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Integrable Systems and Torelli Theorems for the Moduli Spaces of Parabolic Bundles and Parabolic Higgs Bundles

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Abstract. We prove a Torelli theorem for the moduli space of semistable parabolic Higgs bundles over a smooth complex projective algebraic curve under the assumption that the parabolic weight system is generic. When the genus is at least two, using this result we also prove a Torelli theorem for the moduli space of semistable parabolic bundles of rank at least two with generic parabolic weights. The key input in the proofs is a method of J.C. Hurtubise.

1 Introduction

The classical theorem by R. Torelli [CRS] says that a smooth complex algebraic curve is determined by the isomorphism class of its polarized Jacobian up to isomorphism. Similar theorems in many contexts have been worked out, *e.g.*, for moduli spaces of stable vector bundles [Tj, NR, MN] and moduli spaces of stable Higgs bundles [BG]. As far as moduli spaces of parabolic or parabolic Higgs bundles with fixed determinant (see definition below) are concerned, a number of Torelli theorems were proved [BBB, BHK, Seb, GL]. Here we deal with the non-fixed determinant situation.

Hurtubise [Hu] investigated algebraically completely integrable systems satisfying certain conditions. His main result is to extract an algebraic surface out of an integrable system. We observe that a moduli space of parabolic Higgs bundles is an example of the model of completely integrable systems studied in [Hu].

The above mentioned assumption that the determinant is not fixed stems from the fact that in the set-up of [Hu] the Lagrangians in the fibers are required to be Jacobians, while fixing the determinant amounts to making the fibers Prym varieties. To consider the moduli spaces with fixed determinant with our techniques, we would need an analogue of our main tool, namely Theorem 1.11 of [Hu], but for an integrable system in which the fibers are Prym varieties instead of Jacobians. This is planned for future work.

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The Néron–Severi group $\mathrm{NS}(Y)$ of a smooth variety Y is the image of the homomorphism $\mathrm{Pic}(Y) \to H^2(Y,\mathbb{Z})/\mathrm{Torsion}$ that sends a line bundle to its first Chern class. The image of a line bundle L by this map is called the Néron–Severi class of L.

We will prove the following theorems.

Theorem 1.1 (Main Theorem) *Let X and X' be smooth projective curves with genus g and parabolic points D and D', respectively. Let $\mathcal{M}_X(d,r,\alpha)$ (resp., $\mathcal{M}_{X'}(d,r,\alpha)$) be the moduli space of stable parabolic Higgs bundles over X (resp., X') endowed with the usual $\mathbb{C}^*$ action (*cf. (2.2)*) and the determinant line bundle $\mathcal{L}$ (resp., $\mathcal{L}'$) (*cf. (3.4)*). If there is a $\mathbb{C}^*$-equivariant isomorphism between $\mathcal{M}_X(d,r,\alpha)$ and $\mathcal{M}_{X'}(d,r,\alpha)$, preserving the holomorphic symplectic form, such that the pullback of the Néron-Severi class $\mathrm{NS}(\mathcal{L}')$ is $\mathrm{NS}(\mathcal{L})$, then there exists an isomorphism between X and X' inducing a bijection between the parabolic points D and D' whenever the following conditions on the genus and the rank are satisfied:*

(i) $r^2(g-1)+1+nr(r-1)/2 \geq 3$,

(ii) $r(2g-2)+(r-1)n \geq 2g+1$.

Since the moduli space of stable parabolic bundles sits inside the moduli space of stable parabolic Higgs bundles, in all cases where its codimension is greater than two, we get the following extension of the Torelli theorem for the moduli space of stable parabolic bundles given in [BBB].

Theorem 1.2 *Let X and X' be smooth projective curves with genus g and parabolic points D and D' respectively. Let $M_X(d,r,\alpha)$ be the moduli space of stable parabolic bundles over X (resp., $M_{X'}(d,r,\alpha)$), and let $\mathcal{L}$ (resp., $\mathcal{L}'$) be the determinant line bundle (*cf. (3.4)*). If there is an isomorphism between $M_X(d,r,\alpha)$ and $M_{X'}(d,r,\alpha)$ such that the pullback of $\mathrm{NS}(\mathcal{L}')$ is $\mathrm{NS}(\mathcal{L})$, then there exists an isomorphism between X and X' inducing a bijection between the parabolic points D and D', whenever the following conditions on the genus and the rank are satisfied.*

(i) *If $g=2$, then $r \geq 5$.*

(ii) *If $g=3$, then $r \geq 3$.*

(iii) *If $g \geq 4$, then $r \geq 2$.*

2 Preliminaries

Let X be an irreducible smooth projective algebraic curve over $\mathbb{C}$. The holomorphic cotangent bundle of X will be denoted by K. Let $\{p_1,\dots,p_n\}$ be a set of distinct *parabolic points* in X and let $D = p_1+\cdots+p_n$ be the corresponding reduced effective divisor. A *parabolic bundle* on X with parabolic structure over D consists of a holomorphic vector bundle E equipped with a weighted flag over each *parabolic point* $p \in D$ that is a filtration of subspaces $E|_p = E_{p,1} \supset \cdots \supset E_{p,r(p)} \supset E_{p,r(p)+1} = 0$ together with a system of *parabolic weights* $0 \leq \alpha_1(p) < \cdots < \alpha_{r(p)}(p) < 1$. The *parabolic degree* and

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parabolic slope of E are defined by

$$\operatorname{pardeg}(E) := \deg(E) + \sum_{p\in D}\sum_{i=1}^{r(p)} \alpha_i(p)\cdot m_i(p), \quad \operatorname{par}\mu(E) := \frac{\operatorname{pardeg}(E)}{\operatorname{rk}(E)},$$

where $m_i(p) := \dim(E_{p,i}/E_{p,i+1})$ is the multiplicity of the parabolic weight $\alpha_i(p)$. The parabolic bundle is called *stable* (resp., *semistable*) if for all subbundles $0 \neq V \subsetneq E$,

$$\operatorname{par}\mu(V) < \operatorname{par}\mu(E) \qquad (\text{resp., } \operatorname{par}\mu(V) \le \operatorname{par}\mu(E)) \tag{2.1}$$

where V has the induced parabolic structure. Given rank and degree, the system of parabolic weights is called *generic* if every semistable parabolic bundle is stable. We note that the semistability condition describes hyperplanes (or *walls*) in the space of weights. Hence the genericity condition means that the parabolic weights lie in the interior of the chambers defined by the walls.

We denote by $M_X(d,r,\alpha)$ the moduli space of stable parabolic bundles over X with degree d, rank r, and generic weights α. This moduli space is a smooth projective variety with

$$\dim M_X(d,r,\alpha) = r^2(g-1) + 1 + \frac{1}{2}\sum_{p\in D}\sum_{i=1}^{r(p)}\left(r^2 - m_i(p)^2\right).$$

For notational convenience we assume that the flag is *full*, that is, $m_i(p) = 1$ for all p and i, so $r(p) = r$ for all p, but all the results generalize to non the full flags case. Henceforth, we will only consider full flags. Therefore,

$$\dim M_X(d,r,\alpha) = r^2(g-1) + 1 + \frac{1}{2}nr(r-1).$$

An endomorphism of a parabolic bundle E is called *non-strongly parabolic* if for all $p \in D$ and i, $\phi(E_{p,i}) \subset E_{x,i}$ and it is called *strongly parabolic* if $\phi(E_{p,i}) \subset E_{p,i+1}$. The sheaves of non-strongly and strongly parabolic endomorphisms are denoted by $\operatorname{ParEnd}(E)$ and $\operatorname{SParEnd}(E)$, respectively.

A *parabolic Higgs bundle* is a pair (E,Φ) where E is a parabolic bundle and

$$\Phi\colon E \to E \otimes K(D) = E \otimes K \otimes \mathcal{O}_X(D)$$

is a *strongly parabolic* homomorphism, *i.e.*, $\Phi(E_{x,i}) \subset E_{x,i+1} \otimes K(D)_x$ for each point $x \in D$ and all i. A parabolic Higgs bundle is *stable* (resp., *semistable*) if the inequality (2.1) is satisfied for those V with $\Phi(V) \subset V \otimes K(D)$.

Let $\mathcal{M}_X(d,r,\alpha)$ denote the moduli space of stable parabolic Higgs bundles with degree d, rank r, and generic weights α. It is a smooth quasiprojective variety that satisfies $\dim \mathcal{M}_X(d,r,\alpha) = 2r^2(g-1) + 2 + nr(r-1) = 2\cdot\dim M_X(d,r,\alpha)$ (recall that the quasiparabolic flags are full).

For any $E \in M_X(r,d,\alpha)$, the tangent space at E is

$$T_E M_X(r,d,\alpha) = H^1(\operatorname{ParEnd}(E)).$$

Also, the parabolic version of Serre duality gives an isomorphism

$$H^1\big(\operatorname{ParEnd}(E)\big)^* \cong H^0(\operatorname{SParEnd}(E)\otimes K(D)).$$

Therefore, the total space of the cotangent bundle $T^*M_X(r,d,\alpha)$ is a Zariski open subset of $\mathcal{M}_X(r,d,\alpha)$.

The moduli space of parabolic Higgs bundles is endowed with a $\mathbb{C}^*$ action, where $t \in \mathbb{C}^*$ acts as scalar multiplication on the Higgs field

$$(E, \Phi) \longmapsto (E, t \cdot \Phi). \tag{2.2}$$

The total space of the cotangent bundle $T^* M_X(r, d, \alpha)$ also has a canonical $\mathbb{C}^*$ action given by scalar multiplication on the fibers. Both actions are compatible in the sense that the inclusion of the cotangent in the moduli space of Higgs bundles is $\mathbb{C}^*$ equivariant.

3 The Hitchin System

Let $\mathcal{K}(\mathcal{D})$ denote the total space of the line bundle $K(D)$ over X, and let $\gamma\colon \mathcal{K}(\mathcal{D}) \to X$ be the natural projection. Let $\widetilde{x} \in H^0\big(\mathcal{K}(\mathcal{D}), \gamma^* K(D)\big)$ be the tautological section whose evaluation at any point z is z itself. The characteristic polynomial of a Higgs field Φ is

$$\det(\widetilde{x} \cdot \mathrm{Id} - \gamma^* \Phi) = \widetilde{x}^r + \widetilde{s}_1 \widetilde{x}^{r-1} + \widetilde{s}_2 \widetilde{x}^{r-2} + \cdots + \widetilde{s}_r. \tag{3.1}$$

The sections $\widetilde{s}_i$, descend to X, meaning there are sections $s_i \in H^0(X, K^i(iD))$ such that $\widetilde{s}_i = \gamma^* s_i$. Since Φ is strongly parabolic, its residue at each parabolic point is nilpotent, and hence $s_i \in H^0(X, K^i((i-1)D))$. Therefore, there is a morphism, called the *Hitchin map*,

$$H\colon \mathcal{M}_X(d, r, \alpha) \to \mathcal{U} := \bigoplus_{i=1}^{r} H^0(X, K^i((i-1)D)). \tag{3.2}$$

This morphism is surjective and proper, and it induces an isomorphism on globally defined algebraic functions [Hi]; *i.e.*, the lower arrow in the following commutative diagram is an isomorphism:

$$\begin{array}{ccc} \mathcal{M}_X(d, r, \alpha) & \xrightarrow{\quad H \quad} & \mathcal{U} \\ \Big\downarrow a & & \Big\| \\ \operatorname{Spec} \Gamma(\mathcal{M}_X(d, r, \alpha)) & \xrightarrow{\;\cong\;} & \operatorname{Spec} \Gamma(\mathcal{U}) \end{array} \tag{3.3}$$

The variety $\mathcal{M}_X(d, r, \alpha)$ has a natural holomorphic symplectic structure, and the Hitchin map defines an algebraically complete integrable system, in particular, the fibers of H are Lagrangians (these are explained in [GL]).

When the parabolic set is empty ($n = 0$), Hausel proved that the nilpotent cone $H^{-1}(0)$ coincides with the downwards Morse flow on $\mathcal{M}_X(d, r, \alpha)$ giving a deformation retraction of $\mathcal{M}_X(d, r, \alpha)$ to $H^{-1}(0)$ [Hau, Theorem 5.2]. The proof in [Hau] can be translated into the parabolic situation word by word.

The fiber of H over a point $u \in \mathcal{U}$ is canonically isomorphic to the Jacobian of a curve called the *spectral curve*; we now recall its construction.

Given a point $u = (s_1, \ldots, s_r) \in \mathcal{U}$, consider the curve $X_u \subset \mathcal{K}(\mathcal{D})$ defined by the equation $\widetilde{x}^r + s_1 \widetilde{x}^{r-1} + s_2 \widetilde{x}^{r-2} + \cdots + s_r = 0$ (compare it with (3.1)). Note that when X_u is reduced, the projection $\rho := \gamma|_{X_u}\colon X_u \to X$ is a ramified covering of X of degree r which is completely ramified over the parabolic points. Indeed, from the definition of

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$\mathcal{U}$ it follows immediately that ρ is completely ramified over the divisor D. Denote by R_u the ramification divisor on X_u. Denote by $\mathcal{S}$ the family of spectral curves over $\mathcal{U}$.

Proposition 3.1 *For any $u \in \mathcal{U}$ such that the corresponding spectral curve X_u is smooth, the fiber $H^{-1}(u)$ is identified with* $\mathrm{Pic}^{d+r(r-1)(2g-2+n)/2}(X_u)$.

Proof It follows from the proof of Proposition 3.6. in [BNR]. ∎

In general, there is no universal bundle on $X \times \mathcal{M}_X(d,r,\alpha)$. However $\mathcal{M}_X(d,r,\alpha)$ can be covered by finitely many Zariski open subsets $\{V_i\}_{i=1}^b$ and each V_i has a finite étale Galois covering $\delta_i \colon \widetilde{V}_i \to V_i$ such that there is a universal bundle on $X \times \widetilde{V}_i$. Let $\mathcal{E}_i$ be a universal bundle on $X \times \widetilde{V}_i$ and let $q_i \colon X \times \widetilde{V}_i \to \widetilde{V}_i$. Fix a point $x \in X$ of the curve. Let $\chi = \chi(E)$ (since we have fixed the rank and degree, this does not depend on the particular E chosen and can be calculated by the Riemann–Roch formula). There is a line bundle $\mathcal{L}_i^x \to \widetilde{V}_i$ defined as follows (see [KM]):

$$\mathcal{L}_i^x = \det(Rq_{i*}\mathcal{E}_i)^{-r} \otimes \Big(\bigwedge^r \mathcal{E}|_{x\times\mathcal{M}}\Big)^{\chi};$$

the presence of the second factor is a normalization guaranteeing that this $\mathcal{L}_i^x$ does not depend on the choice of universal bundle $\mathcal{E}_i$. The action of the Galois group $\mathrm{Gal}(\delta_i)$ on $\widetilde{V}_i$ lifts to an action of $\mathrm{Gal}(\delta_i)$ on the line bundle $\mathcal{L}_i^x$. Therefore, $\mathcal{L}_i^x$ descends to a line bundle on V_i; this descended line bundle on V_i will be denoted by $\widehat{\mathcal{L}}_i^x$. Two such line bundles $\widehat{\mathcal{L}}_i^x$ and $\widehat{\mathcal{L}}_i^x$ have a natural identification over $V_i \cap V_j$. Therefore, these line bundles $\{\widehat{\mathcal{L}}_i^x\}_{i=1}^b$ patch together compatibly to produce a line bundle

$$\mathcal{L}^x \longrightarrow \mathcal{M}_X(d,r,\alpha). \tag{3.4}$$

Note that this determinant line bundle can also be defined for the moduli space

$$M_X(d,r,\alpha)$$

without Higgs bundle.

We remark that this line bundle $\mathcal{L}^x$ is invariant under the standard $\mathbb{C}^*$ action in (2.2) and we can choose a lift of this $\mathbb{C}^*$ action to $\mathcal{L}^x$.

The fiber of this line bundle over a point corresponding to a Higgs bundle (E,Φ) is canonically isomorphic to

$$\Big[\Big(\bigwedge^{\mathrm{top}} H^0(X,E)\Big)^* \otimes \Big(\bigwedge^{\mathrm{top}} H^1(X,E)\Big)\Big]^{\otimes r} \otimes (\wedge E_x)^{\chi}.$$

Since the curve X is connected, the Néron-Severi class $\mathrm{NS}(\mathcal{L}^x)$ of the line bundle does not depend on the choice of the point $x \in X$.

Lemma 3.2 *If $u \in \mathcal{U}$ is a point in the Hitchin space corresponding to a smooth curve, then the restriction of the line bundle $\mathcal{L}^x$ to the fiber*

$$H^{-1}(u) = \mathrm{Pic}^{d+r(r-1)(2g-2+n)/2}(X_u)$$

is a multiple of the principal polarization of the Jacobian $J(X_u)$ of the spectral curve X_u.

Proof Let (E,Φ) be a point in the moduli space $\mathcal{M}$. If it is in the fiber $H^{-1}(u)$, then there is a line bundle η on the spectral curve $\pi: X_u \to X$ such that $E = \pi_*\eta$. Then the fiber of $\mathcal{L}^x$ over this point is canonically isomorphic to

$$\Big[\Big(\bigwedge^{\text{top}} H^0(X,\pi_*\eta)\Big)^* \otimes \Big(\bigwedge^{\text{top}} H^1(X,\pi_*\eta)\Big)\Big]^{\otimes r} \otimes (\wedge(\pi_*\eta)_x)^\chi$$
$$= \Big[\Big(\bigwedge^{\text{top}} H^0(X_s,\eta)\Big)^* \otimes \Big(\bigwedge^{\text{top}} H^1(X_s,\eta)\Big)\Big]^{\otimes r} \otimes (\wedge\eta_{\pi^{-1}(x)})^\chi.$$

This is the fiber of a line bundle defining a multiple of a principal polarization of the Jacobian. The last factor is just a normalization and the Néron-Severi class of the line bundle does not depend on the choice of the point. ∎

Hurtubise [Hu] considered (local) integrable systems $\mathbb{H}: \mathbb{J} \to \mathbb{U}$, where $\mathbb{U}$ is an open subset of $\mathbb{C}^m$ and $\mathbb{J}$ is a $2m$-dimensional symplectic variety with holomorphic symplectic form Ω, such that the fibers of $\mathbb{H}$ are Lagrangian. Furthermore, suppose there is a family of curves $\mathbb{H}': \mathbb{S} \to \mathbb{U}$ such that for each $u \in \mathbb{U}$, the fiber $J_u = \mathbb{H}^{-1}(u)$ is isomorphic to the Jacobian of $S_u = \mathbb{H}'^{-1}(u)$. For a point $u \in \mathbb{U}$, if we fix a point $s \in \mathbb{S}_u = (\mathbb{H}')^{-1}(u)$, then we have the Abel map

$$\mathbb{S}_u \longrightarrow J(\mathbb{S}_u) = \mathrm{Pic}^0(\mathbb{S}_u), \quad y \longmapsto \mathcal{O}_{\mathbb{S}_u}(y-s).$$

Therefore, to define the Abel map $I: \mathbb{S} \to \mathbb{J}$ we need a section of $\mathbb{H}'$. Such sections can be constructed locally on $\mathbb{U}$. Under the assumption that

$$I^*\Omega \wedge I^*\Omega = 0,$$

Hurtubise proved that for the embedding I the variety $\mathbb{S}$ is co-isotropic, and the quotienting of $\mathbb{S}$ by the null foliation results in a complex algebraic surface Q. The form $I^*\Omega$ descends to Q, and the descended form on Q, which we will denote by ω, is a holomorphic symplectic form [Hu, Theorem 1.11]. He also proved that choosing a different Abel map I' with $I'^*\Omega \wedge I'^*\Omega = 0$, we have $I^*\Omega = I'^*\Omega$ when $m \geq 3$, so that the surface Q depends only on $\mathbb{S}$ and it is independent of the Abel map.

Theorem 3.3 ([Hu, Theorem 1.11 (i) and (ii)]) *For an integrable system*

$$\mathbb{H}: \mathbb{J} \to \mathbb{U} \subset \mathbb{C}^m,$$

with maps $\mathbb{H}': \mathbb{S} \to \mathbb{U}$, and $I: \mathbb{S} \to \mathbb{J}$, as described above, there is an invariant surface Q which only depends on $\mathbb{S}$ and not on the Abel map I whenever $m \geq 3$.

Hurtubise [Hu, Example 4.3] showed that all these conditions are satisfied for the usual moduli space of Higgs bundles, *i.e.*, no parabolic points, but restricted to the open subset U of the Hitchin space $\mathcal{U}$ corresponding to smooth spectral curves

$$\mathbb{H}: \mathcal{M}_X|_U \to U.$$

Let q be the projection $q: \mathbb{S} \to \mathcal{K}$ sending each point on a spectral curve to the total space of the cotangent bundle and let ω be the natural symplectic form on the cotangent. Hurtubise showed that $I^*\Omega = q^*\omega$. It follows that the surface Q is $\mathcal{K}$.

The conditions of the theorem also hold for the moduli space of strongly parabolic Higgs bundles equipped with the Hitchin map, and in this case the surface Q is the image of $\mathbb{S} \to \mathcal{K}(\mathcal{D})$. Note that all spectral curves go through zero on the fibers over the parabolic points, because the eigenvalues of the residues are zero. Therefore, we obtain the following corollary, which will be our main tool in the proof of Theorem 1.1.

Note that the integer m in the statement of Theorem 3.3 is the genus of the spectral curve, which is equal to $\dim M_X(d,r,\alpha)$ and hence, under the assumptions on genus and rank of Theorems 1.1 and 1.2 we always have $m \geq 3$ and hence can apply the Theorem of Hurtubise.

Corollary 3.4 *Let $\mathbb{H}\colon \mathcal{M}_X(d,r,\alpha)|_U \to U$. be the restriction of the Hitchin map on the moduli space of parabolic Higgs bundles with generic weights α to the open set U corresponding to nonsingular curves (*cf. *Lemma (4.1)). Then this integrable system satisfies the conditions of the Theorem of Hurtubise and the surface Q is the image of $\mathcal{K}$ in $\mathcal{K}(\mathcal{D})$ under the injective morphism of sheaves $K \to K(D)$.*

4 Proof of the Theorems

Let $h\colon T^*M_X(d,r,\alpha) \to \mathcal{U} = \bigoplus_{i=1}^r H^0\big(X, K^i((i-1)D)\big)$ be the restriction to the cotangent bundle of the moduli space of stable bundles of the Hitchin integrable system in (3.2). To each point $u \in \mathcal{U}$ we have the associated spectral curve $X_u \subset \mathcal{S}$.

Lemma 4.1 *If $g \geq 2$ or $r(2g-2)+(r-1)n \geq 2g+1$, then the Zariski open subset U of the variety $\mathcal{U}$ that parametrizes the smooth spectral curves is non-empty.*

Proof If $K^r\big((r-1)D\big)$ has a section without multiple zeros, then the above open subset U is nonempty (*cf.* [BNR, Remark 3.5]). A holomorphic line bundle on X of degree at least $2g+1$ is very ample (*cf.* [Har, IV Corollary 3.2]), and hence U is non-empty whenever $r(2g-2)+(r-1)n \geq 2g+1$ and this holds when $g \geq 2$. ∎

Define $\mathcal{J} := H^{-1}(U)$, where H is the Hitchin map for the moduli of Higgs bundles (3.2) and U is the open subset in Lemma 4.1. Let $H_{\mathcal{J}}\colon \mathcal{J} \to U$ be the restriction of H. Let $H_{\mathcal{S}}\colon \mathcal{S} \to U$ be the total space for the family of spectral curve over U, so that the fiber of $H_{\mathcal{S}}$ over any $u \in U$ is the spectral curve X_u.

As we have seen in Corollary 3.4, the surface Q given by the theorem of Hurtubise in this setting is the image of $\mathcal{K}$ in $\mathcal{K}(\mathcal{D})$ under the injective morphism of sheaves $K \to K(D)$. In particular, Q is singular.

The moduli space of parabolic Higgs bundles is known to be a Kähler manifold provided with a $\mathbb{C}^*$ action whose restriction to an S^1 action preserves the Kähler structure

$$\begin{aligned} \tau\colon \mathbb{C}^* \times \mathcal{M}_X(r,d,\alpha) &\to \mathcal{M}_X(r,d,\alpha) \\ \big(t,(E,\Phi)\big) &\mapsto (E, t\cdot\Phi). \end{aligned} \tag{4.1}$$

This $\mathbb{C}^*$ action is compatible with scalar multiplication in the fibers of the cotangent bundle $T^*M_X(r,d,\alpha)$ under the inclusion of this cotangent bundle in the moduli of parabolic Higgs bundles. It induces a $\mathbb{C}^*$ action on $\mathcal{S}$:

$$\begin{aligned} \mathbb{C}^* \times \mathcal{S} &\to \mathcal{S} \\ (t, x \in X_u) &\mapsto (tx \in X_{t\cdot u}), \end{aligned} \tag{4.2}$$

where $t \cdot (s_1, \dots, s_r) = (ts_1, t^2 s_2, \dots, t^r s_r)$ (see (3.1)) and the multiplication tx is defined using the embedding of the spectral curve X_u in the total space of $K(D)$. This action of $\mathbb{C}^*$ on $\mathcal{S}$ evidently produces an action of $\mathbb{C}^*$ on the quotient surface Q. Let $Q^{\mathbb{C}^*} \subset Q$ be the fixed point locus for the above $\mathbb{C}^*$ action on Q.

Lemma 4.2 *The subset $Q^{\mathbb{C}^*}$ is the zero section of the fibration $\mathcal{K}(\mathcal{D}) \to X$.*

Proof Since the natural inclusion $K \hookrightarrow K(D)$ of $\mathcal{O}_X$-modules commutes with the multiplicative action of $\mathbb{C}^*$, the surface Q, which is the image of the total space of K in $\mathcal{K}(\mathcal{D})$, is preserved by the action of $\mathbb{C}^*$ on $\mathcal{K}(\mathcal{D})$. Therefore, the action of $\mathbb{C}^*$ on $\mathcal{K}(\mathcal{D})$ produces an action of $\mathbb{C}^*$ on Q. This action of $\mathbb{C}^*$ on Q coincides with the action on Q induced by (4.2). The lemma follows from this. ∎

Corollary 4.3 *The curve X coincides with $Q^{\mathbb{C}^*}$.*

Proposition 4.4 *The set of parabolic points coincides with the subset of $Q^{\mathbb{C}^*}$ through which every spectral cover passes.*

Proof Since the residue of Φ on the parabolic points is nilpotent, all spectral curves X_u totally ramify over the parabolic points and they intersect the fiber over the parabolic points at zero.

Conversely, let $x \in X$ be a point which is not parabolic. There exists a section $s_r \in H^0\big(K^r((r-1)D)\big)$ which does not vanish at x since this linear system is base point free (*cf.* Lemma 4.1). Furthermore, this section still has no zero on x when considered as a section of $H^0\big(K^r(rD)\big)$ because x is not a parabolic point. Therefore, the spectral curve $\widetilde{x}^r + s_r = 0$ on $\mathcal{K}(\mathcal{D})$ intersects the fiber over x away from zero and the spectral curve $\widetilde{x}^r = 0$ intersects it only at zero, so there is no point over the fiber of x through which every spectral cover passes. ∎

4.1 Proof of Theorem 1.1

We are given the moduli space $\mathcal{M}$ as an abstract algebraic variety with a holomorphic symplectic form, a line bundle $\mathcal{L}$ and a algebraic $\mathbb{C}^*$ action on $\mathcal{M}$ with a linearization on $\mathcal{L}$. Looking at global functions on $\mathcal{M}$, $\alpha\colon \mathcal{M} \to \operatorname{Spec} \Gamma(\mathcal{M})$, we obtain a morphism α which is isomorphic to the Hitchin fibration (*cf.* (3.3)) and the fibers are Lagrangians with respect to the given holomorphic symplectic form. The subset $U \subset \operatorname{Spec} \Gamma(\mathcal{M})$ of points corresponding to smooth spectral curves can be recovered as the points whose fibers are abelian varieties. Let β be the restriction of α to U:

$$\begin{array}{ccc} \mathcal{J} & \hookrightarrow & \mathcal{M} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ U & \hookrightarrow & \operatorname{Spec} \Gamma\mathcal{M}. \end{array}$$

The line bundle $\mathcal{L}$ restricts to (a multiple of) a principal polarization on these abelian varieties, and then the classical Torelli theorem gives us a family of curves $\mathcal{S} \to U$, such

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that the fiber $\mathcal{J}_u$ over $u \in U$ is the Jacobian of $\mathcal{S}_u$. Locally on U there is an Abel–Jacobi map $I: \mathcal{S} \to \mathcal{J}$.

The $\mathbb{C}^*$ action on $\mathcal{M}$ restricts to a $\mathbb{C}^*$ action on the family of Jacobians $\mathcal{J}$. This family of Jacobians has a family of principal polarizations given by the line bundle $\mathcal{L}$. The $\mathbb{C}^*$ action has a lift to $\mathcal{L}$; hence we have an action on the family of principal polarized Jacobians.

By the proof given by Weil of the Torelli theorem [We, Hauptsatz, p. 35], an isomorphism $\psi: (\mathcal{J}_u, \theta_u) \to (\mathcal{J}_{u'}, \theta_{u'})$ of principal polarized Jacobians induces an isomorphism $f: \mathcal{S}_u \to \mathcal{S}'_u$ of the corresponding curves, and this provides an action of $\mathbb{C}^*$ on the family of curves $\mathcal{S}$.

Now we apply Corollary 3.4 to obtain a surface Q as a quotient of $\mathcal{S}$. The action on $\mathcal{S}$ that we have just clearly defined coincides with the action given in (4.2). Therefore by Corollary 4.3 we recover X and by Proposition 4.4 we recover the parabolic points D, thus proving our main theorem.

4.2 Proof of Theorem 1.2

We are given the moduli space as a smooth algebraic variety M with a line bundle $\mathcal{L}$. We consider the total space of the cotangent bundle T^*M. This has a canonical holomorphic symplectic structure and a $\mathbb{C}^*$ given by scalar multiplication on the fibers. The pullback of the line bundle to T^*M is trivial along the fibers, so there is a canonical lift of the $\mathbb{C}^*$ action to the pullback of the line bundle $\mathcal{L}$ to T^*M.

We claim that the generic fiber of the morphism given by global sections

$$h: T^*M \to \operatorname{Spec}\big(\Gamma(T^*M)\big)$$

is an open subset of an abelian variety. Indeed, we know that M is the moduli space for some algebraic curve X (which we want to find), so we know that T^*M is an open subset of a moduli space of parabolic Higgs bundles $\mathcal{M}$ and by Corollary 5.11 we know that the codimension of the complement of this open set is at least two. Therefore, global sections on T^*M extend uniquely to global sections on $\mathcal{M}$ and the morphism h is the restriction of the morphism of global sections of some moduli space of Higgs bundles $\mathcal{M}$:

$$\begin{array}{ccc} T^*M & \xrightarrow{\;h\;} & \operatorname{Spec}\Gamma(T^*M) \\ \downarrow & & \| \\ \mathcal{M} & \xrightarrow{\;H\;} & \operatorname{Spec}\Gamma(\mathcal{M}). \end{array}$$

The compactification of the fiber over u to an abelian variety is unique because birational abelian varieties are isomorphic. Therefore, the isomorphism class of $\mathcal{J} := H^{-1}(U)$ is uniquely defined by the isomorphism class of M and does not depend on the choice of $\mathcal{M}$.

Since the codimension of the complement of the inclusion $T^*M|_U \subset \mathcal{J}$ is at least two, all the structure that we have on T^*M extends uniquely to $\mathcal{J}$, namely the determinant line bundle $\mathcal{L}$, the $\mathbb{C}^*$ action with the lift to $\mathcal{L}$, and the holomorphic symplectic form. Therefore we can now use the same arguments as in the proof of the main theorem to recover the curve X and the parabolic points.

5 Codimension Computation

In this section we compute the codimension of the complement of $T^*M(d,r,\alpha)$ inside $\mathcal{M}(d,r,\alpha)$ fiber-wise following the arguments in [BGL, Section 5]. This complement is

$$\mathcal{V} = \{(E,\Phi) \in \mathcal{M}(d,r,\alpha) \mid E \text{ is unstable}\} .$$

Recall from (4.1) that the moduli space of parabolic Higgs bundles is known to be a Kähler manifold provided with a $\mathbb{C}^*$-action, whose restriction to an S^1-action preserves the Kähler structure.

This action provides us with two stratifications of the moduli space. The first one is the Białynicki–Birula stratification consisting of subsets of $\mathcal{M}(d,r,\alpha)$ such that

$$U_\lambda^+ := \{p \in \mathcal{M}_X(d,r,\alpha); \lim_{t\to 0} tp \in F_\lambda\}$$

and

$$U_\lambda^- := \{p \in \mathcal{M}_X(d,r,\alpha); \lim_{t\to \infty} tp \in F_\lambda\},$$

where F_λ are the disjoint connected components of the fixed pointed set F for the $\mathbb{C}^*$-action on $\mathcal{M}(d,r,\alpha)$.

The second one is known as the Morse stratification and comes from the restriction of the $\mathbb{C}^*$-action to an S^1-action. The last also preserves the Kähler form, hence it gives us a circle Hamiltonian action on $\mathcal{M}(d,r,\alpha)$ with associated moment map

$$\begin{aligned} \mu\colon \mathcal{M}_X(r,d,\alpha) &\to \mathbb{R} \\ (E,\Phi) &\mapsto \|\Phi\|^2 \end{aligned}$$

which is proper, bounded below, and has a finite number of critical submanifolds. So this map is a Morse–Bott map.

For any component F_λ, we recall the definition of the upwards Morse strata, $\widetilde{U}_\lambda^+$, and the downwards Morse strata, $\widetilde{U}_\lambda^-$, that is,

$$\widetilde{U}_\lambda^+ := \{p \in \mathcal{M}_X(d,r,\alpha); \lim_{t\to -\infty} \psi_t(p) \in F_\lambda\}$$

and

$$\widetilde{U}_\lambda^- := \{p \in \mathcal{M}_X(d,r,\alpha); \lim_{t\to +\infty} \psi_t(p) \in F_\lambda\},$$

where ψ_t is the gradient flow for μ.

Recall that these stratifications were proved to be equal $U^+ = \widetilde{U}^+$ and $U^- = \widetilde{U}^-$ by Kirwan [Ki, Theorem 6.18]. We remark that Kirwan stated and proved the theorems for compact manifolds, but as she pointed (*cf.* [Ki, Chapter 9]), these results remain valid if every path of steepest descent for the function μ is contained in some compact subset. In our case this holds, because the Hitchin map is proper.

The union $N = \bigcup_\lambda \widetilde{U}_\lambda^-$ is known as *downwards Morse flow*.

The inverse over the 0 point of the Hitchin map $H^{-1}(0)$ is called the *nilpotent cone* and it coincides with the downwards Morse flow, *i.e.*, $N = H^{-1}(0)$ [GGM, Theorem 3.13]. The following proposition takes the same steps as Proposition 5.1 [BGL].

Proposition 5.1 *Let $\mathcal{V}$ be the complement of the cotangent bundle of $M_X(d,r,\alpha)$ in $\mathcal{M}_X(d,r,\alpha)$ and let $\mathcal{V}'$ be the Białynicki–Birula flow which does not converge to $M_X(d,r,\alpha)$.*

$$\mathcal{V} = \{(E,\Phi) \in \mathcal{M}_X(d,r,\alpha) : E \text{ is unstable}\},$$
$$\mathcal{V}' = \{(E,\Phi) \mid \lim_{t\to 0}(E, t\cdot\Phi) \notin M_X(d,r,\alpha)\}.$$

Then $\mathcal{V}' = \mathcal{V}$.

Proof If $(E,\Phi) \notin \mathcal{V}$, then E is semistable, and in fact stable, since we are assuming that the weights α are generic. Then $\lim_{t\to 0}(E, t\cdot\Phi) = (E,0) \in M_X(d,r,\alpha)$, so this proves that $\mathcal{V}' \subset \mathcal{V}$.

To prove the converse, take (E,Φ) where E is unstable. The standard action of $\mathbb{C}^*$ produces a morphism γ

$$\gamma: \mathbb{C}^* \to \mathcal{M}(d,r,\alpha)$$
$$t \mapsto (E, t\cdot\Phi)$$

The composition with the Hitchin map is such that $\lim_{t\to 0} h(E, t\cdot\Phi) = 0$, and this implies, by properness of the Hitchin map that γ extends to a morphism $\widetilde{\gamma}$ on $\mathbb{C}$.

If the moduli space admits a universal parabolic Higgs bundle, the morphism $\widetilde{\gamma}$ induces a family of stable parabolic Higgs bundles parametrized by the affine line $\mathbb{C}$. In general, there is an open neighborhood of the limiting point $\widetilde{\gamma}(0)$ which has a universal bundle, and this produces a family of stable parabolic Higgs bundles (E_t, Φ_t) parametrized by an open neighborhood U of the origin of $\mathbb{C}$. If $t \neq 0$, then the underlying parabolic bundle E_t is E, hence unstable. Semistability is an open condition ([Ni, Proposition 1.7]), therefore E_0 is also unstable. Note that $\lim_{t\to 0}(E, t\cdot\Phi) = (E_0, \Phi_0)$, hence this limiting point is not in $M_X(d,r,\alpha)$ which means that $(E,\Phi) \in \mathcal{V}'$. ∎

The following facts are recovered from the literature on parabolic Higgs bundles. Let (E,Φ) be a fixed point for the circle action. We have an isomorphism $(E,\Phi) \cong (E, e^{i\theta}\Phi)$ for $\theta \in [0, 2\pi)$ yielding the following commutative diagram.

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E \otimes K(D) \\ \downarrow{\scriptstyle \psi_\theta} & & \downarrow{\scriptstyle \psi_\theta \otimes 1_{K(D)}} \\ E & \xrightarrow{e^{i\theta}\Phi} & E \otimes K(D). \end{array}$$

Proposition 5.2 ([Si, Theorem 8]) *If (E,Φ) belongs to a critical subvariety F_λ for the circle action on $\mathcal{M}_X(d,r,\alpha)$, then E splits $E = \bigoplus_{l=0}^{m} E_l$ and*

$$\Phi \in H^0(\mathrm{SParHom}(E_l, E_{l+1}) \otimes K(D)).$$

The parabolic Higgs bundle in this case (E,Φ) is called Hodge bundle.

The deformation theory of the moduli space of parabolic Higgs bundles was worked out in [Yo]. It is given by the following complex of bundles,

$$C^\bullet(E): \mathrm{ParEnd}(E) \xrightarrow{\Phi := [\,\cdot\,,\Phi]} \mathrm{SParEnd}(E) \otimes K(D).$$

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The tangent space of the moduli space $\mathcal{M}_X(d,r,\alpha)$ at a stable point (E,Φ) is then the first hypercohomology group $\mathbb{H}^1(C^\bullet(E))$ of this complex. Hence for a fixed point (E,Φ) of the $\mathbb{C}^*$-action, the decomposition in Proposition 5.2 induces a decomposition of the deformation complex and of the tangent space at the fixed point. That is, we define

$$C_k := \bigoplus_{j-i=k} \mathrm{ParHom}(E_i,E_j) \quad \text{and} \quad \widehat{C}_{k+1} := \bigoplus_{j-i=k} \mathrm{SParHom}(E_i,E_j)$$

so then

$$C^\bullet(E)_k\colon C_k \xrightarrow{\Phi_k} \widehat{C}_{k+1}\otimes K(D) \quad \text{and} \quad C^\bullet(E) = \bigoplus_{k=-m-1}^{k=m} C^\bullet(E)_k.$$

For this deformation complex, there is a long exact sequence

$$\begin{aligned}
0 &\longrightarrow \mathbb{H}^0(C^\bullet(E)_k) \longrightarrow H^0\Big(\bigoplus_{j-i=k}\mathrm{ParHom}(E_i,E_j)\Big)\\
&\longrightarrow H^0\Big(\bigoplus_{j-i=k}\mathrm{SParHom}(E_i,E_j)\otimes K(D)\Big) \longrightarrow \mathbb{H}^1(C^\bullet(E)_k)\\
&\longrightarrow H^1\Big(\bigoplus_{j-i=k}\mathrm{ParHom}(E_i,E_j)\Big) \longrightarrow H^1\Big(\bigoplus_{j-i=k}\mathrm{SParHom}(E_i,E_j)\otimes K(D)\Big)\\
&\longrightarrow \mathbb{H}^2(C^\bullet(E)_k) \longrightarrow 0.
\end{aligned}$$

Therefore, the tangent space $T_{(E,\Phi)}\mathcal{M}_X(r,d,\alpha)$ decomposes as follows.

Theorem 5.3 ([GGM, Theorem 3.8]) *The function $\mu\colon \mathcal{M}_X(r,d,\alpha)\to\mathbb{R}$ defined by $\mu(E,\Phi)=\|\Phi\|^2$ is a perfect Morse–Bott function. A parabolic Higgs bundle represents a critical point of μ if and only if it is a parabolic complex variation of Hodge structure,* i.e., *$E=\bigoplus_{k=0}^m E_k$ with $\Phi_k=\Phi|_{E_k}\colon E_k\to E_{k+1}\otimes K(D)$ strongly parabolic (where $\Phi=0$ if and only if $m=0$). The tangent space to $\mathcal{M}_X(r,d,\alpha)$ at a critical point (E,Φ) decomposes as $T_{(E,\Phi)}\mathcal{M}_X(r,d,\alpha)=\bigoplus_{k=-m}^{m+1} T_{(E,\Phi)}\mathcal{M}_X(r,d,\alpha)_k$, where the eigenvalue k subspace of the Hessian of μ is $T_{(E,\Phi)}\mathcal{M}_X(r,d,\alpha)_k\cong\mathbb{H}^1(C^\bullet(E)_{-k})$.* ■

Proposition 5.4 ([GGM, Proposition 3.9])

(i) *There is a natural isomorphism*

$$\mathbb{H}^1(C^\bullet(E)_k)\simeq\mathbb{H}^1(C^\bullet(E)_{-k-1})^*$$

and hence a natural isomorphism

$$T_{(E,\Phi)}\mathcal{M}_X(r,d,\alpha)_k\simeq\big(T_{(E,\Phi)}\mathcal{M}_X(r,d,\alpha)_{1-k}\big)^*$$

(ii) *If (E,Φ) is stable, then we have*

$$\mathbb{H}^0(C^\bullet(E)_k)=\begin{cases}\mathbb{C} & \text{if } k=0,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\mathbb{H}^2(C^\bullet(E)_k)=\begin{cases}\mathbb{C} & \text{if } k=-1,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 5.5

$$\dim \mathbb{H}^1(C^\bullet(E)_k) = \begin{cases} 1 - \chi(C^\bullet(E)_k) & \text{if } k = 0, -1, \\ -\chi(C^\bullet(E)_k) & \text{otherwise.} \end{cases}$$

■

For a critical point (E, Φ) of μ we denote by $T_{(E,\Phi)}\mathcal{M}_X(r, d, \alpha)_{<0}$ the subspace of the tangent space on which the Hessian of μ has negative eigenvalues. The real dimension of this subspace is called the Morse index at the point (E, Φ).

Proposition 5.6 *The codimension of the complement of $T^* M_X(d, r, \alpha)$ in $\mathcal{M}_X(d, r, \alpha)$ is equal to half of the minima of the Morse indexes at points $(E, \Phi) \in F_\lambda$ for $\lambda \neq 0$.*

Proof The complement of $T^* M_X(d, r, \alpha)$ is equal to $\mathcal{V}$ which is also equal to $\mathcal{V}'$ from Proposition 5.1. Morse–Bott theory gives us that $\mathcal{V}' = \bigcup_{\lambda \neq 0} U_\lambda^+$, so we conclude that $\operatorname{codim}(\mathcal{V}) = \min_{\lambda \neq 0} \operatorname{codim} U_\lambda^+$. From Morse–Bott theory we also know that the dimension of the upwards Morse flow is such that

$$\dim U_\lambda^+ + \dim T_{(E,\Phi)}\mathcal{M}_X(d, r, \alpha)_{<0} = \dim \mathcal{M}_X(d, r, \alpha),$$

where $T_E\mathcal{M}_X(d, r, \alpha)_{<0}$ is the negative eigenspace for the Hessian of the perfect Morse–Bott function μ for some $E \in U_\lambda^+$. Since the Morse index μ_λ is equal to $2 \dim T_E\mathcal{M}_X(d, r, \alpha)_{<0}$,

$$\operatorname{codim} U_\lambda^+ = \frac{1}{2}\mu_\lambda.$$

Our statement is then

$$\operatorname{codim}(\mathcal{V}) = \min_{\lambda \neq 0} \frac{1}{2}\mu_\lambda,$$

that is, $\operatorname{codim}(\mathcal{V}) = \min_{\lambda \neq 0} \dim T_E\mathcal{M}_X(d, r, \alpha)_{<0}$. ■

Lemma 5.7

$$T_{(E,\Phi)}\mathcal{M}_X(d, r, \alpha)_{<0} = \sum_{k>0} -\chi(C^\bullet(E)_k.$$

■

So we need to bound the Euler characteristic for any k.

Proposition 5.8 *The following inequality holds:*

$$-\chi(C^\bullet(E)_k) \geq (g-1)\big(\operatorname{rk}(C_k) - \operatorname{rk}(\widehat{C}_{k+1})\big).$$

Proof Recall that

$$\begin{aligned}\chi(C^\bullet(E)_k) &= \dim H^0(C_k) - \dim H^1(C_k) - \dim H^0(\widehat{C}_{k+1} \otimes K(D)) \\ &\quad + \dim H^1(\widehat{C}_{k+1} \otimes K(D)) \\ &= \deg(C_k) - \deg(\widehat{C}_{k+1}) - \operatorname{rk}(\widehat{C}_{k+1}) \deg(K(D)) + (\operatorname{rk}(C_k) \\ &\quad - \operatorname{rk}(\widehat{C}_{k+1}))(1-g).\end{aligned}$$

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We first bound $\deg(C_k)-\deg(\widehat{C}_{k+1})$. Consider the following short exact sequences of bundles,

$$0 \longrightarrow \ker(\Phi_k) \longrightarrow C_k \longrightarrow \operatorname{im}(\Phi_k) \longrightarrow 0$$
$$0 \longrightarrow \operatorname{im}(\Phi_k) \longrightarrow \widehat{C}_{k+1} \otimes K(D) \longrightarrow \operatorname{coker}(\Phi_k) \longrightarrow 0.$$

Then

$$\deg(C_k) - \deg(\widehat{C}_{k+1}) = \deg(\ker(\Phi_k)) + \deg(K(D))\operatorname{rk}(\widehat{C}_{k+1}) - \deg(\operatorname{coker}(\Phi_k)).$$

The $\ker(\Phi_k) \subset \operatorname{ParEnd}(E)$ is a subbundle of the bundle of parabolic endomorphisms of E, which we claim is semistable whenever E is stable (see Lemma 5.9). Hence $\operatorname{pardeg}(\ker(\Phi_k)) \leq 0$ and this implies $\deg(\ker(\Phi_k)) \leq 0$.

Hence $\deg(C_k) - \deg(\widehat{C}_{k+1}) \leq \deg(K(D))\operatorname{rk}(\widehat{C}_{k+1}) - \deg(\operatorname{coker}(\Phi_k))$. We also get that

$$-\deg(\operatorname{coker}(\Phi_k)) \leq (2-2g)(\operatorname{rk}(\widehat{C}_{k+1}) - \operatorname{rk}(\Phi_k)), \tag{5.1}$$

so that

$$\deg(C_k) - \deg(\widehat{C}_{k+1}) \leq n\operatorname{rk}(\widehat{C}_{k+1}) + (2g-2)\operatorname{rk}(\Phi_k) \tag{5.2}$$

in the following way.

Note that for any two parabolic bundles E and F,

$$\operatorname{ParHom}(E,F)^* = \operatorname{SParHom}(F,E) \otimes \mathcal{O}(D).$$

So then

$$C_k^* = \Big(\bigoplus_{j-i=k} \operatorname{ParHom}(E_i,E_j)\Big)^* = \bigoplus_{j-i=k} \operatorname{SParHom}(E_j,E_i) \otimes \mathcal{O}(D) = \widehat{C}_k \otimes \mathcal{O}(D).$$

Consider the adjoint map $\Phi_k^t\colon \big(\widehat{C}_{k+1} \otimes K(D)\big)^* \to (C_k)^*$. Then

$$\ker(\Phi_k^t) \hookrightarrow \big(\widehat{C}_{k+1} \otimes K(D)\big)^* \cong C_{-1-k} \otimes K^{-1}.$$

Dualizing again, we get a surjective homomorphism, $\widehat{C}_{k+1} \otimes K(D) \to \big(\ker(\Phi_k^t)\big)^*$.

Define the homomorphism $f\colon \operatorname{coker}(\Phi_k) \to \ker(\Phi_k^t)^*$ which makes the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{im}(\Phi_k) & \longrightarrow & \widehat{C}_{k+1} \otimes K(D) & \longrightarrow & \operatorname{coker}(\Phi_k) & \longrightarrow & 0 \\
 & & & & \| & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & (\operatorname{im}(\Phi_k^t))^* & \longrightarrow & \widehat{C}_{k+1} \otimes K(D) & \longrightarrow & (\ker(\Phi_k^t))^* & \longrightarrow & 0.
\end{array}$$

Note that f is surjective and $\ker(f)$ is a torsion subsheaf. Hence,

$$0 \longrightarrow \ker(f) \longrightarrow \operatorname{coker}(\Phi_k) \longrightarrow (\ker(\Phi_k^t))^* \longrightarrow 0,$$

and $\deg(\operatorname{coker}(\Phi_k)) \geq \deg(\ker(\Phi_k^t)^*)$. As $\ker(\Phi_k^t)$ is a sub bundle of $C_{k+1} \otimes K$,

$$-\deg(\operatorname{coker}(\Phi_k)) \leq \deg(\ker(\Phi_k^t)) \tag{5.3}$$

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Note that there are isomorphisms making the following diagram commutative

$$\begin{array}{ccc} (\widehat{C}_{k+1}\otimes K(D))^* & \xrightarrow{\Phi_k^t} & (C_k)^* \\ \downarrow \cong & & \downarrow \cong \\ C_{-1-k}\otimes K^{-1} & \xrightarrow[\Phi_{-1-k}\otimes 1_{K^{-1}}]{} & \widehat{C}_{-k}\otimes \mathcal{O}(D), \end{array}$$

therefore $\Phi_k^t \cong \Phi_{-1-k}\otimes 1_{K^{-1}}$ so $\ker(\Phi_k^t) = \ker(\Phi_{-1-k})\otimes K^{-1}$ and

$$\deg(\ker(\Phi_k^t)) = \deg(\ker(\Phi_{-1-k})) + (2-2g)\operatorname{rk}(\ker(\Phi_{-1-k}))$$

Notice that $\operatorname{rk}(\Phi_{-1-k}) = \operatorname{rk}(\Phi_k^t) = \operatorname{rk}(\Phi_k)$ and $\operatorname{rk}(\widehat{C}_{k+1}) = \operatorname{rk}((\widehat{C}_{k+1})^*) = \operatorname{rk}(C_{-1-k})$. Then $\operatorname{rk}(\ker(\Phi_{-1-k})) = \operatorname{rk}(\widehat{C}_{k+1}) - \operatorname{rk}(\Phi_k)$, so that equation (5.3) becomes

$$-\deg(\operatorname{coker}\Phi_k) \le \deg(\ker(\Phi_{-1-k})) + (2-2g)\big(\operatorname{rk}(\widehat{C}_{k+1}) - \operatorname{rk}(\Phi_k)\big)$$

and finally, by stability (see Lemma 5.9),

$$-\deg(\operatorname{coker}\Phi_k) \le (2-2g)(\operatorname{rk}(\widehat{C}_{k+1}) - \operatorname{rk}(\Phi_k)).$$

This provides equation (5.1).

Putting together equations (5.1) and (5.2) we get

$$\chi(C^\bullet(E)_k) \le (1-g)(\operatorname{rk}(C_k) - \operatorname{rk}(\widehat{C}_{k+1})),$$

hence, $-\chi(C^\bullet(E)_k) \ge (g-1)(\operatorname{rk}(C_k) - \operatorname{rk}\big(\widehat{C}_{k+1})\big)$, as we wanted. ∎

Lemma 5.9 *If (E,Φ) is a stable parabolic Higgs bundle, then* $(\operatorname{ParEnd}(E), \operatorname{ad}(\Phi))$ *is semistable.*

Proof The proof follows the arguments in [GLM, Proposition 6.7] adapted to the parabolic situation. That is, the vector bundle $\operatorname{ParEnd}(E)$ has a natural parabolic structure induced by the parabolic structure of E. In fact $\operatorname{ParEnd}(E)$ as a parabolic bundle is the parabolic tensor product of the parabolic bundle E and the parabolic dual of E (see [Yo]), and hence its parabolic degree is 0. With respect to this parabolic structure $(\operatorname{ParEnd}(E), \operatorname{ad}(\Phi))$, where $\operatorname{ad}(\Phi)\colon \operatorname{ParEnd}(E) \to \operatorname{SParEnd}(E)\otimes K(D)$ is, again, a parabolic Higgs bundle. Now, the stability of (E,Φ) implies the polystability of $(\operatorname{ParEnd}(E), \operatorname{ad}(\Phi))$. ∎

Proposition 5.10 $\operatorname{codim}(\mathcal{V}) \ge \frac{1}{2}(r-1)(g-1)$.

Proof Propositions 5.6 and 5.8 give

$$\begin{aligned}\operatorname{codim}(\mathcal{V}) &= \min_{F_\lambda}\Big\{\sum_{k>0} -\chi(C^\bullet(E)_k)\Big\} \\ &\ge \min_{F_\lambda}\Big\{\sum_{k>0}(g-1)(\operatorname{rk}(C_k - \operatorname{rk}(\widehat{C}_{k+1}))\Big\} = \operatorname{rk}(C_1)(g-1),\end{aligned}$$

where $\operatorname{rk}(C_1) = \operatorname{rk}(\oplus_{j-i=1}\operatorname{ParHom}(E_i,E_j)$. So if we denote $r_i = \operatorname{rk}(E_i)$, the rank of each piece is $\operatorname{rk}(\operatorname{ParHom}(E_i,E_j) = \frac{1}{2}r_ir_j$, then $\operatorname{rk}(C_1) = \frac{1}{2}(r_1r_2+\cdots+r_{m-1}r_m)$ which is definitely $\operatorname{rk}(C_1) \ge \frac{1}{2}(r-1)$. ∎

Corollary 5.11 *For $g = 2$ and $r \geq 5$ or $g = 3$ and $r \geq 3$ $g \geq 5$ and $r \geq 2$, and u a generic point in $\mathcal{U}$, the codimension of the complement of the fiber $h^{-1}(u)$ in $H^{-1}(u)$ is greater than or equal to 2.*

Remark 5.12. The codimension does not depend on the number of marked points, as in [BGL] it did not depend on the degree of the line bundle L, which was twisting the Higgs bundle and in this case is $K(D)$.

We also obtain the following.

Corollary 5.13 *For $g = 2$ and $r \geq 5$ or $g = 3$ and $r \geq 3$ $g \geq 5$ and $r \geq 2$, the moduli space of parabolic bundles has the same number of irreducible components as the moduli space of parabolic Higgs bundles.*

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References

[AB] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces.* Phil. Trans. R. Soc. London **308**(1982) 523–615. http://dx.doi.org/10.1098/rsta.1983.0017

[BBB] V. Balaji, S. del Baño, and I. Biswas, *A Torelli type theorem for the moduli space of parabolic vector bundles over curves.* Math. Proc. Cambridge Philos. Soc. **130**(2001), 269–280. http://dx.doi.org/10.1017/S0305004100004916

[BG] I. Biswas and T. L. Gómez, *A Torelli theorem for the moduli space of Higgs bundles on a curve.* Quart. Jour. Math. **54**(2003), 159–169. http://dx.doi.org/10.1093/qmath/hag006

[BGL] I. Biswas, P. B. Gothen, and M. Logares, *On moduli spaces of Hitchin pairs.* Math. Proc. Cambridge Philos. Soc. **151**(2011), 441–457. http://dx.doi.org/10.1017/S0305004111000405

[BHK] I. Biswas, Y. Holla, and C. Kumar, *On moduli spaces of parabolic vector bundles of rank 2 over* $\mathbb{CP}^1$. Michigan Math. Jour. **59**(2010), 467–479. http://dx.doi.org/10.1307/mmj/1281531467

[BNR] A. Beauville, M. S. Narasimhan and S. Ramanan, *Spectral curves and the generalized theta divisor.* J. reine angew. Math. **398**(1989), 169–179.

[CRS] C. Ciliberto, P. Ribenboim, and E. Sernesi, *Collected papers of Ruggiero Torelli,*. Queen's Papers in Pure and Applied Mathematics, 101. Queen's University, Kingston, 1995.

[GGM] O. García-Prada, P. B. Gothen, and V. Muñoz, *Betti numbers for the moduli space of rank 3 parabolic Higgs bundles*. Mem. Amer. Math. Soc. **187**(2007), no. 879.

[GLM] O. García-Prada, M. Logares, and V. Muñoz, *Moduli spaces of parabolic* $\mathrm{U}(p,q)$*-Higgs bundles*. Quart. Jour. Math. **60**(2009), 183–233. http://dx.doi.org/10.1093/qmath/han001

[GL] T. L. Gómez and M. Logares, *Torelli theorem for the moduli space of parabolic Higgs bundles*. Adv. Geom. **11**(2011), 429–444.

[Har] R. Hartshorne, *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York, 1977.

[Hau] T. Hausel, *Compactification of the moduli of Higgs bundles*. J. Reine Angew. Math. **503**(1998), 169–192.

[Hi] N.J. Hitchin, *Stable bundles and integrable systems.* Duke Math. J. **54**(1987), 91–114. http://dx.doi.org/10.1215/S0012-7094-87-05408-1

[Hu] J. C. Hurtubise, *Integrable systems and algebraic surfaces*. Duke Math. J. **83**(1996), 19–49. http://dx.doi.org/10.1215/S0012-7094-96-08302-7

[Ki] F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*. Mathematical Notes 31, Princeton University Press, 1984.

[KM] F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves I: preliminaries on "det" and "div"*. Math. Scand. **39**(1976), 19–55.

[LM] M. Logares and J. Martens, *Moduli of parabolic Higgs bundles and Atiyah algebroids*. J. reine angew. Math. **649**(2010), 89–116.

[MN] D. Mumford and P. Newstead, *Periods of a modulispace of bundles on curves* Amer. Jour. Math. **90**(1968), 1200–1208. http://dx.doi.org/10.2307/2373296

[Ni] N. Nitsure, *Cohomology of the moduli of parabolic vector bundles*. Proc. Indian Acad. Sci, (Math. Sci.) **95**(1986), 61–77. http://dx.doi.org/10.1007/BF02837250

[NR] M. S. Narasimhan and S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve*. Ann. Math. **101**(1975), 391–417. http://dx.doi.org/10.2307/1970933

[Seb] R. Sebastian, *Torelli theorems for moduli of logarithmic connections and parabolic bundles*. Manuscr. Math. **136** (2011), 249–271. http://dx.doi.org/10.1007/s00229-011-0446-9

[Ses] C.S. Seshadri, *Fibrés vectorieles sur les courbes algébriques*. Asterisque 96, 1982.

[Si] C. Simpson, *Harmonic bundles on non compact curves*, Jour. Amer. Math. Soc. **3**(1990), 713–770. http://dx.doi.org/10.1090/S0894-0347-1990-1040197-8

[Tj] A. N. Tjurin, *An analogue of the Torelli theorem for two-dimensional bundles over an algebraic curve of arbitrary genus*. Izv. Akad. Nauk SSSR Ser. Mat. **33**(1969), 1149–1170.

[We] A. Weil, *Zum beweis des Torelli satzes*. Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 2(1957) 33–53.

[Yo] K. Yokogawa, *Infinitesimal deformation of parabolic Higgs sheaves*. Inter. Jour. Math. **6**(1995), 125–148. http://dx.doi.org/10.1142/S0129167X95000092

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