eta .$$

This proves (3).

Now $\log \int_0^{2\pi} |p(re^{i\theta})|^2 d\theta$ is a convex function of $\log r$, and therefore for $1 \leqq R \leqq k^2$

$$\left(\int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta\right)^{\log k^2}$$

$$\leqq \left(\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right)^{\log \frac{k^2}{R}} \left(\int_0^{2\pi} |p(k^2 e^{i\theta})|^2 d\theta\right)^{\log R}$$

$$\leqq \left(\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right)^{\log \frac{k^2}{R}} (k^{2n})^{\log R} \left(\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right)^{\log R}$$

$$= (k^{2n})^{\log R} \left(\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right)^{\log k^2}$$

or

$$\left(\int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta\right) \leqq (k^{2n})^{\log R/\log k^2} \left(\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right)$$

$$= R^{\eta} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta .$$

Hence the result.

THEOREM 2. *If the geometric mean of the moduli of the zeros of* $p(z)$ *is at least equal to* $k$, *then*

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta$$

$$\leqq \left(\frac{(1+k^{2n})^{\eta} + k^{2\eta}}{1 + k^{2n}}\right)^{2 \log R/\log (1+k^{2n})} \left(\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta\right)$$



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*for* $1 \leqq R \leqq \sqrt{(1+k^{2n})}$, *and*

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 d\theta \leqq \frac{R^{2n} + k^{2n}}{1 + k^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta$$

for $R \geqq \sqrt{(1+k^{2n})}$.

Theorem 2 can be proved on the same lines as Theorem 2 of [2].

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## REFERENCES

1. G. Pólya and G. Szegö: Aufgaben und Lehrsätze aus der Analysis. Vol. I. Berlin 1925.

2. Q. I. Rahman: Some Inequalities for Polynomial and Related Entire Functions. Illinois Journal of Mathematics. Vol. 5 (1961), pp. 144-151.

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