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NECESSARY AND SUFFICIENT CONDITIONS FOR THE DISCRETENESS OF THE SPECTRUM OF CERTAIN SINGULAR DIFFERENTIAL OPERATORS

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1. Introduction. Let $P(x)$ be an $m \times m$ matrix-valued function that is continuous, real, symmetric, and positive definite for all x in an interval J, which will be further specified. Let $w(x)$ be a positive and continuous weight function and define the formally selfadjoint operator l by

$$l(y) = (-1)^n w(x)^{-1}(P(x)y^{(n)}(x))^{(n)}$$

where $y(x)$ is assumed to be an m-dimensional vector-valued function. The operator l generates a minimal closed symmetric operator L_0 in the Hilbert space $\mathscr{L}_m^2(J; w)$ of all complex, m-dimensional vector-valued functions y on J satisfying

$$\int_J w(x)(y(x), y(x))dx < \infty$$

with inner product

$$[y, z] = \int_J w(x)(y(x), z(x))dx$$

where $(y, z) = \sum_{i=1}^m y_i \bar{z}_i$ for $z(x) = (z_i(x))$ and $y(x) = (y_i(x))$. All selfadjoint extensions of L_0 have the same essential spectrum ([**5**] or [**19**]). As a consequence, the discreteness of the spectrum $S(L)$ of one selfadjoint extension L will imply that the spectrum of every selfadjoint extension is entirely discrete.

We will say that l has property **BD** provided every selfadjoint extension of L_0 has a spectrum which is discrete and bounded below.

In this paper we are primarily concerned with requirements for $P(x)$ and $w(x)$ which will be necessary and sufficient in order that l have property **BD**. When the singularity is infinite we let $J = [1, \infty)$, and if the singularity is finite, in which case we may assume that it is at zero, we let $J = (0, 1]$.

V. A. Tkachenko [**6**] showed that when the singularity is at ∞, $m = 1$, and $w(x) \equiv 1$ in order for l to have property **BD** it is sufficient that

$$\lim_{x\to\infty} x^{2n-1}\int_x^\infty P(s)^{-1}ds = 0.$$

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G. A. Kalyabin [**10**] and R. T. Lewis [**13**] showed independently that this condition is necessary. Kalyabin's result also applied in the vector-matrix case.

It is interesting to compare these results with a result for the "polar" equation. M. Sh. Birman ([**4**] or [**6**, p. 93]) showed that the "polar" equation

$$(-1)^n y^{(2n)}(x) = \lambda P(x)^{-1} y(x) \quad (m = 1)$$

has property **BD** if and only if

$$\lim_{x\to\infty} x^{2n-1} \int_x^\infty P(s)^{-1} ds = 0.$$

V. V. Martynov [**16**] extended this result to the more general vector-matrix operator. We show in this paper that the "polar" operator has property **BD** if, and only if, l has property **BD** $(w(x) \equiv 1)$. Hence, the two results above are equivalent.

In the case of a finite singularity at 0, $m = 1$, and $w(x) \equiv 1$, D. B. Hinton and R. T. Lewis [**9**] showed that l has property **BD** if and only if

$$\lim_{x\to 0^+} x^{1-2n} \int_0^x s^{4n-2} P(s)^{-1} ds = 0.$$

Using some of the well-known methods of oscillation theory, we extend the above criteria to include weights other than $w(x) \equiv 1$ and the more general vector-matrix differential operators. In addition, we show that, for some weight functions, l may have property **BD** even when the integral

$$\nu \int_J P(s)^{-1} ds$$

does not exist, where νA will denote the maximum eigenvalue of a matrix A.

2. Oscillation theory and discreteness of the spectrum. In this section we outline some of the basic ideas of oscillation theory and its connection with spectral theory. Let $Q(x)$ denote a continuous, $m \times m$, symmetric matrix on J.

Distinct points c and d of (a, b), $-\infty \leqq a < b \leqq \infty$, are said to be *conjugate* with regard to the equation $l(y) = Qy$ if there exists a nontrivial solution y such that

$$(2.1) \quad y^{(i-1)}(c) = 0 = y^{(i-1)}(d), \quad i = 1, 2, \ldots, n.$$

In this case, the equation $l(y) = Qy$ is said to be *oscillatory on* (a, b). Otherwise, the equation is said to be *disconjugate* or *nonoscillatory on* (a, b). If the numbers a and b are real, the above definitions hold when

(a, b) is replaced by $[a, b]$. (There are other definitions of disconjugacy in the literature which are not equivalent to the one contained herein.) If the equation $l(y) = Qy$ is oscillatory in every neighborhood (N, ∞) of infinity then the equation is said to be *oscillatory at* ∞. Otherwise, the equation is *nonoscillatory* (or *disconjugate*) at ∞. Similarly, in the case of a finite singularity which we may assume to be at zero, the equation $l(y) = Qy$ is said to be *oscillatory at* 0 if it is oscillatory in every interval of the form $(0, \delta)$. Otherwise, the equation is *nonoscillatory at* 0.

Let $\mathscr{A}_m{}^n(a, b)$ denote the set of all complex, m-dimensional vector-valued functions $y(x)$ that have compact support interior to (a, b) with the first $n - 1$ derivatives absolutely continuous, and $y^{(n)} \in \mathscr{L}_m{}^2((a, b); w)$. When the numbers a and b are real, then we define $\mathscr{A}_m{}^n[a, b]$ as above with (a, b) replaced by $[a, b]$.

For $y \in \mathscr{A}_m{}^n(a, b)$, define

$$\Phi(y) = \int_a^b (P(x)y^{(n)}(x), y^{(n)}(x))dx.$$

If $y \in \mathscr{A}_m{}^n(a, b)$ is also in the domain of the minimal closed symmetric operator L_0 generated by l, then an n-fold integration by parts shows that

$$[L_0(y), y] = [l(y), y] = \Phi(y).$$

The following results are fundamental to our study.

THEOREM 2.1. *The equation $l(y) = Qy$ is nonoscillatory on the interval $[c, d]$ if and only if $\Phi(y) - [Qy, y] > 0$ for all $y \in \mathscr{A}_m{}^n[c, d]$, $y \not\equiv 0$.*

THEOREM 2.2. *In order that $(-\infty, \lambda) \cap S(L)$, the part of the spectrum of a selfadjoint extension L of L_0 lying to the left of a given point λ, be finite it is necessary and sufficient that there exists a number N such that $\Phi(y) - \lambda[y, y] \geqq 0$ for all $y \in \mathscr{A}_m{}^n(N, \infty)$.*

If the singularity is at 0, $J = (0, 1]$, then the analogous result to Theorem 2.2 holds.

THEOREM 2.3. *In order that $(-\infty, \lambda) \cap S(L)$ be finite it is necessary and sufficient that there exists a number $\delta > 0$ such that $\Phi(y) - \lambda[y, y] \geqq 0$ for all $y \in \mathscr{A}_m{}^n(0, \delta)$.*

For the proofs of these theorems and associated discussions the reader is referred to [**6**, **5**, **20**, **18**]. The connection between oscillation theory and the discreteness of the spectrum of selfadjoint extensions of L_0 is given by the next corollary whose proof follows immediately from the above theorems.

COROLLARY 2.1. *When the singularity is at ∞ (at 0), a necessary and sufficient condition that l have property* **BD** *is that $l(y) = \lambda y$ be nonoscillatory at ∞ (at 0) for all λ.*

Our next objective is to generalize to the vector-matrix case a theorem of Ahlbrandt [**1**, Theorem 3.1] which was a critical step in the proof of Theorem 1 of Lewis [**13**, Theorem 3] cited above. We first outline some of the well-known facts needed for this proof. For the remainder of this section we assume that Q is positive definite on J.

Associated with the equation

$$(2.2)\quad (-1)^n (Py^{(n)})^{(n)} = Qy$$

is the reciprocal equation

$$(2.3)\quad (-1)^n (Q^{-1}y^{(n)})^{(n)} = P^{-1}y.$$

We wish to show that the nonoscillation at ∞ or 0 of one of the equations is equivalent to the nonoscillation of the other equation. When $n = m = 1$, a substitution of $z = Py'$ will yield the result easily. Otherwise, this substitution does not work.

Boundary problems of the form (2.2)–(2.1) are equivalent to boundary value problems of the form

$$(2.4)\quad \begin{aligned} u' &= Au + Bv \\ v' &= Cu - A^*v \end{aligned}$$

$$(2.5)\quad u(c) = 0 = u(d)$$

where B and C are Hermitian with B and $-C$ positive semidefinite on (a, b). We let A^* denote the conjugate transpose of A. In case $n = 1$, A is the $m \times m$ zero matrix, $B = P^{-1}$, and $C = -Q$. If $n > 1$, regard A, B, and C as matrices with elements consisting of $m \times m$ matrix-valued functions with $A_{i,i+1} = I_m$, $B_{nn} = P^{-1}$, $C_{11} = -Q$, and all other entries being the $m \times m$ zero matrix where I_m denotes the $m \times m$ identity matrix. For basic disconjugacy results concerning (2.4) the reader should consult [**20**, pp. 337, 338].

If y is an m-vector function for which the derivatives indicated in (2.2) exist and if u and v are n-vectors whose entries are m-vector-valued functions $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ such that

$$(2.6)\quad \begin{aligned} u_k &= y^{(k-1)}, && k = 1, \ldots, n \\ v_{n-k} &= (-1)^k (Py^{(n)})^{(k)}, && k = 0, \ldots, n-1 \end{aligned}$$

then y is a solution of (2.2) if and only if $v_1' = -Qu_1$ if and only if $(u; v)$ is a solution of (2.4). In addition, for solutions (2.2) and (2.4) so related the boundary conditions (2.1) are equivalent to the boundary conditions (2.5) and equation (2.2) is disconjugate on an interval if and only if (2.4) is disconjugate on that interval [**1**].

The system corresponding to (2.3) is

$$(2.7)\quad \begin{aligned} u' &= -A^*u + (-C)v \\ v' &= -Bu + Av \end{aligned}$$

where A, B, and C have the above definitions. System (2.7) is called the *reciprocal* system to system (2.4). For more details, the reader should consult [**1**].

THEOREM 2.4. *Equation* (2.2) *is nonoscillatory at* ∞ (*at* 0) *if and only if equation* (2.3) *is nonoscillatory at* ∞ (*at* 0).

Proof. As in the proof of Theorem 3.1 of Ahlbrandt [**1**] the assumption that equation (2.2) is nonoscillatory in a neighborhood of ∞ (of 0) produces a Hermitian solution W of the matrix Riccati equation.

(2.8) $\quad W' = F + WGW$

in that same neighborhood, where $F = -D^*CD$, $G = D^{-1}BD^{*-1}$ and D is a fundamental solution of $D' = AD$. The crucial step of that proof is establishing that W is nonsingular in some neighborhood of ∞ (of 0). From the assumption that P and Q are positive definite, it follows that F and G are positive semidefinite and the eigenvalues of W are nondecreasing. In the present setting the eigenvalues of W are increasing since $x_1 < x_2$ implies that $\int_{x_1}^{x_2} F(t)dt$ is positive definite by Lemma 2.1 of [**2**] as a consequence of the "identical normality" of system (2.4); i.e., if u is zero throughout any nondegenerate interval, then v is also zero on that interval. (The condition of identical normality is not required for these results. This hypothesis has been removed in the recent Ph.D. Dissertation of Carl H. Rasmussen who was a student at the University of Connecticut.) Thus, there exists a neighborhood of ∞ (of 0) on which W is a solution of (2.8) with all mn eigenvalues of W nonzero; hence W is nonsingular there. Thus $W_1 = W^{-1}$ is a solution of the Riccati equation $W' = -G - WFW$ in that neighborhood of ∞ (of 0) and the reciprocal system (2.7) is nonoscillatory. The converse argument is analogous.

Define $\hat{l}(y)$ by

$$\hat{l}(y) = (-1)^n P(w^{-1}y^{(n)})^{(n)}.$$

The operator $\hat{l}$ generates a minimal closed symmetric operator $\hat{L}_0$ in the Hilbert space $\mathscr{L}_m^2(J; P^{-1})$ of all complex, m-dimensional vector-valued functions y on the interval J satisfying

$$\int_J (P(x)^{-1}y(x), y(x))dx < \infty$$

with inner product $[y, z] = \int_J (P^{-1}y, z)dx$. As for l, we will consider conditions which imply that $\hat{l}$ has property **BD**. As a consequence of Corollary 2.1 and Theorem 2.4, the next result can be established easily when the singularity is at ∞ with $J = [1, \infty)$ or at 0 with $J = (0, 1]$.

THEOREM 2.5. *Property* **BD** *holds for* l *on* $\mathscr{L}_m^2(J; w)$ *if and only if it holds for* $\hat{l}$ *on* $\mathscr{L}_m^2(J; P^{-1})$.

This theorem shows that the "polar operator" $\hat{l}$, [**6**, **16**], has property **BD** if and only if the one-term operator l has property **BD**.

Finally, we establish a needed lemma.

LEMMA 2.1. *Let $P(x)$ be a continuously differentiable, symmetric matrix function on $[a, b]$. Let y be an absolutely continuous m-vector on $[a, b]$ with $y(a) = 0 = y(b)$ and $y' \in \mathscr{L}_m^2([a, b]; 1)$. If P' is positive definite on $[a, b]$, then for $y \not\equiv 0$*

$$\int_a^b (P'(x)y(x), y(x))dx < 4 \int_a^b (P(x)(P'(x))^{-1}P(x)\, y'(x), y'(x))dx.$$

Proof. Let $(P')^{1/2}$ denote the unique positive definite square root of P'. We shall denote the inverse of $(P')^{1/2}$ by $(P')^{-1/2}$. By integrating by parts and using the Cauchy-Schwarz inequality we obtain the following series of inequalities:

$$\begin{aligned}\int_a^b (P'y, y)dx &\leqq 2 \int_a^b |(Py, y')|dx \\ &= 2 \int_a^b |((P')^{1/2}y, (P')^{-1/2}Py')|dx \\ &\leqq 2\left[\int_a^b (P'y, y)dx\right]^{1/2} \left[\int_a^b |(P(P')^{-1}Py', y')|dx\right]^{1/2}.\end{aligned}$$

The inequalities above can now be obtained by squaring both sides and dividing.

By Theorem 2.1, we see that Lemma 2.1 actually shows that the equation

$$(P(P')^{-1}Py')' + \tfrac{1}{4}P'y = 0$$

is nonoscillatory on $[a, b]$ provided P is nonsingular.

We consider a method to transform the singularity at 0 to one at infinity. In

$$(2.9) \quad l(y) = (-1)^n w(x)^{-1}(P(x)y^{(n)})^{(n)}, \quad 0 < x \leqq 1,$$

we make the substitution $t = 1/x$, $y(x) = x^{n-1}z(t)$. Then in [**3**], it is proved that $l(y) = \lambda y$ if and only if

$$(-1)^n(\tilde{P}(t)z^{(n)})^{(n)} = \lambda\tilde{w}(t)z, \quad 1 \leqq t < \infty,$$

where $\tilde{P}(t) = t^{2n}P(1/t)$ and $\tilde{w}(t) = t^{-2n}w(1/t)$. Since this transformation preserves the order of zeros, we see that $l(y) - \lambda y$ is oscillatory at 0 if and only if $\tilde{l}(z) - \lambda z$ is oscillatory at infinity where

$$\tilde{l}(z) = (-1)^n\tilde{w}(t)^{-1}(\tilde{P}(t)z^{(n)})^{(n)}.$$

Hence by Corollary 2.1 and the above remarks, to study property **BD** for (2.9), it is sufficient to consider only equations with a singularity at infinity. Thus we consider hereafter only the case $J = [1, \infty)$; the results obtained have as immediate corollaries results for $J = (0, 1]$ although we will not formally state such theorems.

3. Oscillation and nonoscillation criteria. Corollary 2.1 and the accompanying theorems delineate the connection between the oscillation theory of differential equations and the spectral theory of the associated differential operators. To obtain sufficient conditions for property **BD,** nonoscillation criteria are needed, and to obtain necessary conditions for property **BD**, oscillation criteria are needed. We develop in this section some new results in oscillation theory which will be applied in Section 4.

Let $p(x)$ be a positive continuous (scalar) function on J. For $J = [1, \infty)$ define $r_0(x) = p(x)^{-1}$ and

$$r_k(x) = \begin{cases} \int_x^\infty r_{k-1}(t)dt & \text{when } \int_1^\infty r_{k-1}(t)dt < \infty \\ \int_1^x r_{k-1}(t)dt & \text{when } \int_1^\infty r_{k-1}(t)dt = \infty \end{cases}$$

for $k = 1, 2, \ldots$.

We wish to consider the vector-matrix equation

$$(3.1) \quad (-1)^n (p(x) I_m y^{(n)})^{(n)} = Q(x) y$$

on J where $p(x) > 0$ and $Q(x)$ is positive semidefinite ($Q(x) \geqq 0$) on J.

THEOREM 3.1. *When $n > 1$, assume that for some positive constant b_k the scalar equation*

$$(3.2) \quad (r_k y')' + b_k r_{k-2} y = 0$$

is nonoscillatory at ∞ for each $k \in \{2, 4, \ldots, 2(n-1)\}$. Let $b_0 = 1$ and

$$C_n = \left(\prod_{i=0}^{n-1} b_{2i}\right) \Big/ 4.$$

Equation (3.1) *is nonoscillatory at ∞ provided one of the following two conditions is satisfied:*

(1) *The integral $\int_1^\infty Q(t)dt$ exists and for all x in some neighborhood of ∞*

$$r_{2n-1}(x) \ \nu \int_x^\infty Q(t)dt \leqq C_n.$$

(2) *For all x in some neighborhood of ∞*

$$r_{2n-1}(x) \ \nu \int_1^x Q(t)dt \leqq C_n.$$

Proof. In either case (1) or (2) assume that the condition holds for all

$x \in (N, \infty)$. For $N < b < \infty$ and each $y \in \mathscr{A}_m{}^n(a, b)$ we have that

$$\int_a^b (Qy, y)dx = 2\int_a^b \left(\int_x^\infty Qdty, y'\right)dx$$

in case (1) and

$$\int_a^b (Qy, y)dx = -2\int_a^b \left(\int_1^x Qdty, y'dx\right)$$

in case (2). In either case we can conclude that

$$\int_a^b (Qy, y)dx \leqq 2C_n \int_a^b r_{2n-1}^{-1}\|y\|\ \|y'\|dx$$

by the Cauchy-Schwarz inequality. (Note that $\nu A = \|A\|$, the operator norm, when A is positive definite.) Applying the Cauchy-Schwarz inequality in $\mathscr{L}_m{}^2([a, b]; 1)$ we obtain the inequality

$$\int_a^b (Qy, y)dx \leqq 2C_n\left[\int_a^b (r_{2n-2}r_{2n-1}^{-2}y, y)dx\right]^{1/2}\left[\int_a^b (r_{2n-2}^{-1}y', y')\right]^{1/2}$$
$$\leqq 4C_n \int_a^b (r_{2n-2}^{-1}y', y')dx$$

by an application of Lemma 2.1 with $P(x) = \pm r_{2n-1}^{-1}I_m$ where the sign is chosen in order that $P' > 0$. Note that for $n = 1$ the proof is complete by Theorem 2.1. If $n > 1$ then the vector-matrix equation

$$(r_kI_my')' + b_kr_{k-2}I_my = 0$$

is nonoscillatory at ∞ for $k \in \{2, 4, \ldots, 2(n-1)\}$ and hence,

(3.3) $\quad (r_{k-2}^{-1} I_my')' + b_kr_k{}^{-1}I_my = 0$

is nonoscillatory at ∞ for $k \in \{2, 4, \ldots, 2(n-1)\}$ by Theorem 2.4. Therefore, for N sufficiently large Theorem 2.1 applied to (3.3) implies that

$$\int_a^b (Qy, y)dx \leqq 4C_nb_{2n-2}^{-1}\int_a^b (r_{2n-3}^{-1}y'', y'')dx$$
$$\vdots$$
$$\leqq \int_a^b (r_0{}^{-1}y^{(n)}, y^{(n)})dx$$
$$= \int_a^b (py^{(n)}, y^{(n)})dx.$$

Consequently, by Theorem 2.1 equation (3.1) is nonoscillatory on every subinterval $[a, b]$ of (N, ∞) and it is therefore nonoscillatory at ∞. The proof is complete.

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For $m = n = 1$ Theorem 3.1 is the well-known theorem of Hille [**7**], with $p(x) \equiv 1$, and Moore [**17**], for general $p(x)$.

LEMMA 3.1. *Let $F(x)$ be a continuously differentiable, $m \times m$, positive definite, symmetric matrix on $[a, b]$. If $F'(x)$ is nonsingular, $F''(x)$ is positive definite, and $F(x) - 4cF'(x)(F''(x))^{-1}F'(x)$ is positive semi-definite on $[a, b]$ for some positive constant c, then*

$$(3.4) \quad (Fy')' + cF''y = 0$$

is nonoscillatory on $[a, b]$.

Proof. The proof follows immediately from Lemma 2.1 and Theorem 2.1.

For $m = 1$, $a > 0$, and $F(x) = x^\alpha$, $\alpha \notin [0, 1]$, the hypothesis of Lemma 3.1 holds for $c \leqq (\alpha - 1)/4\alpha$. Since (3.4) reduces to an Euler equation in this case, it is clear that the bound on c is sharp.

COROLLARY 3.1. *If $p'(x) \leqq 0$ for all x in some neighborhood of ∞ and $\int_1^\infty Q(t)dt$ exists, then*

$$r_{2n-1}(x) \ \nu \int_x^\infty Q(t)dt \leqq (4 \cdot 8^n)^{-1}$$

is sufficient for the nonoscillation of (3.1) at ∞.

Proof. By Lemma 3.1 with $m = 1$, $c = 1/8$ and Theorem 3.1 it will suffice to show that for $k \in \{2, 4, \ldots, 2(n - 1)\}$

$$r_k(x) - r_{k-1}(x)^2/2r_{k-2}(x) \geqq 0$$

for all x sufficiently large. Since $p'(x) \leqq 0$, then $r_i(x)$ is increasing, for $i = 0, 1, 2, \ldots$, and

$$2r_{k-2}(x)r_k(x)/r_{k-1}(x)^2 \geqq 2\int_1^x r_{k-2}(t) \times \int_1^t r_{k-2}(s)dsdt \Big/ \left(\int_1^x r_{k-2}(t)dt\right)^2 = 1$$

which completes the proof.

COROLLARY 3.2. *If $p'(x) \geqq 0$ for all x in some neighborhood of ∞ and $\int_1^\infty r_k(t)dt < \infty$ for $k = 0, 1, \ldots, 2n - 3$,*

$$r_{2n-1}(x) \ \nu \int_1^x Q(t)dt \leqq (4 \cdot 8^n)^{-1}$$

is sufficient for the nonoscillation of (3.1) at ∞.

Proof. Proceeding as in the proof of Corollary 3.1 we have that $r_i(x)$ is

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decreasing for $i = 0, 1, 2, \ldots, 2(n-1)$, and

$$2r_{k-2}(x)r_k(x)/r_{k-1}(x)^2 \geqq 2\int_x^\infty r_{k-2}(t) \times \int_t^\infty r_{k-2}(s)dsdt \Big/ \left(\int_x^\infty r_{k-2}(s)ds\right)^2 = 1$$

which completes the proof.

For $p(x) = x^\alpha$, Corollaries 3.1 and 3.2 apply when either $\alpha \leqq 0$ or $\alpha > 2n - 3$, respectively. In fact, when $r_k(x)$ and $r_{k-2}(x)$ are powers of x, equation (3.2) reduces to an Euler equation for which the test for the constant b_k is well-known. However, when $p(x) = x^\alpha$ with $\alpha = 2i - 1$ $i = 1, 2, \ldots, n - 1$, then $r_{k-2}(x) = x^{-1}$ and

$$r_k(x) = \int_1^x \ln t dt \quad \text{for } k = 2i.$$

Since

$$\int^\infty r_k(x)^{-1}dx = \int^\infty r_{k-2}(x)dx = \infty,$$

then, by the Leighton-Wintner Theorem [**11**, **21**], equation (3.2) is oscillatory at ∞ for all positive constants b_k and, consequently, the theorem does not apply. The theorem does apply, however, for any $\alpha \notin \{1, 3, 5, \ldots, 2n - 3\}$. A portion of the next theorem is proved in [**9**, Theorem 2.2] for $m = 1$ (the proof is analogous for general m) and it does apply when $p(x) = x^\alpha$ and $\alpha \in \{1, 3, 5, \ldots, 2n - 3\}$. Otherwise, it follows as a corollary of Theorem 3.1.

THEOREM 3.2. *Suppose that $p(x) = x^\alpha$ for some constant α. There is a positive constant $k(\alpha, n)$ such that equation* (3.1) *is nonoscillatory at ∞ provided one of the following conditions is satisfied:*

(1) *The integral $\int_1^\infty Q(t)dt$ exists, $\alpha \notin \{1, 3, \ldots, 2n - 1\}$, and as $x \to \infty$*

$$x^{2n-1-\alpha} \nu \int_x^\infty Q(t)dt \leqq k(\alpha, n).$$

(2) *The integral $\int_1^\infty Q(t)dt$ exists, $\alpha \in \{1, 3, \ldots, 2n - 1\}$, and as $x \to \infty$*

$$x^{2n-1-\alpha}|\ln x|^\eta \nu \int_x^\infty Q(t)dt \leqq k(\alpha, n)$$

where $\eta = 1$ for $\alpha = 2n - 1$ and $\eta = 2$ for $\alpha \in \{1, 3, \ldots, 2n - 3\}$.

(3) *As $x \to \infty$, $\nu \int_1^x Q(t)dt \to \infty$ and*

$$x^{2n-1-\alpha} \nu \int_1^x Q(t)dt \leqq k(\alpha, n).$$

Proof. The proof of parts (1) and (2) is essentially contained in

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[**9**, Theorem 2.2] and hence, we do not repeat it here. Part (1) also follows as a corollary of Theorem 3.1.

If the inequality of part (3) holds then $\alpha > 2n - 1$. Equation (3.2) reduces to an Euler equation and the proof follows easily as a corollary of Theorem 3.1.

Our need of Theorem 3.2 does not require knowledge of the value of the constant $k(\alpha, n)$. For discussions in this regard, we refer the reader to [**6**, **8**]. We do note that the constants can be shown to be sharp in most cases.

When $m = 1$ and $\alpha = 0$, condition (1) of Theorem 3.2 is the Hille-Glazman criterion [**6**] which was cited for general m by Martynov [**16**] and for general α by Lewis and Wright [**15**].

We now consider oscillation criteria at ∞ for equation (3.1).

LEMMA 3.2. *Suppose p in* (3.1) *is continuously differentiable and for some constants $M > 0$ and $\mu \geqq 0$ satisfies $|p'(x)| \leqq Mp(x)^{\mu}$ on $[1, \infty)$. Then if $K = [2M(3/2)^{\mu}]^{-1}$, we have for all t, $s \geqq 1$ satisfying $|t - s| \leqq Kp(s)^{1-\mu}$ that $1/2 < p(t)/p(s) < 3/2$.*

Proof. For a fixed s let $g(t) = p(t)/p(s)$. If $|g(t) - 1| \geqq 1/2$ for some t such that $|t - s| \leqq Kp(s)^{1-\mu}$, then $g(s) = 1$ implies that there is a t^*, $|t^* - s| \leqq Kp(s)^{1-\mu}$, such that $|g(t^*) - 1| = 1/2$ and $|g(t) - 1| < 1/2$ for all t between s and t^*. Hence by the Mean-value Theorem there is a t_0 between s and t^* such that

$$\begin{aligned}\tfrac{1}{2} &= |g(t^*) - 1| = |g'(t_0)||t^* - s| \\ &= \frac{|p'(t_0)|}{p(s)}|t^* - s| \leqq \frac{Mp(t_0)^{\mu}}{p(s)} Kp(s)^{1-\mu} \\ &= MKg(t_0)^{\mu} < MK(3/2)^{\mu} = 1/2.\end{aligned}$$

This contradiction establishes the lemma.

Note that with the above conditions on p, for $\mu \neq 1$,

$$\left| \frac{p(x)^{1-\mu} - p(1)^{1-\mu}}{1 - \mu} \right| \leqq M(x - 1)$$

and hence

$$Kp(x)^{1-\mu} - x \leqq [KM|1 - \mu| - 1]x + K[p(1)^{1-\mu} - M|1 - \mu|].$$

It is an easy exercise to show $KM|1 - \mu| < 1$ for $\mu \geqq 0$; hence we have

(3.5) $\quad \lim_{x\to\infty} Kp(x)^{1-\mu} - x = -\infty.$

The limit (3.5) is obvious for $\mu = 1$.

THEOREM 3.3. *Let p in (3.1) be as in Lemma 3.2. Then there is a positive constant $c(p)$ such that (3.1) is oscillatory at infinity if either of the following hold* ($\Delta = 1 - (1 - \mu)(2n - 1)$):

(1) *The integral $\int_1^\infty Q(t)dt$ exists and*

$$\limsup_{x\to\infty} p(x)^{-\Delta} \ \nu \int_x^\infty Q(t)dt > C(p).$$

(2) *As $x \to \infty$, $\nu \int_1^x Q(t)dt \to \infty$ and*

$$\limsup_{x\to\infty} p(x)^{-\Delta} \ \nu \int_1^x Q(t)dt > C(p).$$

Moreover, for $p(x) \equiv 1$, the same result holds if we replace $p(x)^{-\Delta}$ in (1) and (2) by x^{2n-1}.

Proof. By Theorem 2.1, it is sufficient to prove that for each N there is a member $y \in A_m{}^n(N, \infty)$ such that

$$\text{(3.6)} \quad I(y) \equiv \int_N^\infty p(s)(y^{(n)}(s), y^{(n)}(s))ds - \int_N^\infty (Q(s)y(s), y(s))ds < 0.$$

Let f be a $C^\infty(-\infty, \infty)$ function such that $f(x) = 0$ for $x \leqq 0$ and $f(x) = 1$ for $x \geqq 1$. For positive numbers r_1 and R_1, $N < r_1 < R_1$ to be determined, define (with K and μ as in Lemma 3.2),

$$g(x) = \begin{cases} f\left(\dfrac{x - r}{r_1 - r}\right), & r \leqq x \leqq r_1, \quad r = r_1 - Kp(r_1)^{1-\mu}, \\ 1, & r_1 \leqq x \leqq R_1, \\ f\left(\dfrac{x - R}{R_1 - R}\right), & R_1 \leqq x \leqq R, \quad R = R_1 + Kp(R_1)^{1-\mu}. \end{cases}$$

Let ξ be a constant, unit m-vector to be further specified and define $y(x) = g(x)\xi$; thus $y \in A_m{}^n(N, \infty)$.

By the above definitions and Lemma 3.2,

$$\int_r^{r_1} p(x)(y^{(n)}(s), y^{(n)}(s))ds = \int_r^{r_1} p(s)f^{(n)}\left(\frac{s - r}{r_1 - r}\right)^2 \frac{ds}{(r_1 - r)^{2n}}$$
$$= \int_0^1 p(s)f^{(n)}(u)^2 \frac{du}{(r_1 - r)^{2n-1}}, \quad u = \frac{s - r}{r_1 - r} \leqq$$
$$\frac{(3/2)p(r)}{(r_1 - r)^{2n-1}} \int_0^1 f^{(n)}(u)^2 du = c_1 p(r)^\Delta, \quad c_1 = (3/2)K^{1-2n} \int_0^1 f^{(n)}(u)^2 du.$$

In a similar manner we have that

$$\int_{R_1}^R p(s)(y^{(n)}(s), y^{(n)}(s))ds \leqq c_1 p(R_1)^\Delta.$$

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Using these bounds in (3.6) and the fact $Q(x) \geqq 0$ yields that

$$(3.7) \quad I(y) \leqq c_1 p(r)^{\Delta} + c_1 p(R_1)^{\Delta} - \int_{r_1}^{R_1} (Q(s)\xi, \xi)ds.$$

We now show that we may take $c(p) = 2^{|\Delta|}c_1$.

In case (1) we write (3.7) as

$$I(y) \leqq p(r_1)^{\Delta}\left[c_1\frac{p(r)^{\Delta}}{p(r_1)^{\Delta}} + c_1\frac{p(R_1)^{\Delta}}{p(r_1)^{\Delta}} - p(r_1)^{-\Delta}\int_{r_1}^{R_1} (Q(s)\xi, \xi)ds\right].$$

From $1/2 \leqq p(r)/p(r_1) \leqq 3/2$, we have $p(r)^{\Delta}/p(r_1)^{\Delta} \leqq 2^{|\Delta|}$; hence

$$(3.8) \quad I(y) \leqq p(r_1)^{\Delta}\left[c(p) + c_1\frac{p(R_1)^{\Delta}}{p(r_1)^{\Delta}} - p(r_1)^{-\Delta}\int_{r_1}^{R_1} (Q(s)\xi, \xi)ds\right].$$

By (1) we have a number $x > N$ so that

$$p(x)^{-\Delta}\ \nu\int_x^{\infty} Q(s)ds > c(p) + \epsilon$$

for some $\epsilon > 0$. By (3.5) it is sufficient also with $r_1 = x$ to assume $r > N$. Let ξ be such that

$$\nu\int_x^{\infty} Q(s)ds = \int_x^{\infty} (Q(s)\xi, \xi)ds;$$

hence we can choose $R_1 > r_1 = x$ so that

$$p(x)^{-\Delta}\int_x^{R_1} (Q(s)\xi, \xi)ds > c(p) + \epsilon/2$$

and $c_1 p(R_1)^{\Delta}/p(r_1)^{\Delta} < \epsilon/2$ (note that (1) implies that $p(x)^{-\Delta} \to \infty$ as $x \to \infty$). Thus the right-hand side of (3.8) is negative and the proof is complete.

In case (2) we write (3.7) as (using $c_1 \leqq c(p)$)

$$(3.9) \quad I(y) \leqq \left(\int_{r_1}^{R_1} (Q(s)\xi, \xi)ds\right)\left[c_1 p(r)^{\Delta}\left(\int_{r_1}^{R_1} (Q(s)\xi, \xi)ds\right)^{-1} + c(p)p(R_1)^{\Delta}\left(\int_{r_1}^{R_1} (Q(s)\xi, \xi)ds\right)^{-1} - 1\right].$$

Let r_1 now be such that $r > N$. If $p(x)^{-\Delta} \to 0$ as $x \to \infty$, then

$$\limsup_{x\to\infty} p(x)^{-\Delta}\ \nu\int_{r_1}^{x} Q(s)ds = \limsup_{x\to\infty} p(x)^{-\Delta}\ \nu\int_1^{x} Q(s)ds > c(p)$$

while if $p(x)^{-\Delta}$ does not converge to zero, then

$$\limsup_{x\to\infty} p(x)^{-\Delta}\ \nu\int_{r_1}^{x} Q(s)ds = \infty.$$

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In either case we can choose $R_1 > r_1$ so that

$$c(p)p(R_1)^{\Delta}\left(\nu\int_{r_1}^{x} Q(s)ds\right)^{-1} < 1 - \epsilon$$

for some $\epsilon > 0$ and also so that

$$c_1 p(r)^{\Delta}\left(\nu\int_{r_1}^{R_1} Q(s)ds\right)^{-1} < \epsilon/2.$$

Choosing ξ so that

$$\nu\int_{r_1}^{R_1} Q(s)ds = \int_{r_1}^{R_1} (Q(s)\xi, \xi)ds$$

ensures that the right-hand side of (3.9) is negative; the proof for (2) is now complete.

In case $p(x) \equiv 1$, we take $r = r_1/2$ and $R = 2R_1$, and repeat the above two arguments for choosing r_1, R_1, and ξ.

For $p(x) = x^{\alpha}$, $\alpha \neq 0$, we choose $\mu = 1 - \alpha^{-1}$ and then

$$p(x)^{-\Delta} = x^{2n-1-\alpha}.$$

With $p(x) = e^{\sigma x}$, we take $\mu = 1$; hence $p(x)^{-\Delta} = e^{-\sigma x}$. The restrictions on p imposed by Lemma 3.2 are only that it grow sufficiently regular.

When $p(x) = x^{\alpha}, \alpha > 2n - 1$, and $\int_1^{\infty} Q(t)dt$ exists, we have established in Theorem 3.2 the fact that (3.1) is nonoscillatory. When $\alpha = 2n - 1$ and $\int_1^{\infty} Q(t)dt$ exists, it appears to be an open question for $n > 1$ as to whether there is a constant C such that

$$\limsup_{x\to\infty} \ln x \ \nu\int_x^{\infty} Q(t)dt > C$$

implies the oscillation of (3.1) at ∞. Theorem 3.2 provides the associated nonoscillation criteria. When $n = 1$, the next theorem, a Hille-type oscillation theorem, answers the above question. For $m = 1$, the theorem was proved by Moore [**17**] and for general m, part (1) of the theorem is similar to a result of Ahlbrandt [**2**, Theorem 4.1]. However, the proofs are different. We let μA denote the minimum eigenvalue of A.

THEOREM 3.4. *If* $n = 1$, $Q(x) \geqq 0$ *for* $x \in [1, \infty)$ *and either*

(1) $\displaystyle\int_1^{\infty} Q(t)dt$ *exists and* $\displaystyle\limsup_{x\to\infty} \mu\int_1^{x} P(t)^{-1}dt \ \nu\int_x^{\infty} Q(t)dt > 1$

or

(2) $\displaystyle\int_1^{\infty} P(t)^{-1}dt$ *exists and* $\displaystyle\limsup_{x\to\infty} \mu\int_x^{\infty} P(t)^{-1}dt \ \nu\int_1^{x} Q(t)dt > 1$

then equation (2.2) *is oscillatory at* ∞.

Proof. For ξ a constant unit m-vector and constants $0 < a < b < c < d$, we define

$$y(x) = \begin{cases} \int_a^x P(t)^{-1}dt\left[\int_a^b P(t)^{-1}dt\right]^{-1}\xi & \text{for } x \in [a, b] \\ \xi & \text{for } x \in (b, c] \\ \int_x^d P(t)^{-1}dt\left[\int_c^d P(t)^{-1}dt\right]^{-1}\xi & \text{for } x \in (c, d]. \end{cases}$$

Since

$$\int_a^d (Py', y')dx \leqq \left\|\left[\int_a^b P(t)^{-1}dt\right]^{-1}\right\| + \left\|\left[\int_c^d P(t)^{-1}dt\right]^{-1}\right\|,$$

then

$$\begin{aligned} (3.4)\quad & \int_a^d [(Py', y') - (Qy, y)]dx \\ & \leqq \left\|\left[\int_a^b P^{-1}\right]^{-1}\right\|\left[1 + \left\|\left[\int_c^d P(t)^{-1}dt\right]^{-1}\right\| \left\|\left[\int_a^b P^{-1}\right]^{-1}\right\|^{-1} \right. \\ & \qquad \left. - \left\|\left[\int_a^b P^{-1}\right]^{-1}\right\|^{-1}\left(\int_b^c Qdx\xi, \xi\right)\right] \end{aligned}$$

or

$$\begin{aligned} (3.5)\quad & \int_a^b [(Py', y') - (Qy, y)]dx \\ & \leqq \left\|\left[\int_c^d P^{-1}\right]^{-1}\right\|\left[\left\|\left[\int_a^b P^{-1}\right]^{-1}\right\| \left\|\left[\int_c^d P^{-1}\right]^{-1}\right\|^{-1} + 1 \right. \\ & \qquad \left. - \left\|\left[\int_c^d P^{-1}\right]^{-1}\right\|^{-1}\left(\int_b^c Qdx\xi, \xi\right)\right]. \end{aligned}$$

In case (1), we can proceed as in the proof of Theorem 3.3 with inequality (3.4) using the fact that $\|A^{-1}\|^{-1} = \mu A$ for a positive definite matrix A. In case (2), if we proceed with inequality (3.5), then the proof will follow.

4. Discreteness of the spectrum. Our attention can now be given to the primary purpose of this paper and that is to establish requirements for $P(x)$ and $w(x)$ which are sufficient and in some cases necessary in order that l have property **BD**. All of our results in this section will be applications of the oscillation and nonoscillation criteria in Section 3.

According to Corollary 2.1 and Theorem 2.4, in order for l to have property **BD** it is necessary and sufficient that the equation

$$(4.1)\quad (-1)^n(w(x)^{-1}y^{(n)})^{(n)} = \lambda P(x)^{-1}y$$

be nonoscillatory at ∞ for all λ when the singularity is at ∞.

As in Section 3, we let $W_0 = w(x)$ and define

$$W_k(x) = \begin{cases} \int_x^\infty W_{k-1}(t)dt & \text{when } \int_1^\infty W_{k-1}(t)dt < \infty \\ \int_1^x W_{k-1}(t)dt & \text{when } \int_1^\infty W_{k-1}(t)dt = \infty \end{cases}$$

for k a positive integer.

THEOREM 4.1. (i) *When $n > 1$, assume that for $k \in \{2, 4, \ldots, 2(n-1)\}$ the scalar equation*

$$(4.2)\quad (W_k y')' + b_k W_{k-2} y = 0$$

is nonoscillatory at ∞ for some constant b_k. If either

$$(4.3)\quad \lim_{x\to\infty} W_{2n-1}(x)\ \nu\int_x^\infty P(t)^{-1}dt = 0$$

or

$$(4.4)\quad \lim_{x\to\infty} W_{2n-1}(x)\ \nu\int_1^x P(t)^{-1}dt = 0$$

then l has property **BD**.

(ii) *When $n = 1$ and $\int_1^\infty P(t)^{-1}dt$ exists, then (4.3) is also a necessary condition for property* **BD**. *When $n = 1$ and $\nu(\int_1^x P(t)^{-1}dt) \to \infty$ as $x \to \infty$, then (4.4) is also a necessary condition for property* **BD**.

(iii) *If $p = w^{-1}$ satisfies the hypothesis of Lemma 3.2, then ($\Delta = 1 - (1-\mu)(2n-1)$)*

$$(4.5)\quad \lim_{x\to\infty} w(x)^\Delta\ \nu\int_x^\infty P(t)^{-1}dt = 0$$

is a necessary condition for property **BD** *provided $\int_1^\infty P(t)^{-1}dt$ exists, and*

$$(4.6)\quad \lim_{x\to\infty} w(x)^\Delta\ \nu\int_1^x P(t)^{-1}dt = 0$$

is a necessary condition for property **BD** *if $\nu\int_1^x P(t)^{-1}dt \to \infty$ as $x \to \infty$.*

Proof. By Theorem 3.1, equation (4.1) is nonoscillatory at ∞ for all λ when either of the conditions (4.3) and (4.4) are satisfied. Hence, l has property **BD** by Corollary 2.1 and Theorem 2.4.

If $n = 1$ and l has property **BD**, then equation (4.1) is nonoscillatory at ∞ for all λ. By Theorem 3.4,

$$\limsup_{x\to\infty} \int_1^x w(t)dt\ \nu\int_x^\infty P(t)^{-1}dt \leqq \lambda^{-1}$$

for all $\lambda > 0$ if $\int_1^\infty P(t)^{-1}dt$ exists and

$$\limsup_{x\to\infty} \int_x^\infty w(t)dt\ \nu\int_1^x P(t)^{-1}dt \leqq \lambda^{-1}$$

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for all $\lambda > 0$ when $\int_1^\infty w(t)dt$ exists. Consequently, either

$$\lim_{x\to\infty} \int_1^x w(t)dt \ \nu \int_x^\infty P(t)^{-1}dt = 0$$

or

$$\lim_{x\to\infty} \int_x^\infty w(t)dt \ \nu \int_1^x P(t)^{-1}dt = 0,$$

which completes the proof of (ii).

Similarly, we may argue that Theorem 3.3 yields the necessary conditions stated in (iii).

For a certain class of weights w, (4.3) and (4.5) are equivalent, and (4.4) and (4.6) are equivalent. Two such functions are $w(x) = x^\gamma$, $\gamma \notin \{-1, -3, \ldots, 1 - 2n\}$, and $w(x) = e^{\sigma x}$, $\sigma \neq 0$, in which case $w(x)^\Delta = x^{2n-1+\gamma}$ and $w(x)^\Delta = e^{\sigma x}$, respectively.

Similarly, the corollaries of Theorem 3.1 produce corollaries of Theorem 4.1.

COROLLARY 4.1. *Property* **BD** *holds for l if one of the following conditions are satisfied:*

(1) *The integral $\int_1^\infty P(t)^{-1}dt$ exists, $w'(x) \geqq 0$, and* (4.3) *is valid.*

(2) *The integrals $\int_1^\infty W_k(t)dt$ exist for $k = 0, 1, \ldots, 2n - 3$, $w'(x) \leqq 0$, and* (4.4) *is valid.*

To consider the case $w(x) = x^\gamma, \gamma \in \{-1, -3, \ldots, 1 - 2n\}$, we state the following theorem which follows as a direct application of Theorems 3.2 and 3.4.

THEOREM 4.2. *If $w(x) = x^\gamma$, $\gamma \in \{-1, -3, \ldots, 1 - 2n\}$, and $\int_1^\infty P(t)^{-1}dt$ exists, then*

$$\lim_{x\to\infty} x^{2n-1+\gamma}|\ln x|^\eta \ \nu \int_x^\infty P(t)^{-1}dt = 0 \tag{4.7}$$

implies that l has property **BD**, *where $\gamma = 1$ for $\eta = 1 - 2n$ and $\eta = 2$ for $\gamma \in \{-1, -3, \ldots, 3 - 2n\}$. When $n = 1$* and *$\gamma = -1$,* (4.7) *is necessary for property* **BD**.

Note that if $w(x) = x^\gamma$ and $\nu \int_1^x P(t)^{-1}dt \to \infty$ as $x \to \infty$ then the necessary condition (4.6) is

$$\lim_{x\to\infty} x^{2n-1+\gamma} \ \nu \int_1^x P(t)^{-1}dt = 0$$

which implies $\gamma < 1 - 2n$.

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