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ANTIPODAL COINCIDENCE SETS AND STRONGER FORMS OF CONNECTEDNESS

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A new notion of α-connectedness (α-path connectedness) in general topological spaces is introduced and it is proved that for a real-valued function defined on a space with this property, the cardinality of the antipodal coincidence set is at least as large as the cardinal number α . In particular, in linear topological spaces, the antipodal coincidence set of a real-valued function has cardinality at least that of the continuum. This could be regarded as a treatment of some Borsuk-Ulam type results in the setting of general topology.

It is implicit in the work of Yang [*19*, *20*, *21*] that if a continuous real-valued function is defined on a sphere in a real normed linear space of dimension greater than 2 then the cardinality of the antipodal coincidence set is at least that of the continuum. In fact, this result and a similar one for functions defined on linear spaces of dimension greater than 1 also hold for topological vector spaces. These are special cases of "intermediate value" theorems which are proved by considering the wealth of sets connecting arbitrary points in these spaces. This extension of the antipodal coincidence and "intermediate value" theories is the theme of this paper. To reduce the extent of repetition in

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the proofs the notion of "wealth of sets connecting arbitrary points" is formally defined for arbitrary topological spaces and the consequences are deduced in the general case. The theory for path connectedness is parallel to that for ordinary connectedness. Although Borsuk [1], Bourgin [2], Conner and Floyd [3], Connett [4], Dyson [5], Granas [7], Holm and Spanier [8], Jaworowski [9, 10, 11, 12], Joshi [13, 14], Kakutani [15], Livesay [16], Spiez [17], Yamabe and Yujobo [18] and many others have worked on similar problems, their methods and results are different from those in this paper.

1. Definition

Let X be a topological space, A be a set, and $\alpha = |A|$ be the cardinality of A. Elements x, y of X are *α-connected* (respectively *α-path connected*) *in* X if, and only if, there is a family $\{C_a : a \in A\}$ of connected (respectively path connected) subsets of X containing x and y with $C_a \cap C_b = \{x, y\}$ for a, b distinct elements of A. Such a family *α-connects* (respectively *α-path connects*) x and y *in* X.

X is *α-connected* (*α-path connected*) if, and only if, all points x, y of X are α-connected (α-path connected) in X. Note that α-connectedness and α-path connectedness are both topological properties.

2. Examples

(i). Every space is 0-path connected.

(ii). A space is 1-connected (1-path connected) if, and only if, it is connected (path connected).

Proof. It is a well-known fact that a space is connected if, and only if, any points x, y are contained in a connected subset of the space, that is the space is 1-connected.

If X is path connected and x, y are elements of X then the singleton family $\{X\}$ 1-path connects x and y.

Conversely, if X is 1-path connected and x, y are points in X, then by 1-path connectedness, $x, y \in D$, where D is a path connected subset of X. Thus there is a path p from x to y in D. Clearly p joins x to y in X. □

(iii). A figure eight is path connected but can be made disconnected by removing one point. In view of Theorem 4 below this shows that the figure eight is not 2-connected.

(iv). A circle is 2-path connected but not 3-connected since it can be disconnected by removing 2 points.

(v). Every connected two point space is α-path connected for every cardinal α.

Proof. Let $\{x, y\}$ be a connected two point space, A be a set and $\alpha = |A|$. Then $\{x, y\}$ does not have the discrete topology so we may assume $\{x\}$ is not an open set. Then p defined by

$$p(\lambda) = \begin{cases} x , & \lambda = 0 , \\ y , & 0 < \lambda \leq 1 , \end{cases}$$

is a path from x to y. So $\{x, y\}$ is path connected and hence $\{\{x, y\} : a \in A\}$ α-path connects x and y. A point is clearly α-path connected to itself. $\square$

The relationship between α-path connectedness and α-connectedness is as one would expect.

3. Theorem

Let x, y be α-path connected in a topological space X. Then x, y are α-connected in X. The converse is false. In fact, for every cardinal α there is a metric space, X, two of whose points, x_0 and x_1, are α-connected, but not path connected, in X.

Proof. Let $\{C_a : a \in A\}$ α-path connect x and y. Each C_a is path connected, and hence connected, so the family $\{C_a : a \in A\}$ α-connects x and y.

To prove the assertions regarding the converse let α be any cardinal number and A be a set of cardinality α. If $\alpha = 0$ the result is trivial since every space is 0-connected; so assume $\alpha > 0$ and let (S, d) be a connected metric space which is not path connected (for example, the topologist's sine curve [6, p. 362]). Let s_0 and s_1 be two points of S which cannot be joined by any path in S .

Define an equivalence relation $\sim$ on $S \times A$ by $(s, a) \sim (t, b) \Leftrightarrow s = t$ and either $a = b$ or $s = s_0$ or s_1 . Let $[s, a]$ denote the equivalence class of (s, a) and $X = S \times A/\sim$, the set of equivalence classes.

Define $\rho : X \times X \to [0, \infty)$ by

$$\rho([s, a], [t, b]) = \begin{cases} d(s, t) & \text{if } a = b , \\ \min\{d(s, s_i)+d(s_i, t) : i = 0, 1\} & \text{if } a \neq b . \end{cases}$$

We claim that ρ is a metric and that (X, ρ) has the desired properties.

ρ is well defined. The only case that needs attention occurs when one of s, t equals one of s_0, s_1 . Assume without loss of generality that $s = s_0$ (possibly also $t = s_0$ or s_1). Then

$$d(s, s_0) + d(s_0, t) = 0 + d(s, t) = d(s, t)$$

and

$$d(s, s_1) + d(s_1, t) \geq d(s, t) .$$

So

$$\min\{d(s, s_i)+d(s_i, t) : i = 0, 1\} = d(s, t)$$

and the two formulae in the definition give the same result.

ρ is a metric. Let $[s, a], [t, b], [u, c]$ be points in X . Clearly

$$\rho([s, a], [s, a]) = d(s, s) = 0$$

and

$$\rho([s, a], [t, b]) = \rho([t, b], [s, a]) .$$

Next observe that since $d(s, s_i) + d(s_i, t) \geq d(s, t)$,

$$\rho([s, a], [t, b]) \geq d(s, t) .$$

If $[s, a] \neq [t, b]$ then either $s \neq t$ so

$$\rho([s, a], [t, b]) \geq d(s, t) > 0 ,$$

or $s = t$, $a \neq b$, $s \neq s_0$ or s_1 so $d(s, s_i) + d(s_i, t) = 2d(s, s_i)$ and hence $\rho([s, a], [t, b]) = 2 \min\{d(s, s_i) : i = 0, 1\} > 0$.

To prove the triangle inequality we consider the three possible cases.

(i) If $a = c$ then

$$\begin{aligned}\rho([s, a], [u, c]) &= d(s, u)\\ &\leq d(s, t) + d(t, u)\\ &\leq \rho([s, a], [t, b]) + \rho([t, b], [u, c]) .\end{aligned}$$

(ii) If $a \neq c$ and $c \neq b$ then

$$\begin{aligned}\rho([s, a], [u, c]) &\leq d(s, s_i) + d(s_i, u)\\ &\leq d(s, t) + d(t, s_i) + d(s_i, u)\\ &\leq \rho([s, a], [t, b]) + d(t, s_i) + d(s_i, u) \quad \text{for } i = 0, 1 .\end{aligned}$$

But $\rho([t, b], [u, c]) = \min\{d(t, s_i)+d(s_i, u) : i = 0, 1\}$ so this shows

$$\rho([s, a], [u, c]) \leq \rho([s, a], [t, b]) + \rho([t, b], [u, c]) .$$

(iii) If $c \neq a$ and $a \neq b$ then the calculation is similar to that in (ii).

Next we find two points in X which are α-connected but not path connected. Let $x_0 = [s_0, a]$, $x_1 = [s_1, a]$. (Recall that $[s_i, a] = [s_i, b]$ for $i = 0, 1$ and $a, b \in A$ and that $A \neq \emptyset$.) Let $C_a = \{[s, a] : s \in S\}$ for each a in A . By the definition of ρ , C_a is isometric to S by $f_a([s, a]) = s$. Hence C_a is connected and thus x_0 and x_1 are α-connected by $\{C_a : a \in A\}$.

Assume there is a path p from x_0 to x_1 in X . Define $f : X \to S$ by $f[s, a] = s$; clearly f is well defined. Also f is continuous since, as we have already observed, $\rho([s, a], [t, b]) \geq d(s, t)$ for s, t in S , a, b in A . Thus $f \circ p$ is a path from $fx_0 = s_0$ to $fx_1 = s_1$ in S . This contradicts the choice of s_0 and s_1 ; hence there is no path in X joining x_0 to x_1 . □

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The general result on the stronger forms of connectedness is the following.

4. Theorem

Let X *be a topological space,* α *a cardinal number.*

(a) If x, y *are* α*-connected (*α*-path connected) in* X *then* x, y *cannot be disconnected (path disconnected) except by removing at least* α *points, that is if* D *is a subset of* X *not containing* x, y *such that* x, y *are not connected (not path connected) in* $X\backslash D$ *then* $\alpha \le |D|$.

(b) If X *is* α*-connected (*α*-path connected) then* X *cannot be made disconnected (path disconnected) except by removing at least* α *points, that is if* D *is a subset of* X *with* $X\backslash D$ *not connected (not path connected) then* $\alpha \le |D|$.

Proof. *(a)* Let x, y, D be as stated in the hypothesis and let $\{C_a : a \in A\}$ α-connect (α-path connect) x and y where, of course, $|A| = \alpha$. If a is an element of A with $C_a \cap D = \emptyset$ then, since C_a is a connected (path connected) subset of $X\backslash D$ containing x and y , these points are connected (path connected) in $X\backslash D$ which contradicts the hypothesis. Thus (by the Axiom of Choice) there is a function $f : A \to D$ with $f(a) \in C_a \cap D$ for each $a \in A$. Since D does not contain x or y and $C_a \cap C_b = \{x, y\}$ for $a \ne b$, it follows that f is injective. Therefore $\alpha = |A| \le |D|$.

(b) If D is a subset of X , $X\backslash D$ not connected (not path connected), then there are points x, y of $X\backslash D$ not connected (not path connected) in $X\backslash D$. But x, y are α-connected (α-path connected) in X so $\alpha \le |D|$ by *(a)*. □

The "intermediate value" result follows easily, as does the result on antipodal coincidence.

5. Corollary

Let X *be an* α*-connected space,* $f : X \to \mathbb{R}$ *be continuous,* x, y *be elements of* X , t *be a real number and suppose that* $f(x) < t < f(y)$. *Then there are at least* α *points* z *in* X *with* $f(z) = t$. *(If* $\alpha = 1$

this reduces to the familiar fact that the image of a connected space under a continuous real-valued function is an interval.)

Proof. Let $D = f^{-1}(t) = \{z \in X : f(z) = t\}$. Then $f(X\backslash D)$ is disconnected since it contains $f(x)$ and $f(y)$ but not t . But the continuous image of a connected set is again connected; so $X\backslash D$ must be disconnected. By the foregoing theorem, this implies $\alpha \leq |D|$. □

6. Corollary

Let T be a continuous involution of an α-connected space X (with or without fixed points), $f : X \to \mathbb{R}$ a continuous function. Then either $f(x) = f(Tx)$ for each x in X , or $f(x) = f(Tx)$ for at least α values of x in X . (If X contains a discrete pair, that is distinct points x and y where the subspace $\{x, y\}$ is discrete then $\alpha \leq |X|$ so the first alternative implies the second.)

Proof. Assume there is a point x_0 of X with $f(x_0) \neq f(Tx_0)$. Define $g : X \to \mathbb{R}$ by $g(x) = f(x) - f(Tx)$. Then

$$\begin{aligned} g(Tx_0) &= f(Tx_0) - f(T(Tx_0)) \\ &= f(Tx_0) - f(x_0) \quad \text{since } T \text{ is an involution} \\ &= -g(x_0) . \end{aligned}$$

Thus $g(x_0) < 0 < g(Tx_0)$ or $g(Tx_0) < 0 < g(x_0)$ so by Corollary 5, there are at least α points x in X with $g(x) = 0$, that is with $f(x) = f(Tx)$. □

For large cardinal numbers we have a partial converse of Theorem 4.

7. Theorem

Let X be a topological space, and let α be a cardinal number greater than c , the cardinality of the continuum.

(a) If x, y cannot be path disconnected in X except by the removal of at least α points then x, y are α-path connected in X .

(b) If X cannot be path disconnected except by the removal of at least α points then X is α-path connected.

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Proof. *(a)* Let A be a set of cardinality α. If $\{x, y\}$ is path connected then put $C_a = \{x, y\}$ for each a in A.

Now suppose that the space $\{x, y\}$ is not path connected. We apply a maximality argument to show the existence of at least α paths (in X) from x to y which do not cross.

Let P be the set of all paths in X from x to y, and let

$$\mathcal{P} = \{Q \subset P : \text{if } p \text{ and } q \text{ are distinct elements of } Q \text{ then } p([0, 1]) \cap q([0, 1]) = \{x, y\}\} .$$

$\mathcal{P}$ is non-empty and partially ordered by set inclusion. Let $\mathcal{C}$ be any linearly ordered subset of $\mathcal{P}$.

Certainly $\cup(\mathcal{C}) \subset P$. If p, q are distinct elements of $\cup(\mathcal{C})$ then $p \in R$, $q \in Q$ for some R, Q in $\mathcal{C}$. But $\mathcal{C}$ is linearly ordered so one of these, R say, is a subset of the other so $p, q \in Q$ and thus $p([0, 1]) \cap q([0, 1]) = \{x, y\}$. But p, q were any two distinct elements of $\cup(\mathcal{C})$ so this shows that $\cup(\mathcal{C})$ is an element of $\mathcal{P}$, that is every chain in $\mathcal{P}$ is bounded above; so by Zorn's Lemma, $\mathcal{P}$ has a maximal element P_0. Let $D = \bigcup_{p \in P_0} p\big((0, 1)\big)\backslash\{x, y\}$. Now, if q is a path from x to y in $X\backslash D$ then for each p in P_0, $p([0, 1]) \cap q([0, 1]) = \{x, y\}$ since $q([0, 1]) \cap D = \emptyset$; so $P_0 \cup \{q\}$ is an element of $\mathcal{P}$. This would contradict the maximality of P_0 since $\{x, y\}$ not path connected implies $q \notin P_0$. Thus x, y are path disconnected in $X\backslash D$; so by hypothesis, $|D| \geq \alpha$. But for each p in P_0, $\big|p\big((0, 1)\big)\big| \leq c$. So $\alpha \leq |D| \leq c|P_0| \leq \max\big(c, |P_0|\big)$. Since $c < \alpha$ this means $\alpha \leq |P_0|$ and hence P_0 has a subset A of cardinality α. The family $\{p([0, 1]) : p \in A\}$ α-path connects x and y.

(b) Take any x, y in X. Then x and y cannot be path disconnected in X except by path disconnecting X which, by hypothesis, requires the removal of α points. Thus, by *(a)*, x and y are α-path connected. □

We now apply the above to topological vector spaces and to the spheres in topological vector spaces. In each case we prove that the relevant space is c-path connected (and hence c-connected) and then appeal to Corollaries 5 and 6.

8. Lemma

Any real topological vector space of dimension greater than 1 *is* c*-path connected.*

Proof. Let X be a real topological vector space of dimension greater than 1 and let x, y be elements of X. If $x = y$ then x and y are c-path connected by $C_t = \{x\}$ for each t in $\mathbb{R}$ so assume $x \neq y$ and choose z linearly independent of $y - x$. For each real number t define a path p_t from x to y by $p_t(\mu) = x + \mu(y-x) + \mu(1-\mu)tz$ for each μ in the unit interval $[0, 1]$.

Let s and t be real numbers and let λ, μ be elements of the unit interval with $p_s(\lambda) = p_t(\mu) \neq x$ or y. By linear independence of $(y-x)$ and z the equation $p_s(\lambda) = p_t(\mu)$ $\{$that is $x + \lambda(y-x) + \lambda(1-\lambda)sz = x + \mu(y-x) + \mu(1-\mu)tz$ $\}$ implies that $\lambda = \mu$ and $\lambda(1-\lambda)s = \lambda(1-\lambda)t$. But since $p_s(\lambda) \neq x$ or y, $\lambda \neq 0$ or 1 so $\lambda(1-\lambda)$ is non-zero; hence $s = t$. This shows that $p_s([0, 1]) \cap p_t([0, 1]) = \{x, y\}$ for $t \neq s$. So $\{p_t([0, 1]) : t \in \mathbb{R}\}$ c-path connects x and y. □

9. Theorem

Let X *be any real topological vector space of dimension greater than* 1, *and let* $f : X \to \mathbb{R}$ *be continuous. Then* $f(X)$ *is an interval and if* t *is any interior point of* $f(X)$ *then* f *maps at least* c *points on* t, *that is* $c \leq |f^{-1}(t)|$.

Proof. This is immediate from Corollary 5 and Lemma 8. □

10. Theorem

Let X *be any real topological vector space of dimension greater than* 1, *and let* $f : X \to \mathbb{R}$ *be continuous. Then there are at least* c *points* x *in* X *with* $f(x) = f(-x)$.

Proof. This is immediate from Corollary 6 and Lemma 8 since $|X| \geq c$. □

The unit sphere in a normed linear space is formed by choosing one point from each ray proceeding from the origin (namely, the point with unit norm). Since there is no apparent natural way to do this in a general topological vector space, we define the sphere as the collection of rays proceeding from the origin.

11. Definition

Let X be a real topological vector space. Define $[\ ] : X\backslash\{0\} \to P(X)$, the power set of X , by $[x] = \{tx : t > 0\}$. Then *the sphere of* X is the range, Σ , of $[\ ]$ with the quotient topology, that is a subset U of Σ is open if, and only if, $\{x \in X\backslash\{0\} : [x] \in U\}$ is open in $X\backslash\{0\}$.

We see immediately that the above definition agrees with the usual notion of a sphere.

12. Lemma

Let X *be a real normed linear space,* $x \in X$ *, and* $r > 0$ *. Then the sphere* $S = \{y \in X : \|x-y\| = r\}$ *with centre* x *and radius* r *is homeomorphic to the sphere,* Σ *, of* X *. In fact* $u : S \to \Sigma$ *defined by* $u(y) = [y-x]$ *is a homeomorphism.*

Proof. Clearly u is continuous since translation and $[\ ]$ are continuous. Define $v : \Sigma \to S$ by $v[y] = x + ry/\|y\|$. This is well defined since if $[y] = [z]$ then $z = ay$, $a > 0$ and $\|z\| = a\|y\|$ so $x + ry/\|y\| = x + rz/\|z\|$. Since $v \circ [\ ]$ is continuous on $X\backslash\{0\}$ it follows that v is continuous by the standard property of the quotient topology. For $[y] \in \Sigma$,

$$\begin{aligned} u \circ v[y] &= u\left(x + \frac{ry}{\|y\|}\right) \\ &= \left[\frac{ry}{\|y\|}\right] \\ &= [y] \text{ since } \frac{r}{\|y\|} > 0 . \end{aligned}$$

For $y \in S$,

$$\begin{aligned} v \circ u(y) &= v[y-x] \\ &= x + \frac{r(y-x)}{\|y-x\|} \\ &= x + (y-x) \quad \text{since} \quad \|y-x\| = r \\ &= y \ . \end{aligned}$$

So u and v are mutually inverse continuous functions and hence u is a homeomorphism. □

13. Lemma

The sphere of a real topological vector space of dimension greater than 2 *is c-path connected.*

Proof. Let X be a real topological vector space of dimension greater than 2 and Σ be the sphere of X. Take any $x, y \in X\backslash\{0\}$.

(i) If $[x] = [y]$ then $[x], [y]$ are c-path connected in Σ by $\{\{[x]\} : t \in \mathbb{R}\}$.

(ii) If $[x] = [-y]$ choose w, z so that w, x, z are linearly independent. For t real define $p_t : [0, 1] \to \Sigma$ by

$p_t(\mu) = [(\cos \pi\mu)x+(\sin \pi\mu)(w+tz)]$ – a great circle from $[x]$ to $[y]$.

If s, t are real, $\mu, \lambda \in [0, 1]$ and $p_s(\lambda) = p_t(\mu) \neq [x]$ or $[y]$ then for some $r > 0$,

$$(\cos \pi\lambda)x + (\sin \pi\lambda)(w+sz) = r\big((\cos \pi\mu)x+(\sin \pi\mu)(w+tz)\big)$$

so (by linear independence of w, x, z) $\sin \pi\lambda = r \sin \pi\mu$ and $s(\sin \pi\lambda) = rt \sin \pi\mu$. But $r \neq 0$ and $\sin \pi\mu \neq 0$ (since $p_t(\mu) \neq [x]$ or $[y]$) so these give $s = t$. Thus $\{p_t([0, 1]) : t \in \mathbb{R}\}$ c-path connects $[x]$ and $[y]$.

(iii) If x, y are linearly independent choose z so that x, y, z are linearly independent. For real t define $p_t : [0, 1] \to \Sigma$ by $p_t(\mu) = [(1-\mu)x+\mu y+t\mu(1-\mu)z]$. If $p_s(\lambda) = p_t(\mu) \neq [x]$ or $[y]$ then $(1-\lambda)x + \lambda y + s\lambda(1-\lambda)z = r\big((1-\mu)x+\mu y+t\mu(1-\mu)z\big)$ for some $r > 0$. So $1 - \lambda = r(1-\mu)$, $\lambda = r\mu$ and $s\lambda(1-\lambda) = rt\mu(1-\mu)$. Adding the first two equations gives $r = 1$ so $\mu = \lambda$ and $s\lambda(1-\lambda) = t\lambda(1-\lambda)$. But

$\lambda(1-\lambda) \neq 0$ since $p_s(\lambda) \neq [x]$ or $[y]$ so $s = t$. Thus $\{p_t([0, 1]) : t \in \mathbb{R}\}$ c-path connects $[x]$ and $[y]$. □

14. Theorem

Let Σ be the sphere of a real topological vector space of dimension greater than 2 and let $f : \Sigma \to \mathbb{R}$ be continuous. Then $f(\Sigma)$ is an interval and if t is any interior point of $f(\Sigma)$ then $c \leq |f^{-1}(t)|$.

Proof. This is immediate from Corollary 5 and Lemma 13. □

15. Theorem

Let Σ be the sphere of a real topological vector space of dimension greater than 2 and let $f : \Sigma \to \mathbb{R}$ be continuous. Then there are at least c points $[x]$ in Σ with $f[x] = f[-x]$.

Proof. This is immediate from Corollary 6 and Lemma 13 since $|\Sigma| \geq c$. □

16. Remarks

(i) If the topological vector space in Theorems 9, 10, 14 or 15 has cardinality c the result becomes an equality, that is the conclusions in Theorems 9 and 14 strengthen to $c = |f^{-1}(t)|$ and in Theorems 10 and 15 we have exactly c points with the relevant property; for example, if $f : S^n \to \mathbb{R}$ $(n \geq 2)$ is continuous then $f(x) = f(-x)$ for exactly c points.

(ii) Since a topological vector space over $\mathbb{C}$ is also a topological vector space over $\mathbb{R}$ results for complex spaces are immediate. For example, any nontrivial complex topological vector space is c-path connected. □

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