HOMOTOPY CLASSIFICATION OF MAPPINGS OF A 4-DIMENSIONAL COMPLEX INTO A 2-DIMENSIONAL SPHERE

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Steenrod [1] solved the problem[1] of enumerating the homotopy classes of maps of an $(n+1)$-complex K into an n-sphere S^n utilizing the cup-i-product, the far-reaching generalization of the Alexander-Čech-Whitney cup product [7] and the Pontrjagin $*$-product [5].

Since Steenrod's paper [1] appeared, the efforts to extend the result to the case where an $(n-1)$-connected space takes the place of S^n have been made by Whitney [8], Postnikov [10] in case $n=2$. and by Postnikov [11] in case $n>2$.

On the other hand, the $(n+2)$-homotopy group $\pi_{n+2}(S^n)$ of S^n was recently determined to be cyclic of order 2 by Pontrjagin [6], G. W. Whitehead [13]. then an attempt to enumerate the homotopy classes of maps of an $(n+2)$-complex K into S^n is expected.[2]

In the present paper this problem will be solved in case $n=2$. As a partial result as to the n-dimensional case a theorem concerning the third obstruction was obtained (this was announced in our previous note [20] without proof). Let two maps f, g of an $(n+2)$-complex K into S^n be homotopic to each other on the $(n+1)$-skeleton K^{n+1} then there exists a map g' such that g' is homotopic to $g (g' \sim g)$ and $g'=f$ on K^{n+1}, and hence $f^*S^n = g'^*S^n \backsim g^*S^n$ (where S^n is the generating n-cocycle of S^n and f^*, g^* are the cochain homomorphisms induced by f, g). The separation cocycle $d^{n+2}(f, g')$ with coefficients in $\pi_{n+2}(S^n)$ is readily defined. In case $n=2$, $f \sim g$ on K if and only if there exists a 1-cocycle λ^1 of K such that $2f^*S^2 \smile \lambda^1 \backsim 0$ and the cohomology class

$$\{d^4(f, g')\} \equiv \{v_\lambda^2 \smile v_\lambda^2\} \quad \text{mod} \quad S_{q_0}H^2(K, \pi_3(S^2)),$$

where v_λ^2 is a 2-cochain such that $\delta v_\lambda^2 = 2f^*S^2 \smile \lambda^1$. In case $n>2$, a sufficient (not necessary) condition for f, g to be homotopic is obtained:

$$\{d^{n+2}(f, g')\} \equiv 0 \quad \text{mod} \quad S_{q_{n-2}}H^n(K, \pi_{n+1}(S^n)).$$

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1) The problem in case $n=2$ was solved by Pontrjagin [4] and independently by Whitney (an abstract in Bull. Amer. Math. Soc., 42 (1936), p. 338).

2) Problem 15 in Eilenberg. "On the problems of topology," Ann. of Math., 50 (1949), 247-260.

Here S_{q_i} is the Steenrod i-square. This condition is necessary in the special case when, for example, $H^{n-1}(K, \pi_n(S^n))=0$.

The homotopy classification theorem is obtained as a corollary of an extension theorem[3] in case $n=2$ which states that if a map f of the 2-skeleton K^2 of a complex K into S^2 is extended to a map $\bar{f}$ of K^4 into S^2, then the third obstruction

$$\{C^5(\bar{f})\} \equiv \psi\{f^*S^2\} \quad \text{mod} \quad S_{q_1}H^3(K, \pi_3(S^2)).$$

Here ψ is a new type of squaring operation defined for a 2-cohomology class W^2 such that $W^2 \smile W^2=0$, which determines a coset of the factor group $H^5(K, Z_2)$ mod $S_{q_1}H^3(K, Z)$. The ψ-square is a special case of ψ-product, $\psi(V^2, W^2)$, defined for a pair of 2-cohomology classes V^2, W^2 such that $V^2 \smile W^2=0$, in which $\psi(W^2)=\psi(W^2, W^2)$. The ψ-product is defined in terms of the $\smile_{-1}$-product newly defined and of the $\smile_i$-product, and it has a topological invariant meaning. The relation between the ψ-product and the functional cup product (Steenrod [2]) is given.

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Added after the submission: I was just informed, through a correspondence with Professor N. E. Steenrod, of the thesis of Doctor José Adem,[*] in which he solved the n-dimensional case ($n \geqq 2$) of the classification problem and obtained several results which may be even more important. According to a copy of the announcement of Dr. Adem's results, which Prof. Steenrod was kind enough to send me, the method employed there is far more fruitful than the older one used in the present paper. Dr. Adem's method relies upon the use of the Steenrod's conceptual definition of the squaring operations introduced recently in the Annals of Mathematics, which appeared after the preparation of the present paper.

Part I. Products

Preliminary. Denote by K a finite simplicial complex, by Z the group of integers and by Z_2 the group of integers reduced mod 2. Elements of the p-dimensional cochain group $L^p(K, Z)$ of K with integer coefficients and those of the p-dimensional cochain group $L^p(K, Z_2)$ of K with coefficients in Z_2 are called for simplicity p-integer cochains and p-cochains mod 2 respectively. Similarly we say p-integer cocycles and p-cocycles mod 2. There is a natural reduction $r_2: L^p(K, Z) \to L^p(K, Z_2)$ and we have $r_2\delta=\delta r_2$ for the coboundary operator δ. Since this reduction is onto, on calculation reduced mod 2 we often do not

[3] This problem was proposed by Steenrod (see the last section in [1]).

[*] Added in proof: His note "The Iteration of the Steenrod Squares in Algebraic Topology" appeared in Proc. Nat. Acad. Sci. vol. 38 (1952), 720-726.

distinguish a cochain mod 2 from one of its representative cochain with integer coefficients and we shall exclusively use ordinary cochains even if we call them cochains mod 2 or cocycles mod 2.

If a is a p-cocycle, with coefficients in Z or Z_2, then $\{a\}$ denotes its cohomology class. $a \backsim b$ means that $a-b$ is a coboundary.

An order in K is a partial order of vertices such that the vertices of any simplex are linearly ordered. A fixed order α in K will be assumed until further notice. The array $(A_0 A_1 \cdots A_p)$ of vertices A_i of a p-simplex σ ordered as in α will denote the oriented simplex σ and will be written simply $(01 \cdots p)$ without ambiguity.

In various products of integer cochains, the pairing of the coefficients is defined as the product (reduced mod 2 if necessary) of integers.

§ 1. $\vee_{-1}$-product.

We have the following formula, due to H. Cartan [3], relating to the cup product and the squaring operations:

$$S_{qp}(a^r \smile b^s) \backsim \sum_{i+j=p} (S_{qi}a^r) \smile (S_{qj}b^s) \qquad \text{mod } 2, \tag{1.1}$$

for two cocycles mod 2, a^r and b^s, (superscripts denote the dimension). Let us consider this formula in a simplical complex K with ordered vertices, especially in case $p=r+s-2$. We intend to find explicitly a cochain mod 2 whose coboundary will give the difference of the left and the right-hand sides of (1.1). For this purpose, define the $(r+s+1)$-cochain mod 2, $a^r \vee_{-1} b^s$, for two cocycles mod 2, a^r and b^s, by setting

$$\begin{aligned} a^r \vee_{-1} b^s(0, 1, \ldots, r+s+1) &= \sum_{i:\text{odd}>0} a(0, 1, \ldots, \widehat{r-i}, \ldots, r+1) \cdot \\ a(0, 1, \ldots, r) &\cdot \sum_{j:\text{even}>0} b(r, r+1, \ldots, \widehat{r+j}, \ldots, r+s+1) \cdot \\ b(r+1, \ldots, r+s+1) & \qquad \text{mod } 2, \end{aligned} \tag{1.2}$$

where the symbol "$\wedge$" means the deletion of the marked vertex. Then the following coboundary formula holds for $r+s \leqq 5$:

$$\begin{aligned} \delta(a^r \vee_{-1} b^s) = S_{q_{r+s-2}}(a^r \smile b^s) &+ (S_{q_{r-1}}a^r) \smile (S_{q_{s-1}}b^s) \\ &+ a^r \smile (S_{q_{s-2}}b^s) + (S_{q_{r-2}}a^r) \smile b^s \qquad \text{mod } 2. \end{aligned} \tag{1.3}$$

Indeed this is verified by a direct computation. In the below (1.3) will be used only for the case when both r and s are not greater than 2.

§ 2. ψ-product and ψ-square.

We shall restrict ourselves to the case $r=s=2$. Let a and b be 2-dimensional integer cocycles. Then we have by (1.2)

$$a \vee_{-1} b(012345) = a(023)a(012)b(235)(345) \qquad \text{mod } 2,$$

and by (1.3)

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$$\delta(a \vee_{-1} b) = (a \smile b) \smile_2 (a \smile b) + (a \smile_1 a) \smile (b \smile_1 b) + (a \smile a) \smile b + a \smile (b \smile b) \quad \text{mod } 2. \tag{2.1}$$

If $a \smile b \backsim 0$ (not mod 2) and $a \smile b = \delta u$ for a 3-integer cochain u, we have

$$\delta(\mathfrak{p}_1 u) = (a \smile b) \smile_2 (a \smile b) \quad \text{mod } 2, \tag{2.2}$$

where $\mathfrak{p}_1$ denotes the generalized Pontrjagin square:

$$\mathfrak{p}_1 u = u \smile_1 u + u \smile_2 \delta u.$$

Recalling that $(a \smile a) \smile b = a \smile (a \smile b)$, $a \smile (b \smile b) = (a \smile b) \smile b$ and $a \smile_1 a \backsim 0$ (see Theorem 12.6 in Steenrod [1]), we can define a 5-cocycle mod 2, $\psi(a, b;\ u)$, by setting

$$\psi(a, b;\ u) = a \vee_{-1} b + \tilde{a} \smile (b \smile_1 b) + \mathfrak{p}_1 u + a \smile u + u \smile b \quad \text{mod } 2, \tag{2.3}$$

where $\tilde{a}$ is a 2-integer cochain such that $\delta \tilde{a} = a \smile_1 a$. $\psi(a, b;\ u)$ depends on the choice of $\tilde{a}$, but its cohomology class $\{\psi(a, b;\ u)\}$ is independent of the choice of $\tilde{a}$ because of $b \smile_1 b \backsim 0$.

We shall enumerate some properties of $\psi(a, b;\ u)$.

$$\psi(a, b;\ u + \lambda) - \psi(a, b;\ u) \backsim S_{q_1}\lambda + (a + b) \smile \lambda \quad \text{mod } 2, \tag{2.4}$$

for a 3-integer cocycle λ.

$$\psi(a, b + c;\ u + v) - \psi(a, b;\ u) - \psi(a, c;\ v) \backsim a \smile (b \smile_1 c) + v \smile b + u \smile c \quad \text{mod } 2, \tag{2.5}$$

for a, b, c, u and v such that $a \smile b \backsim 0$, $a \smile c \backsim 0$, $\delta u = a \smile b$ and $\delta v = a \smile c$ (not mod 2).

$$\psi(a + b, c;\ u + v) - \psi(a, c;\ u) - \psi(b, c;\ v) \backsim (a \smile_1 b) \smile c + a \smile v + b \smile u \quad \text{mod } 2, \tag{2.5$'$}$$

for a, b, c, u and v such that $a \smile c \backsim 0$, $b \smile c \backsim 0$, $\delta u = a \smile c$ and $\delta v = b \smile c$ (not mod 2).

$$\psi(a, b + \delta e;\ u + a \smile e) \backsim \psi(a, b;\ u) \quad \text{mod } 2, \text{ and} \tag{2.6}$$

$$\psi(a + \delta e, b;\ u + e \smile b) \backsim \psi(a, b;\ u) \quad \text{mod } 2, \tag{2.6$'$}$$

for a 1-integer cochain e.

$$\psi(a, \delta e;\ a \smile e) \backsim 0 \quad \text{mod } 2, \text{ and} \tag{2.7}$$

$$\psi(\delta e, b;\ e \smile b) \backsim 0 \quad \text{mod } 2, \tag{2.7$'$}$$

for a 1-integer cochain e.

$$\psi(a, b;\ u) \backsim \psi(b, a;\ u + a \smile_1 b) \quad \text{mod } 2. \tag{2.8}$$

Let c be a 2-integer cocycle such that $c \smile c \backsim 0$ (not mod 2) and $c \smile c = \delta u$ for a 3-integer cochain. We can define a 5-cocycle mod 2, $\psi(c;\ u)$, by setting

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(2.9) $$\psi(c;\ u)=c\vee_{-1}c+\tilde{c}\smile(c\smile_1 c)+\mathfrak{p}_1 u \qquad \text{mod } 2,$$

where $\tilde{c}$ is a 2-integer cochain such that $\delta\tilde{c}=c\smile_1 c$. $\psi(c;\ u)$ is independent of the choice of $\tilde{c}$.

We shall enumerate some properties of $\psi(c;\ u)$.

(2.10) $$\psi(c;\ u+\lambda)-\psi(c;\ u)\backsim S_{q_1}\lambda \qquad \text{mod } 2.$$

for a 3-integer cocycle λ.

(2.11) $$\psi(c;\ u)\backsim\psi(c,\ c;\ u) \qquad \text{mod } 2.$$

(2.12) $$\psi(\delta e;\ e\smile\delta e)\backsim 0 \qquad \text{mod } 2, \text{ and}$$

(2.13) $$\psi(c+\delta e;\ u+c\smile e+e\smile c+e\smile\delta e)\backsim\psi(c;\ u) \qquad \text{mod } 2,$$

for a 1-integer cochain e.

(2.14) If $f: K'\to K$ is an order-preserving simplicial map, then

$$f^*\psi(a,\ b;\ u)=\psi(f^*a, f^*b;\ f^*u) \quad \text{and,}$$

$$f^*\psi(c;\ u)=\psi(f^*c;\ f^*u),$$

where $\frac{1}{2}(a\smile_2 a-a)$ is used as $\tilde{a}$.

It is conjectured that the following formula should hold

(2.15) $$\psi(a+b;\ u+v+w+a\smile_1 b)-\psi(a;\ u)-\psi(b;\ v)-w\smile_1 w\backsim 0 \qquad \text{mod } 2,$$

for a, b, u, v and w such that $a\smile a=\delta u$, $b\smile b=\delta v$ and $2\,a\smile b=\delta w$.

Before we prove all these formulae (2.4)-(2.8) and (2.10)-(2.14), we state the conclusions which are deducible from them. Consider the cohomology class $\{\psi(a,\ b;\ u)\}$, then (2.4) shows that $\{\psi(a,\ b;\ u)\}\in H^5(K, Z_2)$ is determined by a and b up to the subgroup $\sigma_*[H^3(K, Z)]$, the image of a homomorphism $\sigma_*: H^3(K, Z)\to H^5(K, Z_2)$, induced by

(2.16) $$\sigma(\lambda)=\lambda\smile_1\lambda+(a+b)\smile\lambda,$$

for a 3-integer cocycle λ. In the definition of σ, the pairing of coefficients is defined as the product, reduced mod 2, of integers. And (2.6), (2.6)$'$, (2.7), (2.7)$'$ show that the coset $\{\psi(a,\ b;\ u)\}$ mod $\sigma_*[H^3(K, Z)]$ depends only on the cohomology classes $\{a\}$ and $\{b\}$. We denote the coset by $\psi(\{a\}, \{b\})$.

THEOREM 2.1. *For $\{a\}$, $\{b\}\in H^2(K, Z)$ such that $\{a\}\smile\{b\}=0\in H^4(K, Z)$, we can determine an element $\psi(\{a\}, \{b\})$ of the factor group $H^5(K, Z_2)$ mod $\sigma_*[H^3(K, Z)]$ and we have $\psi(\{a\}, \{b\})=\psi(\{b\}, \{a\})$.*

This operation is called ψ-product. The latter part of the theorem follows from (2.8).

Similarly, from (2.10), (2.12) and (2.13), we obtain

THEOREM 2.2. *For* $\{c\}\in H^2(K, Z)$ *such that* $\{c\}\smile\{c\}=0\in H^4(K, Z)$, *we can determine an element* $\psi\{c\}$ *of the factor group* $H^5(K, Z_2)$ *mod* $Sq_2H^3(K, Z)$, *and we have* $\psi\{c\}=\phi(\{c\}, \{c\})$.

This operation is called ψ-square.

§ 3. Proofs of the formulae in § 2.

We shall first describe auxiliary formulae. In the following a, b, c, d are 2-integer cocycles and u, v are 3-integer cochains

$$\mathfrak{p}_1(u+v)\backsim\mathfrak{p}_1u+\mathfrak{p}_1v+\delta u\smile_3\delta v \quad \text{mod } 2. \tag{3.1}$$

$$(a\smile_1 b)\smile c\backsim(b\smile_1 a)\smile c \quad \text{mod } 2. \tag{3.2}$$

$$(a\smile_1 b)\smile c+a\smile_1(b\smile c)+b\smile(a\smile_1 c)\backsim 0 \quad \text{mod } 2. \tag{3.3}$$

$$(a\smile_1 b)\smile a\backsim a\smile_1(b\smile a) \quad \text{mod } 2. \tag{3.3$'$}$$

$$c\smile_1(c\smile c)\backsim 0 \quad \text{mod } 2. \tag{3.3$''$}$$

$$(a\smile b)\smile_3(c\smile d)=acdb+cabd+(a\smile c)\smile(b\smile_2 d)+(a\smile_2 c)\smile(b\smile_1 d) \quad \text{mod } 2, \tag{3.4}$$

where $acdb$ denotes a 5-cochain mod 2 defined as $acdb(012345)=a(023)c(012)\cdot d(235)b(345)$ mod 2.

$$\begin{aligned}\delta(a^{\vee-1}e+a\smile e\smile e)&=a^{\vee-1}\delta e+(a\smile_1 a)\smile(e\smile e+e\smile_1\delta e)\\&\quad+\mathfrak{p}_1(a\smile e)+a\smile a\smile e+a\smile e\smile\delta e \quad \text{mod } 2,\end{aligned} \tag{3.5}$$

for a 1-integer cochain e, where $a^{\vee-1}e$ denotes a 4-cochain mod 2 defined as $a^{\vee-1}e(01234)=a(023)a(012)e(23)e(34)$.

$$\begin{aligned}\delta(e^{\vee-1}b+e\smile e\smile b)&=\delta e^{\vee-1}b+(e\smile e+e\smile_1\delta e)\smile(b\smile_1 b)\\&\quad+\mathfrak{p}_1(e\smile b)+\delta e\smile e\smile b+e\smile b\smile b \quad \text{mod } 2,\end{aligned} \tag{3.5$'$}$$

for a 1-integer cochain e, where $e^{\vee-1}b$ denotes a 4-cochain mod 2 defined as $e^{\vee-1}b(01234)=e(01)e(12)b(124)b(234)$.

$$\delta(e^{\vee-1}\delta e)=\delta e^{\vee-1}\delta e+(e\smile e+e\smile_1\delta e)\smile(\delta e\smile_1\delta e)+\mathfrak{p}_1(e\smile\delta e) \quad \text{mod } 2, \tag{3.5$''$}$$

for a 1-integer cochain e.

We shall prove (3.3). This follows from

$$\delta(abc)=(a\smile_1 b)\smile c+a\smile_1(b\smile c)+b\smile(a\smile_1 c) \quad \text{mod } 2, \tag{3.3°}$$

where abc is a 4-cochain mod 2 defined as $abc(01234)=a(024)b(012)c(234)$ mod 2.

Now we begin to prove the formulae in § 2. Among them (2.8) will be proved later (in § 4).

(2.4) and (2.10) immediately follow from (3.1).

(2.5) and (2.5)′ follow from a direct computation by means of (3.4).

(2.6) and (2.6)′ follow from (2.5) and (2.5)′.

(2.7) and (2.7)′ follow from (3.5) and (3.5)′.

(2.11) follows from (3.3)″.

(2.12) follows from (2.11) and (2.7)′ or directly from (3.5)″.

(2.13) follows from (2.11), (2.6), (2.6)′, (2.7) and (2.7)′.

(2.14) is easily seen from the definition of ψ.

§4. Change of order in K, simplicial map, and proof of (2.8).

In this section we shall prove that the ψ-product is independent of the choice of an order in K. This is done in a similar way as in the proof of the independence of the squaring operation (cf. Section 8 in Steenrod [1]).

Let $L = K\times 1$ be the product complex of K and the unit interval [0, 1]. We shall subdivide L simplicially as follows. Let (A_0) and (A_1) be two disjoint sets of vertices of $K\times 0$ and of $K\times 1$ each in a $1-1$ correspondence with the vertices (A) of K. Let $f_0(A) = A_0$, $f_1(A) = A_1$ be the correspondences. The union of (A_0) and (A_1) form the set of vertices of L. Let α be an order in K. A set of vertices $A_0^0 \ldots A_0^k A_1^{k+1} \ldots A_1^p$ are those of a p-simplex in L if, in the order α, $A^0 < A^1 < \ldots < A^k \leqq A^{k+1} < \ldots < A^p$, and these are the vertices of a p- or $(p-1)$-simplex of K. The maps f_0, f_1 define simplicial maps of K into L. The map $g\colon L\to K$, defined by $g(A_0) = g(A_1) = A$ for each A, is a simplicial map and

$$(4.1)\qquad gf_0 = gf_1 = \text{the identity map of } K.$$

If u is a p-cochain of L $(P>0)$, define a $(p-1)$-cochain Du of K by

$$(4.2)\qquad Du(A^0 \ldots A^{p-1}) = \sum_{k=0}^{p-1}(-1)^k u(A_0^0 \ldots A_0^k\, A_1^k \ldots A_1^{p-1}).$$

Then we have

$$(4.3)\qquad \delta Du = f_1^* u - f_0^* u - D\delta u, \text{ for a } p\text{-cochain } u \text{ of } L\ (P>0),$$

$$(4.4)\qquad 0 = f_1^* u - f_0^* u - D\delta u, \text{ for a 0-cochain } u \text{ of } L,$$

$$(4.5)\qquad Dg^* = 0,$$

where f_1^*, f_0^* and g^* are the cochain maps induced by f_1, f_0 and g respectively. (For the proof, see Section 7 in Steenrod [1].)

Let α_0, α_1 be two orders in K. The orders α_0, α_1 define two cup-i-products $\smile_i^0$, $\smile_i^1$ and two ψ-products ψ_0, ψ_1 in K. An order (α_0, α_1) is defined in L as follows. Order (A_0) as their correspondents (A) are ordered by α_0, order (A_1) as their correspondents (A) are ordered by α_1, and agree that, on any simplex of L, a vertex of (A_0) precedes one of (A_1). Then (α_0, α_1) defines products $\smile_i$ and ψ in L. Since $f_0(f_1)$ preserves $\alpha_0(\alpha_1)$, it follows from (2.14) that $f_0^*(f_1^*)$ maps $\smile_i$ into $\smile_i^0(\smile_i^1)$ and ψ into $\psi_0(\psi_1)$ respectively.

Corresponding to the orders α_0, α_1 define a 4-cochain mod 2 of K by

$$(4.6)\qquad a^{\#}b = D\psi(g^*a,\ g^*b;\ v) \qquad \text{mod } 2,$$

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for 2-integer cocycles a, b of K such that $a \smile_0 b \backsim 0$ and for a 3-integer cochain v of L such that $\delta v = g^* a \smile g^* b$. Then from (4.3), (4.1), (2.14) we have

$$\delta(a^{\#} b) = \psi_1(a, b; f_1^* v) - \psi_0(a, b; f_0^* v) \quad \text{mod } 2. \tag{4.7}$$

This proves

THEOREM 4.1. *The ψ-product and the ψ-square are independent of the order used to define them.*

THEOREM 4.2. *If $f: K' \to K$ is simplicial, then ψ-operation commutes with f^*.*

Since, for any order α in K, there exists an order in K' such that f is order-preserving, Theorem 4.2 follows from (2.14) and Theorem 4.1.

We shall prove here (2.8). Take the order α_1 as the inversion of α_0 in (4.6). Then thereby we have

$$\begin{aligned} g^* a \smile g^* b(\bar{4}\bar{3}\bar{2}\bar{1}\bar{0}) &= b(012)a(234), \\ {}'' \qquad (0\bar{3}\bar{2}\bar{1}\bar{0}) &= a(023)b(012), \\ {}'' \qquad (01\bar{3}\bar{2}\bar{1}) &= -a(013)b(123), \\ {}'' \qquad (012\bar{3}\bar{2}) &= 0, \\ {}'' \qquad (0123\bar{3}) &= 0, \\ {}'' \qquad (01234) &= a(012)b(234) \quad \text{etc.} \end{aligned} \tag{4.8}$$

Choose v such as

$$\begin{aligned} v(\bar{3}\bar{2}\bar{1}\bar{0}) &= u(0123) + a(023)b(012) - a(013)b(123), \\ v(0\bar{3}\bar{2}\bar{1}) &= -u(0123) + a(013)b(123), \\ v(0\bar{2}\bar{1}\bar{0}) &= 0, \\ v(01\bar{3}\bar{2}) &= -u(0123), \\ v(012\bar{3}) &= u(0123), \\ v(012\bar{2}) &= 0, \quad v(01\bar{2}\bar{1}) = 0, \\ v(0123) &= u(0123) \quad \text{etc.,} \end{aligned} \tag{4.9}$$

where $i(\bar{i})$ denotes $A_0^i(A_1^i)$ and $\delta u = a \smile_0 b$, then we have $\delta v = g^* a \smile g^* b$, $f_1^* v = u + a \smile_1^0 b$ and $f_0^* v = u$. It follows from (4.7) that

$$\psi_1(a, b; u + a \smile_1^0 b) - \psi_0(a, b, u) = \delta(a^{\#} b) \quad \text{mod } 2. \tag{4.10}$$

Since it is easy to see that $\psi_1(a, b; u + a \smile_1^0 b) \backsim \psi_0(b, a; u + a \smile_1^0 b)$ mod 2, we obtain (2.8).

§ 5. ψ-product in space and topological invariance of ψ-product.

Let X be a topological space. Let $H^p(X, G)$ denote the Cech cohomology group of X with coefficients in G. An element $\{\xi\} \in H^p(X, G)$ is represented by $\xi \in H^p(K, G)$, where K is the nerve of some finite covering of X by closed sets. If $\xi' \in H^p(K', G)$ for a second covering complex K', and $\{\xi\} = \{\xi'\}$, then there exists a common refinement of the two coverings with nerve K'' such that

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$$g^*\xi = g'^*\xi' \quad \text{in} \quad H^p(K'', G) \tag{5.1}$$

where $g: K'' \to K$ and $g': K'' \to K'$ are simplicial projections determined by inclusion relations among the closed sets of the various coverings. From (5.1) and Theorem 4.2 it follows that $\{\psi(\xi, \eta)\} = \{\psi(\xi', \eta')\}$ for $\xi, \eta \in H^2(K, Z)$ and $\xi', \eta' \in H^2(K', Z)$ such that $\xi \smile \eta = 0$, $\xi' \smile \eta' = 0$, $\{\xi\} = \{\xi'\}$, $\{\eta\} = \{\eta'\}$. Therefore

$$\psi(\{\xi\}, \{\eta\}) = \{\psi(\xi, \eta)\} \in H^5(X, Z_2) \quad \text{mod} \quad \sigma_*[H^3(X, Z)] \tag{5.2}$$

defines ψ-product for $\{\xi\}$ and $\{\eta\} \in H^2(X, Z)$ such that $\{\xi\} \smile \{\eta\} = 0 \in H^4(X, Z)$.

If $f: X' \to X$ is continuous, then f induces a homomorphism

$$f^*: H(X, G) \to H(X', G). \tag{5.3}$$

It is determined as follows. Let $\xi \in H^p(K, G)$, represent $\{\xi\} \in H^p(X, G)$, where K is the nerve of the covering $[U]$. Then $[f^{-1}(U)]$ is a covering of X' with nerve K'. Let $f_K: K' \to K$ be the simplicial map which attaches the vertex $f^{-1}(U)$ of K' to the vertex U of K. Then (5.3) is obtained by

$$f^*\{\xi\} = \{f_K^*\xi\}. \tag{5.4}$$

By (5.4), (5.2) and Theorem 4.2

$$f^*\psi = \psi f^*. \tag{5.5}$$

Suppose now that X is the space of a complex K. Then K is the nerve of the covering of X by the closed stars of the vertices of K in the first barycentric subdivision K' of K. If $\xi \in H^p(K, G)$, then $\varphi\xi = \{\xi\} \in H^p(X, G)$ is known to be an isomorphism $\varphi: H^p(K, G) \approx H^p(X, G)$.

Since $\varphi\psi(\xi, \eta) = \{\psi(\xi, \eta)\} = \psi(\{\xi\}, \{\eta\}) - \psi(\varphi\xi, \varphi\eta)$, it follows that $\varphi\psi = \psi\varphi$. Therefore the operation ψ as defined in a complex has a topological invariant meaning.

§ 6. Relation between the ψ-product and the functional cup product.

Let k; $K' \to K$ be an order-preserving simplicial map of a complex K' into another complex K. Let k^* be the cochain map induced by k. Then we have from (2.14)

$$k^*\psi(a, b;\ u) = \psi(k^*a, k^*b;\ k^*u). \tag{6.1}$$

If $k^*a \backsim 0$ in K', then the right hand side vanishes in the sense of mod $\sigma_*[H^3(K', Z)]$. But here we shall compute it mod $k^*\sigma_*[H^3(K, Z)]$, then we have by (6.1)

$$k^*\psi(a, b, u) = k^*a^{\vee_{-1}}k^*b + k^*\tilde{a} \smile (k^*b \smile_1 k^*b) + \mathfrak{p}_1 k^*u + k^*a \smile k^*u + k^*u \smile k^*b. \tag{6.2}$$

Since, from (3.5)′, we have

$$\begin{aligned} k^*a^{\vee_{-1}}k^*b + k^*\tilde{a} \smile (k^*b \smile_1 k^*b) \backsim \mathfrak{p}_1(e \smile k^*b) \\ + k^*a \smile (e \smile k^*b) + (e \smile k^*b) \smile k^*b \qquad \text{mod } 2, \end{aligned} \tag{6.3}$$

where $\delta e = k^*a$, (6.2) and (6.3) give

$$k^*\psi(a, b;\ u) \backsim \mathfrak{p}_1(k^*u - e \smile k^*b) + k^*a \smile (k^*u - e \smile k^*b) + (k^*u - e \smile k^*b) \smile k^*b \quad \text{mod } 2. \tag{6.4}$$

This implies

$$k^*\psi(a, b;\ u) \backsim S_{q_1}(a \smile^k b) + (a \smile^k b) \smile k^*b \quad \text{mod } 2. \tag{6.5}$$

where $a \smile^k b = k^*u - e \smile k^*b$ is the functional cup product of a and b (see Steenrod [2]). The both sides of (6.5) have an invariant meaning and they are determined mod the same group $k^*\sigma_*[H^3(K, Z)]$. Change of order in K or in K' does not effect (6.5) in this sense. Therefore, for any simplicial map k, we have

THEOREM 6.1. *If $k: K' \to K$ is a simplicial map such that $k^*\{a\} = 0$ in K', then we have*

$$k^*\psi(\{a\}, \{b\}) = S_{q_1}(\{a\} \smile^k \{b\}) + (\{a\} \smile^k \{b\}) \smile k^*\{b\} \quad mod \quad k^*\sigma_*[H^3(K, Z)],$$

where $\{a\}, \{b\} \in H^2(K, Z)$ and $\{a\} \smile \{b\} = 0 \in H^4(K, Z)$.

THEOREM 6.2. *If $k^*\{c\} = 0$, $\{c\} \smile \{c\} = 0 \in H^4(K, Z)$, then we have*

$$k^*\psi\{c\} = S_{q_1}(\{c\} \smile^k \{c\}) \quad mod \quad k^*S_{q_1}H^3(K, Z).$$

Remark. These theorems hold for spaces.

PART II. EXTENSION AND HOMOTOPY CLASSIFICATION PROBLEMS

§ 7. Pairing of coefficients and the *i*-square.

In this section. an algebraic preparation is described, which will be made use of in the next section.

Let G be an arbitrary abelian group, and let H be an abelian group, each element of which has order p (p: prime number).

LEMMA 7.1.[4] *If $S: G \to H$ is a homomorphism, then there exists a symmetric homomorphism φ of the p-fold tensor product $G \otimes \ldots \otimes G$ of G into H such that $\varphi(\alpha \otimes \ldots \otimes \alpha) = S(\alpha)$. (The adjective "symmetric" means that $\varphi(\alpha_{i_1} \otimes \ldots \otimes \alpha_{i_p}) = \varphi(\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_p)$ if $(i_1 \ldots i_p)$ is a permutation of $(1 \ldots p)$.)*

Proof. H is considered as a module over the prime field of characteristic p, therefore H has a base and H is a weak direct sum $\sum_\mu H_\mu$ of cyclic groups H_μ of order p. Let $S_\mu(\alpha)$ be the μ-th component of $S(\alpha)$ and set

$$\varphi_\mu(\alpha_1 \otimes \ldots \otimes \alpha_p) = S_\mu(\alpha_1) \circ S_\mu(\alpha_2) \cdots S_\mu(\alpha_p), \tag{7.1}$$

Where "$\circ$" denotes the multiplication in H_μ as the field of p elements. Then φ_μ is a symmetric homomorphism of $G \otimes \ldots \otimes G$ into H and

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[4] This lemma is suggested during discussion with professors Y. Matsushima, M. Kuranishi and Mr. N. Itô.

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$$(7.2)\qquad \varphi_\mu(\alpha\otimes\ldots\otimes\alpha)=[S_\mu(\alpha)]^p\equiv S_\mu(\alpha)\quad \mathrm{mod}\ p.$$

Define

$$(7.3)\qquad \varphi(\alpha_1\otimes\ldots\otimes\alpha_p)=\sum_\mu\varphi_\mu(\alpha_1\otimes\ldots\otimes\alpha_p),$$

then we have the required homomorphism.

By a pairing of G with itself to H, we mean a distributive multiplication "$\circ$" which gives an element $\alpha\circ\beta$ of H for two elements α, β of G.

COROLLARY TO LEMMA 7.1.[5] *Suppose $p=2$, then any homomorphism $S: G\to H$ is extensible to a commutative pairing of G with itself to H.*

Let K be a finite simplicial complex, and let G be an abelian group, and let G' be an abelian group each element of which has order 2.

LEMMA 7.2.[5] *The i-square $S_{q_i}: H^n(K,G)\to H^{2n-i}(K,G')$ is independent of the choice of a commutative pairing of the coefficient group G with itself to G' so far as $\alpha\circ\alpha=S(\alpha)$ for a given homomorphism $S: G\to G'$.*

Proof. Let $c^n=\sum\alpha_i\sigma_i^n$ ($\alpha_i\in G$, σ_i^n: an n-simplex of K) be a cocycle. The coefficients α_i of c^n generate a subgroup B of G. B is a finitely generatsed abelian group and, therefore, has a minimal system $\{\beta_\mu\}$ of generators β_μ. Then c^n is written by β_μ to be $c^n=\sum\beta_\mu\cdot c_\mu^n$, where c_μ^n is a cocycle mod m_μ ($m_\mu\geqq 0$ is the order of β_μ). If either m_μ or m_ν is odd, then $\beta_\mu\circ\beta_\nu=0$. Then $S_{q_i}c^n=(\sum\beta_\mu c_\mu^n)\smile_i(\sum\beta_\nu c_\nu^n)=\sum_\mu S(\beta_\mu)S_{q_i}c_\mu^n+\sum_{\mu>\nu}(\beta_\mu\circ\beta_\nu)(c_\mu^n\smile_i c_\nu^n+c_\nu^n\smile_i c_\mu^n)$, where $c_\mu^n\smile_i c_\nu^n+c_\nu^n\smile_i c_\mu^n\backsim 0$ mod 2 since both m_μ and m_ν are even. Thus we have

$$(7.4)\qquad S_{q_i}c^n\backsim\sum_\mu S(\beta_\mu)\cdot S_{q_i}c_\mu^n.$$

This proves the lemma.

§8. The third obstruction.

For the application of the results in Part I to the problems of extension and homotopy classification, we shall exclusively deal with mappings of a simplicial complex K into a 2-sphere S^2 in the sequal. However, we shall here prove a theorem concerning the extension cocycles of mappings in a generalized case which shows the behavior of the third obstructions. The result was already announced in a previous note [20].

Let K denote a finite simplicial complex, K^r its r-skeleton. Let Y be an $(n-1)$-connected[6] topological space (besides the $(n-1)$-connectedness we assume no restriction to the homotopy groups of Y). If f, g are normal[7] maps

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5) This was stated firstly by Postnikov [11].

6) A space Y is called $(n-1)$-connected if it is arcwise connected and its i-th homotopy group $\pi_i(Y)$ vanishes for $i=1, 2, \ldots, n-1$.

7) A map f of a complex K into an $(n-1)$-connected space Y is called normal when $f(K^{n-1})$ is a point of Y.

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of K^{n+2} into Y which coincide on K^n, then the $(n+3)$-extension cocycles $C^{n+3}(f)$, $C^{n+3}(g)$ with coefficients in $\pi_{n+2}(Y)$ are related as follows.

THEOREM 8.1. *$C^{n+3}(f)-C^{n+3}(g) \backsim S_{q_{n-1}}d^{n+1}$ in case $n>2$,*

$$C^5(f)-C^5(g) \backsim S_{q_1}d^3+c^2 \smile d^3 \quad \text{in case} \quad n=2,$$

where $d^{n+1}=d^{n+1}(f, g)$ is the separation cocycle of f and g, and $c^2=c^2(f)=c^2(g)$ is the characteristic cocycle with coefficients in $\pi_2(Y)$. Let $\eta_*:\pi_{n+1}(Y)\to {}_2[\pi_{n+2}(Y)]$[8] *be the homomorphism induced by the superposition of the elements of $\pi_{n+1}(Y)$ by an essential element η of $\pi_{n+2}(S^{n+1})$. The pairing of coefficients in $S_{q_{n-1}}d^{n+1}$ is an extension of η_* (see Corollary to Lemma 7.1), while the coefficients in the term $c^2 \smile d^3$ are paired by the Whitehead product.*

COROLLARY TO THEOREM 8.1. *Suppose that the space Y is an n-sphese S^n, then in the same notations as in the theorem,*

$$C^{n+3}(f)-C^{n+3}(g) \backsim S_{q_{n-1}}d^{n+1} \quad \text{for} \quad n \geqq 2.$$

Proof of Corollary. The Whitehead product $\alpha\beta$ of $\alpha \in \pi_2(S^2)$ and $\beta \in \pi_3(S^2)$ vanishes and also the term $c^2 \smile d^3$. (cf. (3.72) in G. W. Whitehead [12]).

Proof of Theorem 8.1. Denote K_0^{n+2} the cell complex, the image of a cellular map $h:K^{n+2}\to K_0^{n+2}$ such that h is homeomorphic on an open simplex of dimension r for $r \geqq n$ while $h(K^{n-1})=p_0$ is a point. Since the map $f:K^{n+2}\to Y$ is normal, i.e. $f(K^{n-1})=*$ a fixed point in Y, there exists a map $\varphi:K_0^{n+2}\to Y$ such that the composition

$$\varphi \circ h = f. \tag{8.2}$$

The coefficients $d^{n+1}(\sigma_i^{n+1})$ of $d^{n+1}(f, g)$ generate a subgroup B of $\pi_{n+1}(Y)$. Since B is finitely generated, B has a minimal system $\{\beta_\mu\}$ of generators β_μ with order $m_\mu \geqq 0$. Then $d^{n+1}(f, g)=\sum \beta_\mu \cdot d_\mu^{n+1}$ where d_μ^{n+1} is an integer cochain such that $0 \leqq d_\mu^{n+1}(\sigma^{n+1})<m_\mu$ (if $m_\mu>0$) and $\delta d_\mu^{n+1}\equiv 0 \bmod m_\mu$. Let $B_\mu^{n+2}=S_\mu^{n+1}\smile e_\mu^{n+2}$ be a cell complex composed of an $(n+1)$-sphere S_μ^{n+1} and an $(n+2)$-cell e_μ^{n+2} attached to S_μ^{n+1} by a map of ∂e_μ^{n+2} into S_μ^{n+1} with degree m_μ. Let $B^{n+2}=\sum_\mu B_\mu^{n+2}$ be a collection of B_μ^{n+2} such that $\bigcap_\mu B_\mu^{n+2}=p$ is a point of B^{n+2} and $B_\mu^{n+2} \frown B_\nu^{n+2}=p$. Let $R^{n+2}=K_0^{n+2}+B^{n+2}$ where K_0^{n+2} is attached to B^{n+2} by the identification of the point p_0 to p. The map φ is naturally extended to a map $\varphi:R^{n+2}\to Y$.

Define now a map $k:K^{n+2}\to R^{n+2}$ as follows. Set $k=h$ on K^n. Order linearly the vertices of K. In each $(n+1)$-simplex σ_i^{n+1} $(A_0 \ldots A_{n+1})$, we take an n-sphere S_i^n where the south pole is the first vertex A_0, $S_i^n-A_0 \subset$ the inner of σ_i^{n+1}. S_i^n bounds an $(n+1)$-cell ε_i^{n+1}. Define $k:\sigma_i^{n+1}\to h(\sigma_i^{n+1})+\sum_\mu S_\mu^{n+1}$ such that

[8] ${}_2G$ denotes the subgroup generated by elements with order 2 of an abelian group G.

$k(S_i^n) = p_0$ and $k(\sigma_i^{n+1} - \varepsilon_i^{n+1}) = h(\sigma_i^{n+1}) - p_0$ is homeomorphic and $k(\varepsilon_i^{n+1}) \subset \sum_\mu S_\mu^{n+1}$ has degree $-d_\mu^{n+1}(\sigma_i^{n+1})$ on S_μ^{n+1} for each μ. Thus we have $d^{n+1}(h, k) = \sum \iota_\mu \cdot d_\mu^{n+1}$ (ι_μ: a generator of $\pi_{n+1}(B_\mu^{n+2})$ and

$$d^{n+1}(f, g) = \sum \beta_\mu \cdot d_\mu^{n+1} = d^{n+1}(\varphi \circ h, \varphi \circ k) = -k^* \sum \beta_\mu \cdot S_\mu^{n+1}, \tag{8.3}$$

where S_μ^{n+1} denotes the generating $(n+1)$-cocycle mod m_μ of B_μ^{n+2}. Form (8.3) and (8.2), it follows that

$$d^{n+1}(\varphi \circ k, g) = 0 \tag{8.4}$$

This means that $\varphi \circ k$ and g are homotopic on K^{n+1} rel. K^n. Since $d^{n+1}(h, k)$ is the image of the cocycle $\sum -\iota_\mu \cdot S_\mu^{n+1}$ in B^{n+2} by k^*, we can extend k to a map $k: K^{n+2} \to R^{n+2}$ such that if $\rho: R^{n+2} \to K_0^{n+2}$ be the projection, then

$$\rho \circ k \sim h \tag{8.5}$$

on K^{n+2}. Then (8.4) shows that $C^{n+3}(g) \backsim C^{n+3}(\varphi \circ k)$ (cf. Eilenberg [15]), together with (8.2) we have

$$C^{n+3}(f) - C^{n+3}(g) \backsim C^{n+3}(\varphi \circ h) - C^{n+3}(\varphi \circ k). \tag{8.6}$$

We shall compute the latter. Let $\overline{K}_0^{n+2} = K_0^{n+3} \smile \sum \varepsilon_l^{n+3}$, $B^{n+3} = B^{n+2} \smile \sum e_\mu^{n+3}$ and let $R^{n+3} = \overline{K}_0^{n+3} + B^{n+3}$ in case $n>2$, $R^5 = (\overline{K}_0^5 + B^5) \smile \sum e_{j,\mu}^5$ in case $n=2$, such that each cell ε_l^{n+3} is attached to K_0^{n+2} by a map of $\partial \varepsilon_l^{n+3}$ into K_0^{n+2} representing each generator of $\pi_{n+2}(K_0^{n+2})$, each cell e_μ^{n+3} is attached to B^{n+2} by a map of ∂e_μ^{n+3} into S_μ^{n+1} (m_μ: even) representing a generator of $\pi_{n+2}(B_\mu^{n+2})$ and each cell $e_{j,\mu}^5$ is attached to $\overline{K}_0^5 + B^5$ by a map of $\partial e_{j,\mu}^5$ into $S_j^2 \vee S_\mu^3$ representing the Whitehead product of a generator of $\pi_2(S_j^2)$ and that of $\pi_3(S_\mu^3)$, where $S_j^2 = h(\sigma_j^2)$ for each 2-simplex σ_j^2 of K. Then $\pi_{n+2}(R^{n+3})$ vanishes (cf. Blakers and Massay [14]) and the maps h, $k: K^{n+2} \to R^{n+2}$ are extended to maps of K^{n+3} into R^{n+3}, where $h(K^{n+3}) \subset \overline{K}_0^{n+3}$.

Denote by ξ_l, $\eta_*(\beta_\mu)$, $\alpha_j\beta_\mu$ the elements of $\pi_{n+2}(Y)$ represented by the maps $\varphi(\partial \varepsilon_l^{n+3})$, $\varphi(\partial e_\mu^{n+3})$, $\varphi(\partial e_{j,\mu}^5)$ respectively. Then

$$C^{n+3}(\varphi \circ h) = h^* C^{n+3}(\varphi), \quad C^{n+3}(\varphi \circ k) = k^* C^{n+3}(\varphi) \quad \text{and} \tag{8.7}$$

$$C^{n+3}(\varphi) = \sum \xi_l \cdot \varepsilon_l^{n+3} + \sum_{m_\mu : \, even} \eta_*(\beta_\mu) \cdot e_\mu^{n+3} + \sum \alpha_j \beta_\mu \cdot e_{j,\mu}^5, \tag{8.8}$$

where the last term is added only in case of $n=2$. From the way of construction of h, we have $h^* e_\mu^{n+3} = 0$, $h^* e_{j,\mu}^5 = 0$. From (8.5) we have $k^* \varepsilon_l^{n+3} = h^* \varepsilon_l^{n+3}$. Thus we obtain, by (7.4), (8.3)

$$\begin{aligned} C^{n+3}(\varphi \circ h) - C^{n+3}(\varphi \circ k) &= -k^*\Big(\sum_{m_\mu : \, even} \eta_*(\beta_\mu) e_\mu^{n+3} + \sum \alpha_j \beta_\mu \cdot e_{j,\mu}^5\Big) \\ &\backsim -k^* \sum \eta_*(\beta_\mu) S_{q_{n-1}} S_\mu^{n+1} - k^* \sum \alpha_j \beta_\mu S_j^2 \smile S_\mu^3 \\ &\backsim k^* S_{q_{n-1}}\Big(\sum \beta_\mu S_\mu^{n+1}\Big) - \Big(\sum \alpha_j k^* S_j^2\Big) \smile k^* \sum \beta_\mu S_\mu^3. \end{aligned} \tag{8.9}$$

$$\smile S_{q_{n-1}} d^{n+1}(f, g) + c^2(f) \smile d^3(f, g).$$

Here we used the fact that $S_{q_{n-1}} S_\mu^{n+1} \smile e_\mu^{n+3}$ mod 2 in B^{n+3} (m_μ: even). (8.6) and (8.9) prove the theorem.

§ 9. Extension theorem.

Let K be finite simplicial complex, f a map of K^2 into the 2-sphere S^2 which is extensible to a map $\bar{f}$ of K^4 into S^2. Then from Corollary to Theorem 8.1, we conclude that the third obstruction $\{C^5(\bar{f})\}$ is determined independently of the choice of an extension $\bar{f}: K^4 \to S^2$ of a given map $f: K^2 \to S^2$ up to the subgroup $S_{q_1} H^3(K, \pi_3(S^2))$ of $H^5(K, \pi_4(S^2))$. An algebraic determination of the residue class $\{C^5(\bar{f})\}$ mod $S_{q_1} H^3(K, \pi_3(S^2))$ is furnished by

THEOREM 9.1. *If a map $f: K^2 \to S^2$ is extensible to a map $\bar{f}$ of K^4, i.e., $\delta c^2(f) = 0$ and $c^2(f) \smile c^2(f) \backsim 0$ in K, where $c^2(f) = f^* S^2$ is the characteristic cocycle, then the residue class reprsented by the third obstruction*

$$\{C^5(\bar{f})\} \quad mod \quad S_{q_1} H^3(K, \pi_3(S^2)) = \psi\{c^2(f)\},$$

where $\pi_2(S^2)$ and $\pi_3(S^2)$ are regarded as the group of integers, and $\pi_4(S^2)$ is regarded as the group of integers reduced mod 2 (see § 2).

Proof. Let $P^5 = S^2 \smile e^5$ be a cell complex constructed from a 2-sphere S^2 and a 5-cell e^5 attached to S^2 by an essential map of ∂e^5 into S^2. Let $M^5 = \bar{S}^3 \smile \bar{e}^5$ be a cell complex constructed from a 3-sphere $\bar{S}^3$ and a 5-cell $\bar{e}^5$ attached to $\bar{S}^3$ by an essential map of $\partial \bar{e}^5$ into $\bar{S}^3$ (see Steenrod [1], Section 20). Let $\kappa: (M^5, \bar{S}^3) \to (P^5, S^2)$ be a cellular map such that κ is homeomorphic on the open cell $\bar{e}^5$ and is of Hopf invariant 1 on $\bar{S}^3$. Then the functional cup product $\{S^2\} \smile_\kappa \{S^2\} = \{\bar{S}^3\}$ (see Steenrod [2]) and $S_{q_1}\{\bar{S}^3\} = \{\bar{e}^5\}$ mod 2. By making use of theorem 6.2 and its remark, we have

$$\kappa^* \psi\{S^2\} = S_{q_1}\{\bar{S}^3\} = \{\bar{e}^5\} = \kappa^*\{e^5\} \qquad \text{mod } 2.$$

Since $\kappa^*: H^5(P^5) \to H^5(M^5)$ is isomorphic, we obtain

$$\psi\{S^2\} = \{e^5\} \qquad \text{mod 2 in } P^5. \tag{9.1}$$

The given map $\bar{f}: K^4 \to S^2$ is extensible to a map $\bar{f}: K^5 \to P^5$ and

$$\{C^5(f)\} = \bar{f}^*\{e^5\} = \bar{f}^* \psi\{S^2\} \equiv \psi(f^*\{S^2\}) = \psi\{c^2(f)\} \quad \text{mod } S_{q_1} H^3(K, \pi_3(S^2)). \tag{9.2}$$

This prove the theorem.

§ 10. Classification theorem.

THEOREM 10.1. *Let f, g be two normal map of a 4-dimensional finite simplicial complex K into the 2-sphere S^2 which coincide on the 3-skeleton K^3. Then $f \sim g$ if and only if there exists a 1-cocycle λ^1 with coefficients in $\pi_2(S^2)$ such* *that*

$$2c^2 \smile \lambda^1 \backsim 0 \quad in \quad K, \tag{10.1}$$

for the characteristic cocycle $c^2 = f^* \iota \cdot S^2 = g^* \iota \cdot S^2$, *(in* (10.1) *the pairing of coefficients is defined as* $\iota \cdot \iota = \eta$ *for a generator* ι *of* $\pi_2(S^2)$ *and a generator* η *of* $\pi_3(S^2)$*) and*

$$\{d^4(f, g)\} \equiv \{S_{q_0} v_\lambda^2\} \quad mod \quad S_{q_0} H^2(K, \pi_3(S^2)), \tag{10.2}$$

where $d^4(f, g)$ *is the separation cocycle of* f *and* g, v_λ^2 *is a* 2*-cochain with coefficients in* $\pi_3(S^2)$ *such that*

$$\delta v_\lambda^2 = 2c^2 \smile \lambda^1. \tag{10.3}$$

In defining $S_{q_0} v_\lambda^2 = v_\lambda^2 \smile v_\lambda^2$, *the coefficients are paired as* $\eta \cdot \eta = \xi$ *for a generator* ξ *of* $\pi_4(S^2)$.

Let $\Gamma^1(K, \pi_2(S^2))$ denote the subgroup, generated by classes $\{\lambda^1\}$ of 1-cocycles λ^1 satisfying (10.1), of the 1-cohomology group $H^1(K, \pi_2(S^2))$ of K with coefficients in $\pi_2(S^2)$. Then setting

$$\Phi_{c^2}(\{\lambda^1\}) = \{S_{q_0} v_\lambda^2\} \quad \mathrm{mod} \quad S_{q_0} H^2(K, \pi_3(S^2)), \tag{10.4}$$

we have a homomorphism

$$\Phi_{c^2} : \Gamma^1(K, \pi_2(S^2)) \to H^4(K, \pi_4(S^2)) / S_{q_0} H^2(K, \pi_3(S^2)). \tag{10.5}$$

The homomorphism Φ_{c^2} depends only on the cohomology class $\{c^2\}$ as it is easily seen from the definition (10.4).

THEOREM 10.2. *Let* K *be a* 4*-dimensional finite simplicial complex. Consider the homotopy classes of those mapping of* K *into* S^2 *which are homotopic to one another on* K^3. *All such homotopy classes are in a one to one correspondence with the cosets of the factor group*

$$H^4(K, \pi_4(S^2)) / S_{q_0} H^2(K, \pi_3(S^2)) \Big/ \Phi_{W^2}[\Gamma^1(K, \pi_2(S^2))] \tag{10.6}$$

where $W^2 = \{c^2\}$ *is the characteristic class of these mappings.*

Since Theorem 10.2 follows from Theorem 10.1, we shall prove the letter.

Proof. As it is well known, f and g are homotopic to each other if and only if $d^4(f, g) \backsim O^4(f, f)$, the latter being one of the homotopy obstruction cocycles. Which satisfies the following condition: There exists a map F of the 4-skeleton L^4 of the product complex $L = K \times I$ into S^2 such that $F = f$ on $(K \times 0) \smile (K \times 1)$ and $O^4(f, f) = DC^5(F)$ for the extension cocycle $C^5(F)$ of F (for the operator D, see (4.2)). Thus the problem is reduced to seek such maps F and to compute $C^5(F)$ by making use of Theorem 9.1. We take a simplicial subdivision of L such as described in § 4. Let $F: L^4 \to S^2$ be normal and have the above mentioned property: $F=f$ on $(K\times 0)\smile(K\times 1)$, such a map is called allowable. Then the characteristic cocycle $c^2(F)$ of F satisfies

$$c^2(F)\smile c^2(F)\backsim 0. \tag{10.7}$$

Henceforth we regard cochains with coefficients in $\pi_2(S^2)$ or $\pi_3(S^2)$ integer cochains, cochains with coefficients in $\pi_4(S^2)$ as cochains mod 2. If we put

$$\lambda^1(\sigma)=c^2(F)(\sigma\times I)' \quad \text{for each 1-simplex } \sigma \text{ of } K, \tag{10.8}$$

(where the symbol "$'$" denotes the subdivision) it follows that λ^1 is an integer 1-cocycle of K and from (10.7) that

$$c^2(f)\smile\lambda^1+\lambda^1\smile c^2(f)\backsim 0, \quad \text{or} \quad 2c^2(f)\smile\lambda^1\backsim 0. \tag{10.9}$$

Conversely, for any such 1-cocycle λ^1 we can choose an allowable map F which satisfies (10.7). For the sake of convenience, let us consider the case with the following restriction to allowable maps F without losing the generality. Denote $c^2(F)$ by w^2, $c^2(f)$ by c^2, then we have

$$\begin{aligned} &w(0\bar{1}\bar{2})=c(012),\\ &w(01\bar{2})=c(012)+\lambda(01),\\ &w(01\bar{1})=\lambda(01), \quad w(0\bar{0}\bar{1})=0 \quad \text{etc.,} \end{aligned} \tag{10.10}$$

thereby we have

$$\begin{aligned} &w\smile w(0123\bar{3})=c(012)\cdot\lambda(23)\\ &\quad{}'' \quad (012\bar{2}\bar{3})=0,\\ &\quad{}'' \quad (01\bar{1}\bar{2}\bar{3})=\lambda(01)\cdot c(012),\\ &\quad{}'' \quad (0\bar{0}\bar{1}\bar{2}\bar{3})=0 \quad \text{etc.,} \end{aligned} \tag{10.11}$$

or simply,

$$w\smile w=(c\smile c)\times 0-(c\smile\lambda+\lambda\smile c)\times I+(c\smile c)\times 1.$$

For two allowable maps F, $F':L^4\to S^2$ such that $c^2(F)=c^2(F')=w^2$, we have $C^5(F)-C^5(F')\backsim S_{q_1}d^3(F, F')$. In order to compute $\{C^5(F)\}$ mod $S_{q_1}H^3(L, Z)=\psi\{w^2\}$, we choose a 2-integer cochain d_λ^2 of K and a 3-integer cochain $\bar{u}^3$ of L such that

$$\delta d_\lambda=c\smile\lambda+\lambda\smile c, \tag{10.12}$$

and $\delta\bar{u}=w\smile w$, for example, set

$$\begin{aligned} &\bar{u}(\bar{0}\bar{1}\bar{2}\bar{3})=\bar{u}(0123)=u(0123),\\ &\bar{u}(0\bar{1}\bar{2}\bar{3})=u(0123),\\ &\bar{u}(01\bar{2}\bar{3})=u(0123)+\lambda(01)\cdot c(123),\\ &\bar{u}(012\bar{3})=u(0123)+\lambda(01)\cdot c(123)+d_\lambda(012),\\ &\bar{u}(012\bar{2})=d_\lambda(012),\\ &\bar{u}(0\bar{0}\bar{1}\bar{2})=\bar{u}(01\bar{1}\bar{2})=0 \quad \text{etc.,} \end{aligned} \tag{10.13}$$

where u is a 3-integer cochain of K such that $\delta u = c \smile c$. By means of (10.10), (10.12) and (10.13), we obtain

$$D\psi(w\,;\ \bar{u}) \backsim \Phi_c(\lambda\,;\ d_\lambda) \quad \text{mod } 2. \tag{10.14}$$

where $\Phi_c(\lambda\,;\ d_\lambda)$ is a 4-cocycle mod 2 of K defined as follows.

$$\begin{aligned}\Phi_c(\lambda\,;\ d_\lambda) = {} & c \vee_{-1} \lambda + \lambda \vee_{-1} c + (\lambda \smile \lambda) \smile_1 (c \smile_1 c) \\ & + (c \smile \lambda) \smile_2 (\lambda \smile c) + \lambda \smile_1 (c \smile c) + \mathfrak{p}_0 d_\lambda \quad \text{mod } 2.\end{aligned} \tag{10.15}$$

Here $\mathfrak{p}_0 d_\lambda = d_\lambda \smile d_\lambda + d_\lambda \smile_1 \delta d_\lambda$ is the Pontrjagin square. The residue class $\{\Phi_c(\lambda\,;\ d_\lambda)\}$ mod $S_{q_0}H^2(K, Z)$ is determined independently of the choice of d_λ. The expression (10.15) can be reduced to a simple form:

$$\Phi_c(\lambda\,;\ d_\lambda) \backsim \mathfrak{p}_0 d_\lambda + \mathfrak{p}_0(c \smile_1 \lambda) \backsim S_{q_0} v_\lambda \quad \text{mod } 2. \tag{10.16}$$

where $v_\lambda = d_\lambda + c \smile_1 \lambda$ is a 2-cocycle mod 2 such that $\delta v_\lambda = 2c \smile \lambda$. From the earlier part of the proof and (10.14) and (10.16), we obtain (10.2). This complete the proof.

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