Extension of the Notion of Wave-surface to Space of n Dimensions.

By Professor P. H. SCHOUTE.

1. The following two modes of generation of the wave-surface are pretty generally known.

(*a*) A given ellipsoid $E \equiv x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 - 1 = 0$ (surface of elasticity) is cut by any central plane π along an ellipse of semi-axes λ_1 and λ_2. If π varies, the two pairs of planes π' parallel to π at distances k^2/λ_1, k^2/λ_2 (k = constant) from it envelope the wave-surface W_1 represented by the tangential equation

$$\sum_{i=1}^{i=3} a_i^2 u_i^2/(a_i^2 - k^4 \Sigma u_i^2) = 0,$$

if the tangential coordinates u_i depend on $\Sigma ux + 1 = 0$.

(*b*) A given ellipsoid $V \equiv a_1^2 x_1^2 + a_2^2 x_2^2 + a_3^2 x_3^2 - k^4 = 0$ (surface of velocity) is cut by any central plane π along an ellipse of semi-axes μ_1, μ_2. If π varies, the locus of the two pairs of points P′, situated in the normal to π through the centre O of V and at distances μ_1, μ_2 from 0, is the wave-surface W_2 represented by the equation

$$\sum_{i=1}^{i=3} x_i^2/(k^4 - a_i^2 \Sigma x_i^2) = 0.$$

The surfaces W_1 and W_2 coincide. This result found by Professor Mannheim, who gave a very elegant geometrical demonstration of it (*Annuaire de l'association française, Congrès de Lille*, 1874) is not easily verified by analysis.

2. In the *Nieuw Archief* (series 2, vol. 3, p. 239, 1897) I have extended the two modes of generation stated above to n-dimensional space S^n.

My results are as follows:—

(*a*) A given quadratic space of $n-1$ dimensions represented by $E^{n-1} \equiv \sum_{i=1}^{i=n} x_i^2/a_i^2 - 1 = 0$ is cut by any central linear space of $n-1$ dimensions σ in a quadratic space E^{n-2} of semi-axes $\lambda_1 \; \lambda_2 \; \ldots \; \lambda_{n-1}$.

https://doi.org/10.1017/S0013091500032351 Published online by Cambridge University Press

If σ varies, the $n-1$ pairs of linear spaces σ' parallel to σ at distances k^2/λ_1, k^2/λ_2, ... k^2/λ_{n-1} from it envelope the wave-space W_1^{n-1} represented by

$$\sum_{i=1}^{i=n} a_i^2 u_i^2/(a_i^2 - k^4\Sigma u_i^2) = 0.$$

(β) A given quadratic space represented by

$$V^{n-1} \equiv \sum_{i=1}^{i=n} a_i^2 x_i^2 - k^4 = 0$$

is cut by any central linear space σ in a quadratic space V^{n-2} of semi-axes μ_1 μ_2 ... μ_{n-1}. If σ varies, the locus of the $n-1$ pairs of points, situated in the normal to σ through the centre O of V^{n-1} at distances μ_1 μ_2 ... μ_{n-1} from O, is the wave-space W_2^{n-1} represented by

$$\sum_{i=1}^{i=n} x_i^2/(k^4 - a_i^2\Sigma x_i^2) = 0.$$

The spaces W_1^{n-1} and W_2^{n-1} coincide in the same quadratic space W^{n-1} of degree and class $2(n-1)$.

3. In various publications Professor Mannheim has introduced a third mode of generation of the wave-surface. The object of this paper is to give an analytical demonstration of this third mode of generation, and to solve the question whether it is capable of as simple an extension to n-dimensional space.

According to the mode of generation in view, the wave-surface is the locus of the point P that admits with respect to a given ellipsoid L an enveloping cone, one of the principal sections of which is a right angle.

If we put $L \equiv x_1^2/b_1^2 + x_2^2/b_2^2 + x_3^2/b_3^2 - 1 = 0$, the enveloping cone with the vertex P (y_1, y_2, y_3) is represented by

$$\Sigma\left(\frac{y_2^2}{b_2^2} + \frac{y_3^2}{b_3^2} - 1\right)\frac{x_1^2}{b_1^2} - 2\Sigma\frac{y_2 y_3}{b_2^2 b_3^2}x_2 x_3 = 0$$

referred to parallel axes through P.

This corresponds to the symbolical form $(c_1x_1 + c_2x_2 + c_3x_3)^2 = 0$, if we put

$$c_{11} = \frac{1}{b_1^2}\left(\frac{y_2^2}{b_2^2} + \frac{y_3^2}{b_3^2} - 1\right), \qquad c_{23} = -y_2y_3/b_2^2b_3^2, \text{ etc.}$$

Now, according to the conditions of the problem the equation in S,—

$$D \equiv \begin{vmatrix} c_{11} - S & c_{12} & c_{13} \\ c_{21} & c_{22} - S & c_{23} \\ c_{31} & c_{32} & c_{33} - S \end{vmatrix} = 0$$

must have two roots whose sum is zero.

If D_0 is the value of D for $S = 0$, and C_{11}, C_{22}, C_{33} be the minors of D_0 with respect to c_{11} c_{22} c_{33}, the condition is

$$(c_{11} + c_{22} + c_{33})(C_{11} + C_{22} + C_{33}) = D_0.$$

In this way we find after some calculation

$$\Sigma(b_2^2 + b_3^2)y_1^2 \;\; \Sigma y_1^2 - \;\; \Sigma(b_2^2 + b_3^2)(2b_1^2 + b_2^2 + b_3^2)y_1^2$$
$$+ \Sigma b_1^2 \;\; \Sigma b_2^2 b_3^2 - b_1^2 b_2^2 b_3^2 = 0,$$

which reduces to

$$\overset{3}{\underset{1}{\Sigma}} y_i^2/(k^4 - a_i^2 \Sigma y_i^2) = 0$$

by the substitution

$$b_2^2 + b_3^2 = k^4/a_1^2, \quad b_3^2 + b_1^2 = k^4/a_2^2, \quad b_1^2 + b_2^2 = k^4/a_3^2.$$

So we find that the new surface coincides with W_1 and W_2, if L has the equation

$$x_1^2\Big/\left(\frac{1}{a_2^2} + \frac{1}{a_3^2} - \frac{1}{a_1^2}\right) + x_2^2\Big/\left(\frac{1}{a_3^2} + \frac{1}{a_1^2} - \frac{1}{a_2^2}\right) + x_3^2\Big/\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} - \frac{1}{a_3^2}\right) - k^4 = 0.$$

4. Let us proceed now to the space S^n and seek the locus of the point P that admits with respect to a given quadratic space $L^{n-1} \equiv \sum_{i=1}^{i=n} \frac{x_i^2}{b_i^2} - 1 = 0$ an enveloping cone, one of the principal sections of which is a right angle.

If we take parallel axes passing through $P(y_1, y_2 \ldots y_n)$, the equation of the enveloping cone of vertex P is

$$\Sigma\left(\frac{y^2}{b_2^2} + \frac{y_3^2}{b_3^2} + \;\ldots\; + \frac{y_n^2}{b_n^2} - 1\right)\frac{x_1^2}{b_1^2} - 2\Sigma\frac{y_i y_j}{b_i^2 b_j^2} x_i x_j = 0.$$

This appears in the symbolical form

$$(c_1x_1 + c_2x_2 + \quad . \; . \; . \quad + c_nx_n)^2 = 0$$

by the substitution

$$c_{11} = \frac{1}{b_1^{\,2}}\left(\frac{y_2^{\,2}}{b_2^{\,2}} + \frac{y_3^{\,2}}{b_3^{\,2}} + \quad . \; . \quad + \frac{y_n^{\,2}}{b_n^{\,2}} - 1\right), \quad c_ic_j = -y_iy_j/b_i^2b_j^2, \text{ etc.}$$

Now, according to the condition of the problem, the equation

$$D \equiv \begin{vmatrix} c_{11} - S & c_{12} & \vdots & c_{1n} \\ c_{21} & c_{22} - S & \vdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \vdots & c_{nn} - S \end{vmatrix} = 0$$

must have two roots in S whose sum is zero.

For if $S_1, S_2 \ldots S_n$ be the roots, the equation of the enveloping cone with reference to its axes of symmetry is $\Sigma S_ix_i^2 = 0$ and the principal plane section $S_ix_i^2 + S_jx_j^2 = 0$ situated in the plane X_iOX_j is a right angle $x_i^2 - x_j^2 = 0$ provided $S_i + S_j = 0$.

If we suppose the roots to be

$$S_0, \quad S_0, \quad T_1, \quad T_2, \; . \; . \; . \; T_{n-2},$$

and represent respectively by λ, s_k, t_k the square of S_0 and the sum of the partial products k at a time of all the n roots, and of the $n-2$ roots T, we have

$$\begin{aligned} s_1 &= t_1 \\ s_2 &= t_2 - \lambda \\ s_3 &= t_3 - \lambda t_1 \\ s_4 &= t_4 - \lambda t_2 \\ . \; &. \; . \; . \; . \\ s_{n-2} &= t_{n-2} - \lambda t_{n-4} \\ s_{n-1} &= \qquad - \lambda t_{n-3} \\ s_n \; &= \qquad - \lambda t_{n-2}. \end{aligned}$$

The elimination of $t_1 \; t_2 \; . \; . \; . \; t_{n-2}$ gives the two equations

$$\left.\begin{aligned} s_{n-1} + \lambda s_{n-3} + \lambda^2 s_{n-5} + \; . \; . \; . \; . \; . \; &= 0 \\ s_n \quad + \lambda s_{n-2} + \lambda^2 s_{n-4} + \; . \; . \; . \; . \; . \; &= 0 \end{aligned}\right\} \quad . \quad . \qquad (1)$$

which can be treated by the dialytic method of Sylvester.

https://doi.org/10.1017/S0013091500032351 Published online by Cambridge University Press

We may obtain the equations (1) in an easier manner. For, if the equation in S is written in the form

$$S^n - s_1 S^{n-1} + s_2 S^{n-2} - s_3 S^{n-3} + \ldots\ldots = 0,$$

the conditions for the two solutions $+S_0$, and $-S_0$ are

$$S_0^n \pm s_1 S_0^{n-1} + s_2 S_0^{n-2} \pm \ldots\ldots = 0$$

or

$$\left.\begin{array}{l} S_0^n + s_2 S_0^{n-2} + s_4 S_0^{n-4} + \ldots\ldots = 0 \\ s_1 S_0^{n-1} + s_3 S_0^{n-3} + \ldots\ldots\ldots\ldots = 0 \end{array}\right\},$$

a system which is identical with (1) as is proved by the substitution $S_0^2 = \lambda$.

The general result being too complicated for practical use, we consider the particular case $n = 4$. Then we have

$$s_3 + \lambda s_1 = 0 \qquad s_4 + \lambda s_2 + \lambda^2 = 0$$

and find

$$s_3^2 - s_1 s_2 s_3 + s_1^2 s_4 = 0.$$

Now the determinant

$$D_0 \equiv \begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{vmatrix}$$

leads to the notation

$$s_1 = c_{11} + c_{22} + c_{33} + c_{44}, \quad s_2 = \Sigma \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}, \quad s_3 = \Sigma \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = \Sigma C_{11},$$

$$s_4 = D_0,$$

where the quantities and minors following the Σ's belong to and have principal minors belonging to the principal diagonal of D_0.

Let P denote $y_1^2/b_1^2 + y_2^2/b_2^2 + y_3^2/b_3^2 + y_4^2/b_4^2 - 1$, then the form of D_0,

$$\begin{vmatrix} \frac{1}{b_1^2}\left(P - \frac{y_1^2}{b_1^2}\right), & -\frac{y_1 y_2}{b_1^2 b_2^2}, & -\frac{y_1 y_3}{b_1^2 b_3^2}, & -\frac{y_1 y_4}{b_1^2 b_4^2} \\ -\frac{y_2 y_1}{b_2^2 b_1^2}, & \frac{1}{b_2^2}\left(P - \frac{y_2^2}{b_2^2}\right), & -\frac{y_2 y_3}{b_2^2 b_3^2}, & -\frac{y_2 y_4}{b_2^2 b_4^2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

https://doi.org/10.1017/S0013091500032351 Published online by Cambridge University Press

leads to the following values of s:

$$Bs_1 = \Sigma b_1^2 b_2^2 y_3^2 - \Sigma b_1^2 b_2^2 b_3^2, \qquad Bs_2 = P(\Sigma b_1^2 y_2^2 - \Sigma b_1^2 b_2^2)$$

$$Bs_3 = \quad P^2(\Sigma y_1^2 - \Sigma b_1^2) \quad , \qquad Bs_4 = -P^3, \text{ where } B = b_1^2 b_2^2 b_3^2 b_4^2.$$

Substitution of these values in the relation

$$s_3^2 - s_1 s_2 s_3 + s_1^2 s_4 = 0$$

gives

$$W_3^3 \equiv (\Sigma b_1^2 b_2^2 b_3^2 y_4^2 - b_1^2 b_2^2 b_3^2 b_4^2)(\Sigma y_1^2 - \Sigma b_1^2)^2$$
$$- (\Sigma b_1^2 b_2^2 y_3^2 - \Sigma b_1^2 b_2^2 b_3^2)(\Sigma b_1^2 y_2^2 - \Sigma b_1^2 b_2^2)(\Sigma y_1^2 - \Sigma b_1^2)$$
$$- (\Sigma b_1^2 b_2^2 y_3^2 - \Sigma b_1^2 b_2^2 b_3^2)^2 = 0.$$

This equation represents a tridimensional space W_3^3 of the sixth order, which contains only once the imaginary sphere at infinity common to all hyperspheres in S^4. Therefore it can not coincide with $W_1^3 = W_2^3$ of which that sphere is a double surface.

Of the space W_3^3 the bidimensional surface $s_1 = 0$, $s_3 = 0$ is a double surface, in other words W_3^3 passes two times through the intersection of the quadratic space $s_1 = 0$ and the hypersphere $s_3 = 0$; this double surface is not situated in a linear tridimensional space, as that of $W_1^3 = W_2^3$.

So we have shown that the third method of generation of the wave-surface, found by Professor Mannheim, admits a generalisation to n-dimensional space, but that this extension is of a more complicated character than that of the two ordinary modes of generation.

https://doi.org/10.1017/S0013091500032351 Published online by Cambridge University Press