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On the Square of the First Zero of the Bessel Function $J_\nu(z)$

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Abstract. Let $j_{\nu,1}$ be the smallest (first) positive zero of the Bessel function $J_\nu(z)$, $\nu > -1$, which becomes zero when ν approaches -1. Then $j^2_{\nu,1}$ can be continued analytically to $-2 < \nu < -1$, where it takes on negative values. We show that $j^2_{\nu,1}$ is a convex function of ν in the interval $-2 < \nu \le 0$, as an addition to an old result [Á. Elbert and A. Laforgia, SIAM J. Math. Anal. **15**(1984), 206–212], stating this convexity for $\nu > 0$. Also the monotonicity properties of the functions $\frac{j^2_{\nu,1}}{4(\nu+1)}$, $\frac{j^2_{\nu,1}}{4(\nu+1)\sqrt{\nu+2}}$ are determined. Our approach is based on the series expansion of Bessel function $J_\nu(z)$ and it turned out to be effective, especially when $-2 < \nu < -1$.

1 Introduction and Results

The Bessel function $J_\nu(z)$ of first kind has the representation

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\frac{z}{2})^{2n+\nu}}{\Gamma(n+\nu+1)}, \quad z > 0$$

and has infinitely many positive zeros $j_{\nu,k}$, $k = 1, 2, \ldots$, $0 < j_{\nu,1} < j_{\nu,2} < \cdots$, tending to infinity as $\nu \to \infty$ [10, p. 478]. For $\nu > -1$ all zeros of $J_\nu(z)$ are positive. The first zero $j_{\nu,1}$ can be continued analytically to $\nu = -1$ where it vanishes. Continuing $j_{\nu,1}$ analytically to the interval $(-2, -1)$ we find, according to a theorem of Hurwitz [3], [10, p. 483] that $j_{\nu,1}$ becomes purely imaginary. At the point $\nu = -2$ the function $j_{\nu,1}$ is vanishing again. Concerning the local behavior of $j_{\nu,1}$, R. Piessens [9] has found the following representation

$$j_{\nu,1} = 2(\nu+1)^{1/2}\left[1 + \frac{\nu+1}{4} - \frac{7}{96}(\nu+1)^2 + \cdots\right]$$

in the neighborhood of $\nu = -1$. We shall investigate the function $j^2_{\nu,1}$ for $\nu > -2$ where it is real. Clearly the function

$$\ell(\nu) = \frac{j^2_{\nu,1}}{4(\nu+1)} \tag{1.1}$$

has the local representation

$$\ell(\nu) = 1 + \frac{\nu+1}{2} - \frac{1}{12}(\nu+1)^2 + \cdots \tag{1.2}$$

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which implies

(1.3) $$\lim_{\nu\to-1}\ell(\nu)=1,\quad \lim_{\nu\to-2}\ell(\nu)=0,\quad\text{and}\quad \ell(1)=\frac{j_{1,1}^2}{8}=1.83525\cdots.$$

Recalling the inequalities [6,(5.11)], [6,(5.12)]

$$j_{\nu,1}^2<2(\nu+1)(\nu+3),\quad \nu>-1$$
$$j_{\nu,1}^2>2(\nu+1)(\nu+3),\quad -2<\nu<-1,$$

we have

(1.4) $$\ell(\nu)<1+\frac{1}{2}(\nu+1)=\frac{\nu+3}{2}\quad\text{for } \nu>-2,\quad \nu\neq-1.$$

In [6], [7] one can find the graph of the function $j_{\nu,1}^2$ in the interval $(-2,0)$, indicating the property that $j_{\nu,1}^2$ is a convex function of ν in that interval. This property was proved for $3\le\nu<+\infty$ by J. T. Lewis and M. E. Muldoon [8]. Á. Elbert and A. Laforgia [2] proved this property for $j_{\nu,k}^2$, $k=1,2,\dots$, $\nu\ge0$. Also, they indicated that the function $j_{\nu,k}^2$ can not be convex on the whole interval $(-k,\infty)$ for $k=2,3,\dots$, and conjectured that the function $j_{\nu,1}^2$ is convex for $-1<\nu<0$. In [7] it was proved that $j_{\nu,1}^2$ decreases to a minimum and then increases again to 0 as ν increases from -2 to -1. In this paper we shall prove the convexity of $j_{\nu,1}^2$ in $(-2,0]$. Consequently, by [2] the function $j_{\nu,1}^2$ is convex on $(-2,\infty)$, too, because $dj_{\nu,1}/d\nu$ is continuous function of the variable ν (see [10, Ch. 15.6]). Concerning the function $\ell(\nu)$, two observations were formulated in [6, p. 9]:

(i) the function $\ell(\nu)$ is increasing for $\nu>-2$ (for $\nu>-1$ this fact is already known, see [5, Thm. 2]),

(ii) the function $\frac{\ell(\nu)}{\sqrt{\nu+2}}$ decreases in the interval $(-2,-1)$ and increases for $\nu>-1$.

All these observations turned out to be correct and we are going to prove them.

The main tool is the implicit relation between $\ell=\ell(\nu)$ and ν

(1.5) $$\begin{aligned}H(\ell,\nu)=1-\frac{\ell}{1!}+\frac{\ell^2}{2!}\frac{\nu+1}{\nu+2}-\frac{\ell^3}{3!}\frac{(\nu+1)^2}{(\nu+2)(\nu+3)}+\cdots\\ +\frac{(-1)^k\ell^k}{k!}\frac{(\nu+1)^{k-1}}{(\nu+2)\cdots(\nu+k+1)}+\cdots=0\end{aligned}$$

which comes from the series expansion of Bessel function $J_\nu(z)$. Introducing the notations

(1.6) $$e_0(\nu)=1,\quad e_k(\nu)=\frac{(\nu+1)^k}{(\nu+2)\cdots(\nu+k+1)},\quad k=1,2,\dots,$$

the relation (1.5) is written as follows

(1.7) $$H(\ell,\nu)=1-\frac{\ell}{1!}+\frac{\ell^2}{2!}e_1(\nu)-\frac{\ell^3}{3!}e_2(\nu)+\cdots+\frac{(-1)^k\ell^k}{k!}e_{k-1}(\nu)+\cdots=0.$$

Our statements on the function $H(\ell,\nu)$ are formulated in two lemmas:

Lemma 1 *The partial derivative $\frac{\partial H(\ell,\nu)}{\partial\ell}\equiv H_\ell$ is negative for $-2<\nu\le1$ and $0<\ell<2$.*

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Lemma 2 *The partial derivative* $\frac{\partial H(\ell,\nu)}{\partial \nu} \equiv H_\nu$ *is positive for* $-2 < \nu \le 1$ *and* $0 < \ell < 2$.

These two lemmas yield the following

Theorem 1 *The function* $\ell(\nu)$ *in* (1.1) *increases for* $-2 < \nu \le 1$.

Concerning the derivative $\ell'(\nu)$ of the function $\ell(\nu)$ with respect to ν the next lemma holds.

Lemma 3 *The function* $\ell'(\nu)$ *satisfies the inequalities*

(i) $\ell'(\nu) < \frac{\ell(\nu)}{2(\nu+2)}$ *for* $-2 < \nu < -1$;
(ii) $\ell'(\nu) > \frac{\ell(\nu)}{2(\nu+2)}$ *for* $-1 < \nu \le 1$.

Using this lemma and also the inequalities from [4]

$$\frac{1}{j_{\nu,1}} \frac{dj_{\nu,1}}{d\nu} > \frac{1}{j_{\nu,1}^2}\left[1 + (1 + j_{\nu,1}^2)^{1/2}\right] \quad \text{for } \nu > -1 \tag{1.8}$$

and from [5, (6.10)]

$$j_{\nu,1}^2 < \frac{2(\nu+1)(\nu+5)(5\nu+11)}{7\nu+19}, \quad \nu > -1, \tag{1.9}$$

we are going to prove

Theorem 2 *The function* $\frac{j_{\nu,1}^2}{4(\nu+1)\sqrt{\nu+2}}$ *decreases from* $\sqrt{2}$ *to 1 for* $-2 < \nu < -1$ *and increases for* $\nu > -1$.

From Theorem 2 we obtain the inequalities

$$4\sqrt{2}(\nu+1)\sqrt{\nu+2} < j_{\nu,1}^2 < 4(\nu+1)\sqrt{\nu+2}, \quad -2 < \nu < -1.$$

The right hand side is already known [6, (5.8)]. The lower bound is new and it is sharp when ν approaches -2.

Finally, we formulate our main result.

Theorem 3 *The function* $j_{\nu,1}^2$ *is convex for* $-2 < \nu \le 0$.

Using the convexity of $j_{\nu,1}^2$, we can obtain new inequalities in the interval $(-2, 0)$. For example, $j_{\nu,1} < j_{0,1}\sqrt{\nu+1}$ provided $-1 < \nu < 0$.

We conjecture that $\ell(\nu) = \frac{j_{\nu,1}^2}{4(\nu+1)}$ is a concave function for $\nu > -2$.

The question of convexity of $j_{\nu,1}^2$ is connected with the Putterman-Kac-Uhlenbeck conjecture [1] about a quantum mechanics problem which states that the sequence of the differences $j_{n,1}^2 - j_{n-1,1}^2$ is increasing as n increases where $j_{n,1}$ denotes the first positive zero of Bessel function $J_n(x)$, $n = 1, 2, \ldots$.

In the next section we give the proofs of the above results. Also the inequality (1.4) could be proved by our approach at least for $-2 < \nu \le 1$, but we shall not address ourselves to this problem here.

2 Proofs

During the proofs of the above statements we shall use the following relations. By (1.6) we obtain

$$e_k'(\nu) = \frac{e_k(\nu)}{\nu+1}\alpha_k(\nu), \quad k = 1, 2, \dots \tag{2.1}$$

where

$$\begin{aligned} \alpha_k(\nu) &= \frac{1}{\nu+2} + \frac{2}{\nu+3} + \dots + \frac{k}{\nu+k+1} = k - (\nu+1)c_k(\nu), \\ c_k(\nu) &= \frac{1}{\nu+2} + \frac{1}{\nu+3} + \dots + \frac{1}{\nu+k+1}, \end{aligned} \tag{2.2}$$

moreover

$$\begin{aligned} \bar{c}_k(\nu) &= \frac{1}{(\nu+2)^2} + \frac{1}{(\nu+3)^2} + \dots + \frac{1}{(\nu+k+1)^2}, \\ e_k''(\nu) &= \frac{e_k(\nu)}{\nu+1}\left[\frac{\alpha_k^2(\nu) - \alpha_k(\nu)}{\nu+1} + \alpha_k'(\nu)\right], \\ \alpha_k'(\nu) &= -c_k(\nu) - (\nu+1)c_k'(\nu) = -c_k(\nu) + (\nu+1)\bar{c}_k(\nu). \end{aligned} \tag{2.3}$$

For $\alpha_k(\nu)$ and $\bar{c}_k(\nu)$ we have the inequalities

$$\begin{aligned} \frac{1}{2}k < \alpha_k(\nu) < k \quad &\text{for } -1 < \nu \le 0 \\ \bar{c}_k(\nu) < \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} < \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \quad &\text{for } \nu > -1. \end{aligned} \tag{2.4}$$

Proof of Lemma 1 Partial differentiation of $H(\ell, \nu)$ in (1.5) with respect to the variable ℓ gives

$$\frac{\partial H(\ell,\nu)}{\partial \ell} \equiv H_\ell = -1 + \frac{1}{1!}e_1(\nu)\ell - \frac{1}{2!}e_2(\nu)\ell^2 + \dots + \frac{(-1)^k}{(k-1)!}e_{k-1}(\nu)\ell^{k-1} + \cdots. \tag{2.5}$$

Hence by (1.5)

$$\frac{\partial H(\ell,\nu)}{\partial \ell} = \frac{\partial H(\ell,\nu)}{\partial \ell} + H(\ell,\nu) = -\ell G(\ell,\nu)$$

where

$$G(\ell,\nu) = \frac{1}{\nu+2} - \frac{\ell}{1!}\frac{1}{\nu+3}e_1(\nu) + \dots + \frac{(-1)^k\ell^k}{k!}\frac{1}{\nu+k+2}e_k(\nu) + \cdots. \tag{2.6}$$

We observe first that for $-2 < \nu < -1$ the function $G(\ell,\nu)$ is a sum of positive terms hence $G(\ell,\nu) > 0$ and $H_\ell < 0$.

For $\nu = -1$ we have $\ell(-1) = 1$ and $G(\ell, -1) = 1 > 0$.

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For $-1 < \nu \le 1$ we observe that $G(\ell,\nu)$ is a sum of terms with alternating sign and the first term is positive. We are going to show that the terms of $G(\ell,\nu)$ form a Leibniz type series (*i.e.*, it is a series of the type $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ where $a_n \ge 0$ such that (i) $a_1 \ge a_2 \ge \cdots \ge a_n \ge \cdots$ and (ii) $\lim_{n\to\infty} a_n = 0$. Then this sum is convergent: $s = \sum(-1)^{n-1}a_n$, and $sa_1 \ge 0$. Since

$$\frac{\ell^k}{k!}\frac{e_k(\nu)}{\nu+k+2} - \frac{\ell^{k+1}}{(k+1)!}\frac{e_{k+1}(\nu)}{\nu+k+3} = \frac{\ell^k}{k!}\frac{e_k(\nu)}{\nu+k+2}\left[1 - \frac{\ell}{k+1}\frac{\nu+1}{\nu+k+3}\right]$$

and

$$\frac{\nu+1}{\nu+k+3} \le \frac{2}{k+4} \quad \text{for } -1 < \nu \le 1$$

we have $(k+1)(k+4) \ge 4 \ge 2\ell$. Consequently, we have a Leibniz type series in $G(\ell,\nu)$ which was to be proved.

Proof of Lemma 2 Partial differentiation of $H(\ell,\nu)$ in (1.5) with respect to the variable ν gives

$$\frac{\partial H(\ell,\nu)}{\partial \nu} \equiv H_\nu = \frac{1}{2!}e_1'(\nu)\ell^2 - \frac{1}{3!}e_2'(\nu)\ell^3 + \cdots + \frac{(-1)^k}{k!}e_{k-1}'(\nu)\ell^k + \cdots. \tag{2.7}$$

By (2.1)

$$\operatorname{sign} e_k'(\nu) = \begin{cases} 1 & \nu > -1 \\ (-1)^{k-1} & -2 < \nu < -1, \end{cases}$$

hence it follows from (2.7) that $H_\nu > 0$ for $-2 < \nu < -1$.

Now we prove that $H_\nu > 0$ also for $-1 < \nu < 1$. In this case we observe that $\frac{\ell^2}{2!}e_1'(\nu) > 0$ and that the signs of the consecutive terms of series (2.7) are alternating. So, we are going to show that the series (2.7) is of Leibniz type:

$$\frac{\ell^k}{k!}e_{k-1}'(\nu) > \frac{\ell^{k+1}}{(k+1)!}e_k'(\nu), \quad k = 2,3,\dots$$

or by (2.2)

$$\frac{e_{k-1}(\nu)}{\nu+1}\alpha_{k-1} > \frac{\ell}{k+1}\frac{e_k(\nu)}{\nu+1}\alpha_k,$$

hence by $e_k(\nu) = e_{k-1}(\nu)\frac{\nu+1}{\nu+k+1}$

$$\left[\frac{1}{\nu+2} + \frac{2}{\nu+3} + \cdots + \frac{k-1}{\nu+k}\right]\left[1 - \frac{\ell}{k+1}\frac{\nu+1}{\nu+k+1}\right] > \frac{k}{(\nu+k+1)^2}\frac{\ell(\nu+1)}{k+1}. \tag{2.8}$$

First we examine the case $k = 2$. We have from (2.8)

$$\frac{1}{\nu+2}\left[1 - \frac{\ell}{3}\frac{\nu+1}{\nu+3}\right] > \frac{2}{(\nu+3)^2}\frac{\ell(\nu+1)}{4},$$

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or after some simplifications

$$1 > \frac{\ell(\nu+1)(3\nu+7)}{3(\nu+3)^2}.$$

Since $\frac{(\nu+1)(3\nu+7)}{(\nu+3)^2} < \frac{5}{4}$ for $-1 < \nu < 1$, we have from the above inequality that $1 > \frac{\ell}{3}\frac{5}{4}$ or $\ell < \frac{12}{5}$ which is true.

Next we examine the case $k \geq 3$. We observe that the left hand side of (2.8) is a decreasing function of ν and for the right hand side we have

$$\frac{(\nu+1)}{(\nu+k+1)^2} < \frac{2}{(k+2)^2} \quad \text{in } -1 < \nu < 1.$$

So it is sufficient to show that

$$\left[\frac{1}{3} + \cdots + \frac{k-1}{k+1}\right]\left[1 - \frac{2\ell}{(k+1)(k+2)}\right] > \frac{2\ell k}{(k+1)(k+2)^2}. \tag{2.9}$$

For $k \geq 3$ the left hand side of (2.9) is an increasing function of k and the right hand side is a decreasing one. In particular, for $k = 3$ the inequality in (2.9) is reduced to

$$\left[\frac{1}{3} + \frac{2}{4}\right]\left[1 - \frac{2\ell}{4\cdot 5}\right] > \frac{2\cdot 3\ell}{4\cdot 5^2}$$

or to $\ell < \frac{250}{43}$ which is true and the proof of Lemma 2 is complete.

Proof of Theorem 1 Differentiation of $H\big(\ell(\nu),\nu\big) = 0$ with respect to ν gives

$$H_\ell \frac{d\ell(\nu)}{d\nu} + H_\nu = 0$$

hence

$$\frac{d\ell(\nu)}{d\nu} = \frac{H_\nu}{-H_\ell}. \tag{2.10}$$

According to Lemma 1 and Lemma 2, the relations $H_\nu > 0$, $H_\ell < 0$ hold provided $-2 < \nu \leq 1$, $0 < \ell < 2$, we find $\frac{d\ell(\nu)}{d\nu} > 0$. Since $\ell(1) = \frac{j_{1,1}^2}{8} = 1.83525\cdots < 2$ therefore $\frac{d\ell(\nu)}{d\nu} > 0$ for $-2 < \nu \leq 1$.

Proof of Lemma 3 Since $\ell'(\nu) = \frac{H_\nu}{-H_\ell}$, by Lemma 1 it is sufficient to prove that

$$2(\nu+2)H_\nu + \ell H_\ell < 0 \quad \text{for } -2 < \nu < -1 \tag{2.11}$$

and

$$2(\nu+2)H_\nu + \ell H_\ell > 0 \quad \text{for } -1 < \nu < 1. \tag{2.12}$$

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By (2.5), (2.7) we have

$$\begin{aligned}\frac{2(\nu+2)}{\ell}H_\nu + H_\ell &= -1 + \ell\big[(\nu+2)e_1'(\nu) + e_1(\nu)\big] + \cdots \\ &\quad + \frac{(-1)^{k+1}}{(k+1)!}\ell^k\big[2(\nu+2)e_k'(\nu) + e_k(\nu)\big] + \cdots .\end{aligned} \tag{2.13}$$

Since $(\nu+2)e_1'(\nu)+e_1(\nu) = 1$ and $H\big(\ell(\nu),\nu\big) = 1-\frac{\ell}{1!}+\frac{\ell^2}{2!}e_1(\nu)+\cdots = 0$, the relation (2.13) is rewritten as

$$\frac{2(\nu+2)}{\ell}H_\nu + H_\ell + H = \sum_{k=2}^{\infty}\frac{(-1)^k}{(k+1)!}\ell^k\frac{e_{k-1}(\nu)}{\nu+k+1}A_k(\nu) \tag{2.14}$$

where

$$A_k(\nu) = -2(\nu+2)\left[\frac{1}{\nu+2} + \frac{2}{\nu+3} + \cdots + \frac{k}{\nu+k+1}\right] + k^2 + k. \tag{2.15}$$

Now we are going to prove the relations

$$0 < A_2(\nu) < A_3(\nu) < \cdots < A_{k-1}(\nu) < A_k(\nu) < \cdots, \quad k = 2,3,\ldots. \tag{2.16}$$

The first elements of the sequence $\{A_k(\nu)\}_{k=2}^{\infty}$ are

$$\begin{aligned}A_2(\nu) &= \frac{4}{\nu+3}\\ A_3(\nu) &= \frac{4(4\nu+13)}{(\nu+3)(\nu+4)}\\ A_4(\nu) &= \frac{4(10\nu^2+75\nu+137)}{(\nu+3)(\nu+4)(\nu+5)}.\end{aligned} \tag{2.17}$$

By (2.15) we find

$$A_k(\nu) - A_{k-1}(\nu) = \frac{2k(k-1)}{\nu+k+1} \tag{2.18}$$

which is positive. Thus the relation (2.16) is true. Since $\operatorname{sign}(-1)^k e_{k-1}(\nu) < 0$ for $-2 < \nu < -1$ and $A_k(\nu) > 0$, $k = 2,3,\ldots$, it follows from (2.14) that inequality (2.11) holds.

Now let $-1 < \nu < 1$. Then the terms have alternating signs in (2.14). For $k = 2$ the first term of the infinite sum is positive. We are going to show that

$$\frac{1}{(k+1)!}\ell^k\frac{e_{k-1}(\nu)}{\nu+k+1}A_k(\nu) > \frac{1}{(k+2)!}\ell^{k+1}\frac{e_k(\nu)}{\nu+k+2}A_{k+1}(\nu)$$

or equivalently by (1.6)

$$A_k(\nu) > \frac{1}{k+2}\ell\frac{\nu+1}{\nu+k+2}A_{k+1}(\nu), \quad k = 2,3,\ldots, \quad -1 < \nu \le 1. \tag{2.19}$$

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Let $k = 2$. Then by (2.17) we have the restriction on ℓ

$$\frac{4(\nu+4)^2}{(\nu+1)(4\nu+13)} > \ell \quad \text{for } -1 < \nu \le 1.$$

The function on the left hand side is decreasing on $(-1, 1]$, its minimum on $(-1, 1]$ is $\frac{50}{17}$, which is clearly greater than $\ell = \ell(\nu)$, since $\ell(\nu) < 2$.

Similarly we get for $k = 3$ the upper bound

$$\frac{5(\nu+5)^2(4\nu+13)}{(\nu+1)(10\nu^2+75\nu+137)} > \ell \quad \text{for } -1 < \nu \le 1. \tag{2.20}$$

The left hand side of (2.20) is again a decreasing function of ν, with the minimum $\frac{5\cdot 6^2\cdot 17}{2\cdot 222} = \frac{765}{111} > 2 > \ell$.

When $k > 3$, we rewrite the inequality (2.19) by using (2.18) in the following form

$$A_k(\nu)\left[1 - \frac{\ell(\nu+1)}{(k+2)(\nu+k+2)}\right] > \frac{2\ell(\nu+1)k(k+1)}{(k+2)(\nu+k+2)^2}. \tag{2.21}$$

By (2.16) the left hand side decreases if we replace k by 3, while the right hand side increases. However, for $k = 3$ inequality (2.21) reduces to the case $k = 3$ in (2.19) which was treated before. Hence the inequality (2.12) is proved completing the proof of Lemma 3.

Proof of Theorem 2 Since

$$\frac{d}{d\nu}\left[\frac{\ell^2(\nu)}{\nu+2}\right] = \frac{\ell(\nu)}{(\nu+2)^2}\left[2\ell'(\nu)(\nu+2) - \ell(\nu)\right] \tag{2.22}$$

and $\ell(\nu) > 0$, Lemma 3 yields

$$\frac{d}{d\nu}\left[\frac{\ell^2(\nu)}{\nu+2}\right] < 0 \quad \text{for } -2 < \nu < -1,$$

which proves that the function $\frac{\ell(\nu)}{\sqrt{\nu+2}} = \frac{j_{\nu,1}^2}{4(\nu+1)\sqrt{\nu+2}}$ decreases as ν increases in the interval $(-2, -1)$.

On the other hand, we have by (1.5)

$$1 - \ell(\nu) + \frac{\ell^2(\nu)}{2(\nu+2)}\left[\nu + 1 - \frac{\ell(\nu)}{3}\frac{\nu+1}{\nu+3} + \frac{\ell^2(\nu)}{3\cdot 4}\frac{(\nu+1)^2}{(\nu+3)(\nu+4)} + \cdots\right] = 0$$

and since $\lim_{\nu\to-2} \ell(\nu) = 0$, we obtain

$$\lim_{\nu\to-2} \frac{\ell^2(\nu)}{2(\nu+2)} = 1$$

or

$$\lim_{\nu\to-2} \frac{\ell(\nu)}{\sqrt{\nu+2}} = \sqrt{2},$$

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which completes the proof of the first part of Theorem 2.

Also, from (2.22) and Lemma 2, we obtain

$$\frac{d}{d\nu}\left[\frac{\ell^2(\nu)}{\nu+2}\right] > 0 \quad \text{for } -1 < \nu \le 1,$$

which implies that the function $\frac{\ell(\nu)}{\sqrt{\nu+2}} = \frac{j_{\nu,1}^2}{4(\nu+1)\sqrt{\nu+2}}$ increases as ν increases in the interval $(-1, 1]$.

Next we are going to prove that

$$\frac{d}{d\nu}\left[\frac{j_{\nu,1}^2}{4(\nu+1)\sqrt{\nu+2}}\right] > 0 \quad \text{for } \nu > 1$$

or equivalently

$$\frac{1}{j_{\nu,1}}\frac{j_{\nu,1}}{d\nu} > \frac{3\nu+5}{4(\nu+1)(\nu+2)} = \frac{1}{P(\nu)} \quad \text{for } \nu > 1. \tag{2.23}$$

In (1.8) we have a lower bound for the devivative of $j_{\nu,1}$ in terms of $j_{\nu,1}$. The right hand side of (1.8) will be greater than $\frac{1}{P(\nu)}$ of (2.23) if

$$j_{\nu,1}^2 < P^2(\nu) + 2P(\nu).$$

To check the validity of this inequality we make use of the upper bound for $j_{\nu,1}^2$ in (1.9):

$$\begin{aligned} P^2(\nu) + 2P(\nu) - j_{\nu,1}^2 &> P^2(\nu) + 2P(\nu) - \frac{2(\nu+1)(\nu+5)(5\nu+11)}{7\nu+19} \\ &= \frac{2(\nu+1)^3(11\nu^2+20\nu-7)}{(5+3\nu)^2(7\nu+19)} \end{aligned}$$

which is clearly positive for $\nu > 1$. This proves the inequality (2.23). So the function $\frac{j_{\nu,1}^2}{4(\nu+1)\sqrt{\nu+2}}$ increases as ν increases for $\nu > 1$ which completes the proof of Theorem 2.

Proof of Theorem 3 Since $\frac{1}{4}j_{\nu,1}^2 = \ell(\nu)(\nu+1)$, we have to prove that

$$\frac{d^2}{d\nu^2}\big[\ell(\nu)(\nu+1)\big] = \frac{d^2\ell(\nu)}{d\nu^2}(\nu+1) + 2\frac{d\ell(\nu)}{d\nu} = \ell''(\nu)(\nu+1) + 2\ell'(\nu) > 0.$$

From (2.3) and Lemma 1, we can rewrite this inequality into the following

$$-H_\ell\big[\ell''(\nu)(\nu+1) + 2\ell'(\nu)\big] = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+2)!}\ell^k\frac{e_k(\nu)}{\nu+k+2}B_k(\nu) > 0 \tag{2.24}$$

where

$$B_0(\nu) = \frac{2}{(\nu+2)^2}\big[(\nu+1)(\nu+2)\ell' + \ell\big]^2$$

$$B_k(\nu) = (k+1)(k+2)\left[(\nu+1)\ell' + \frac{\alpha_{k+1}(\nu)}{k+1}\ell\right]^2 + (\nu+1)^2\ell^2\left[\bar{c}_{k+1} - \frac{1}{k+1}c_{k+1}^2(\nu)\right] \quad \text{for } k = 1,2,\dots \tag{2.25}$$

and $\alpha_k(\nu)$, $c_k(\nu)$, $\bar{c}_k(\nu)$ are defined in (2.3).

By the well-known inequality between the quadratic and arithmetic means we have for $k = 2,3,\dots$

$$\frac{1}{k}\bar{c}_k = \frac{\frac{1}{(\nu+2)^2} + \cdots + \frac{1}{(\nu+k+1)^2}}{k} > \left(\frac{\frac{1}{\nu+2} + \cdots + \frac{1}{\nu+k+1}}{k}\right)^2 = \left(\frac{c_k(\nu)}{k}\right)^2.$$

Using this inequality, we conclude from (2.25) that $B_k(\nu) > 0$ for $k = 1,2,\dots$. Concerning the relation $B_0(\nu) > 0$ it is clear that it holds for $\nu \ge -1$ and also for $-2 < \nu < -1$ using inequality (1.8) of Lemma 3. Consequently, we have

$$B_k(\nu) > 0 \quad \text{for } k = 0,1,2,\dots. \tag{2.26}$$

Consider the interval $(-2,-1]$ in (2.24). Since $\operatorname{sign}\big((-1)^k e_k(\nu)\big) > 0$, every term of the infinite sum is positive which implies that $j_{\nu,1}^2$ is convex on $(-2,-1]$.

In the case $-1 < \nu \le 0$ the signs of the terms of the infinite series in (2.24) are alternating. Therefore we are going to show that the series is of Leibniz type, *i.e.*, the inequality

$$\frac{1}{(k+2)!}\ell^k\frac{e_k(\nu)}{\nu+k+2}B_k > \frac{1}{(k+3)!}\ell^k\frac{e_{k+1}(\nu)}{\nu+k+3}B_{k+1}, \quad k = 0,1,2,\dots$$

or

$$(k+3)(\nu+k+3)B_k > \ell(\nu+1)B_{k+1}, \quad k = 0,1,2,\dots \tag{2.27}$$

holds.

Let $k = 0$. Then

$$3(\nu+3)B_0 - \ell(\nu+1)B_1 = U_0(\ell')^2 + 2V_0\ell\ell' + W_0\ell^2 \tag{2.28}$$

where

$$U_0 = 6(\nu+1)^2\big[(\nu+3) - (\nu+1)\ell\big]$$

$$V_0 = 3\frac{\nu+1}{\nu+2}\left[2(\nu+3) - \ell\frac{(\nu+1)(3\nu+7)}{\nu+3}\right]$$

$$W_0 = 6\frac{\nu+1}{(\nu+2)^2} - \frac{3}{2}\ell\frac{(\nu+1)(3\nu+7)^2}{(\nu+3)^2(\nu+2)^2} + \ell\frac{(\nu+1)^3}{2(\nu+2)^2(\nu+3)^2}.$$

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Recalling the inequality (1.4), we get $U_0 > 0$, $V_0 > 0$ for $\nu \geq -1$, and also

$$\begin{aligned} W_0 &= \frac{1}{(\nu+2)^2}\left\{6(\nu+3) - \ell\frac{(\nu+1)}{(\nu+3)^2}(13\nu^2+62\nu+73)\right\} \\ &> \frac{1}{(\nu+2)^2}\left\{6(\nu+3) - \frac{(\nu+1)}{2(\nu+3)}(13\nu^2+62\nu+73)\right\} \\ &= \frac{-13\nu^3-63\nu^2-63\nu+35}{(\nu+2)^2(\nu+3)} > 0, \end{aligned}$$

for $-1 < \nu \leq 0$. Hence by (2.28) the inequality (2.27) is justified for $k = 0$.

Let $k \geq 1$. Using relation (2.25) in (2.27) we obtain

$$(k+3)(\nu+k+3)B_k - \ell(\nu+1)B_{k+1} = U_k(\ell')^2 + 2V_k\ell\ell' + W_k\ell^2 \tag{2.29}$$

where

$$U_k = (k+2)(k+3)(\nu+1)^2\left[(k+1)(\nu+k+3) - (\nu+1)\ell\right]$$
$$V_k = (k+3)(\nu+1)Z_{k+1}$$

and

$$Z_k = (k+1)(\nu+k+2)\alpha_k - \ell(\nu+1)\alpha_{k+1}$$

$$\begin{aligned} W_k = (k+3)(\nu+k+3)&\left[(k+2)\frac{\alpha_{k+1}^2}{k+1} + (\nu+1)^2\left(\bar{c}_{k+1} - \frac{1}{k+1}c_{k+1}^2\right)\right] \\ &- \ell(\nu+1)\left[(k+3)\frac{\alpha_{k+2}^2}{k+2} + (\nu+1)^2\left(\bar{c}_{k+2} - \frac{1}{k+2}c_{k+2}^2\right)\right]. \end{aligned}$$

It is clear that $U_k > U_0 > 0$ for $-1 < \nu \leq 0$, $k = 1, 2, \ldots$. Using the relation $\alpha_{k+1} = \alpha_k + \frac{k+1}{\nu+k+2}$, we have

$$\begin{aligned} Z_k &= \left[(k+1)(\nu+k+2) - \ell(\nu+1)\right]\alpha_k - \frac{\ell(\nu+1)(k+1)}{\nu+k+2} \\ &> \left[(k+1)(\nu+k+2) - \frac{(\nu+1)(\nu+3)}{2}\right]\frac{1}{\nu+2} - \frac{(\nu+1)(\nu+3)}{2} \\ &= \frac{2(k-1)^2 + (k-1)(10+2\nu) + (3+\nu)(1-4\nu-\nu^2)}{2(\nu+2)} > 0 \end{aligned}$$

hence $V_k > 0$ for $-1 < \nu \leq 0$, $k = 1, 2, \ldots$.

Finally, using the inequalities (2.4), we get

$$
\begin{aligned}
W_k &> (k+3)(\nu+k+3)\left[(k+2)\frac{\alpha_{k+1}^2}{k+1}\right] - \ell(\nu+1)\left[(k+3)\frac{\alpha_{k+2}^2}{k+2} + (\nu+1)^2\bar{c}_{k+2}\right] \\
&> (k+3)(\nu+k+3)(k+2)\frac{k+1}{4} - \ell(\nu+1)\left[(k+3)(k+2)^2 + (\nu+1)^2\frac{\pi^2}{6}\right] \\
&> (k+3)(k+2)\left[\frac{(k+1)(\nu+k+3)}{4} - \frac{(\nu+1)(\nu+3)}{2}\right] - \frac{(\nu+3)(\nu+1)^3}{2}\frac{\pi^2}{6}] \\
&\geq 6\big[\nu+4-(\nu+1)(\nu+3)\big] - \frac{\pi^2}{4}(\nu+1)^3 = 6[-\nu^2-3\nu+1] - \frac{\pi^2}{4}(\nu+1)^3 \\
&> 6 - \frac{\pi^2}{4} > 0
\end{aligned}
$$

for $-1 < \nu \leq 0$.

On the right hand side in (2.29) every term is positive hence by (2.27) the terms in the infinite series (2.24) are of Leibniz type and consequently, the $j_{\nu,1}^2$ is convex on $(-1,0]$.

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