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SOME REMARKS ON TALENTI'S SEMIGROUP

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1. **Introduction.** Let X be a Banach space. Consider the family $I(\alpha)$ of linear continuous operators from X into itself, depending on the parameter $\alpha \geq 0$. Suppose that $I(\alpha_1+\alpha_2)=I(\alpha_1)I(\alpha_2)\forall\alpha_1, \alpha_2 \geq 0$ and $I(0)=I$ (semigroupal property). Such a semigroup is said to be strongly continuous for $\alpha>0$, if $\lim_{h\to 0} I(\alpha+h)x =I(\alpha)x$, $\forall x \in X$. We shall study here an example of such semigroup, generalizing the well-known semigroup generated by the Riemann–Liouville Integral (see [2]). The semigroup we are studying here appeared in a paper by G. Talenti [1].

2. Take $X=C[0, 1]$ the space of continuous functions on [0, 1] with the topology of the uniform convergence. Take also $p(t)$ a positive and continuous function in [0, 1] and define, $\forall g \in C[0, 1]$ the operator

$$I(\alpha)g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t g(w)\left(\int_w^t p(s)\,ds\right)^{\alpha-1}p(w)\,dw, \quad \alpha > 0 \tag{2.1}$$

where $I(\alpha)=[p^{-1}(d/dt)]^{-\alpha}$.

Remark. If $p(t)\equiv 1$ then (2.1) reduces to Riemann–Liouville Integral.

$$I(\alpha)g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t g(w)(t-w)^{\alpha-1}\,dw \tag{2.2}$$

which belongs to $C[0, 1]$, and

$$\|I(\alpha)[g]\| \leq \frac{\|g\|}{\Gamma(\alpha+1)}, \quad \alpha > 0. \tag{2.3}$$

It is trivial that $I(\alpha)(g_1+g_2)=I(\alpha)g_1+I(\alpha)g_2$, $\alpha>0$.

Theorem I. *$I(\alpha)$ maps $C[0, 1]$ into itself and*

$$\|I(\alpha)[g]\| \leq \frac{\|g\|}{\Gamma(\alpha+1)}\|p\|^{\alpha}, \quad \alpha > 0. \tag{2.4}$$

Proof. Let $0\leq t<t+h\leq 1$ and $0<\alpha<1$.

$$\begin{aligned} I(\alpha)g(t+h)-I(\alpha)g(t) &= \frac{1}{\Gamma(\alpha)}\int_0^{t+h} g(w)\left(\int_w^{t+h} p(s)\,ds\right)^{\alpha-1}p(w)\,dw \\ &\quad -\frac{1}{\Gamma(\alpha)}\int_0^t g(w)\left(\int_w^t p(s)\,ds\right)^{\alpha-1}p(w)\,dw. \end{aligned} \tag{2.5}$$

We make the substitution

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(2.6)
$$\int_0^s p(s)\,ds = P(s).$$

Then

$$\int_w^t p(s)\,ds = P(t)-P(w).$$

Therefore the R.H. side of (2.5) is equal to

$$\frac{1}{\Gamma(\alpha)}\int_0^{t+h} g(w)(P(t+h)-P(w))^{\alpha-1}P'(w)\,dw-\frac{1}{\Gamma(\alpha)}\int_0^t g(w)(P(t)-P(w))^{\alpha-1}P'(w)\,dw$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^t g(w)(P(t+h)-P(w))^{\alpha-1}P'(w)\,dw+\frac{1}{\Gamma(\alpha)}\int_t^{t+h} g(w)$$
$$\times\ (P(t+h)-P(w))^{\alpha-1}P'(w)\,dw-\frac{1}{\Gamma(\alpha)}\int_0^t g(w)(P(t)-P(w))^{\alpha-1}P'(w)\,dw$$
$$=\frac{1}{\Gamma(\alpha)}\int_0^t g(w)[(P(t+h)-P(w))^{\alpha-1}-(P(t)-P(w))^{\alpha-1}]P'(w)\,dw$$
$$+\frac{1}{\Gamma(\alpha)}\int_t^{t+h} g(w)(P(t+h)-P(w))^{\alpha-1}P'(w)\,dw.$$

$$|I(\alpha)g(t+h)-I(\alpha)g(t)| \le \frac{\|g\|}{\Gamma(\alpha)}\left\{\int_0^t |\,(P(t+h)-P(w))^{\alpha-1}\right.$$
$$\left.-(P(t)-P(w))^{\alpha-1}|P'(w)\,dw+\int_t^{t+h}|(P(t+h)-P(w))^{\alpha-1}|\,P'(w)\,dw\right\}.$$

Since $P(t+h)-P(w)>P(t)-P(w)$ and, for $0<\alpha<1$,

$$(P(t+h)-P(w))^{\alpha-1} < (P(t)-P(w))^{\alpha-1},$$

this implies that

$$|(P(t+h)-P(w))^{\alpha-1}-(P(t)-P(w))^{\alpha-1}|$$
$$=(-1)[(P(t+h)-P(w))^{\alpha-1}-(P(t)-P(w))^{\alpha-1}].$$

Therefore,

$$\int_0^t |(P(t+h)-P(w))^{\alpha-1}-(P(t)-P(w))^{\alpha-1}|P'(w)\,dw$$
$$=\int_0^t [(P(t+h)-P(w))^{\alpha-1}-(P(t)-P(w))^{\alpha-1}](-P'(w)\,dw)$$
$$=\frac{(P(t+h)-P(w))^{\alpha}}{\alpha}-\frac{(P(t)-P(w))^{\alpha}}{\alpha}\Bigg|_0^t$$
$$=\frac{1}{\alpha}[(P(t+h)-P(t))^{\alpha}+P(t)^{\alpha}-P(t+h)^{\alpha}]$$

and

$$\int_t^{t+h}(P(t+h)-P(w))^{\alpha-1}P'(w)\,dw = -\frac{1}{\alpha}(P(t+h)-P(w))^{\alpha}\Big|_t^{t+h}$$
$$=\frac{1}{\alpha}(P(t+h)-P(t))^{\alpha}.$$

Thus,

$$|I(\alpha)g(t+h)-I(\alpha)g(t)|$$

$$\leq \frac{\|g\|}{\Gamma(\alpha+1)}[2(P(t+h)-P(t))^{\alpha}+P(t)^{\alpha}-P(t+h)^{\alpha}] \to 0 \text{ as } h \to 0.$$

This proves continuity on the right and left continuity is proved in a similar manner. Using the same techniques one can estimate that

$$\|I(\alpha)g\| \leq \frac{\|g\|}{\Gamma(\alpha+1)}\|P\|^{\alpha},$$

from which (2.3) follows.

Thus we have proved that $I(\alpha) \in \mathscr{L}(X, X) \forall \alpha > 0$, and the semigroupal property $I(\alpha_1+\alpha_2)=I(\alpha_1)I(\alpha_2)$ is proved in [1].

Next we proceed to show that the above considered semigroup is not only strongly continuous for each $\alpha > 0$, but it is fact continuous in the operator norm.

Proof.

$$I(\alpha+h)g(t)-I(\alpha)g(t) = \int_0^t g(w)\left[\frac{(P(t)-P(w))^{\alpha+h-1}}{\Gamma(\alpha+h)}-\frac{(P(t)-P(w))^{\alpha}}{\Gamma(\alpha)}\right]P'(w)\,dw.$$

$$|I(\alpha+h)g(t)-I(\alpha)g(t)|$$

$$\leq \int_0^t |g(w)|\left|\left(\frac{(P(t)-P(w))^{\alpha+h-1}}{\Gamma(\alpha+h)}-\frac{(P(t)-P(w))^{\alpha-1}}{\Gamma(\alpha)}\right)P'(w)\right|dw$$

from which we have

$$\begin{aligned}\|I(\alpha+h)g-I(\alpha)g\| &= \sup_{0\leq t\leq 1}|I(\alpha+h)g(t)-I(\alpha)g(t)| \\ &\leq \|g\| \sup_{0\leq t\leq 1}\left|\frac{(P(t)-P(w))^{\alpha+h}}{(-)(\alpha+h)\Gamma(\alpha+h)}+\frac{(P(t)-P(w))^{\alpha}}{\alpha\Gamma(\alpha)}\Big|_0^t\right| \\ &= \|g\| \sup_{0\leq t\leq 1}\left|\frac{P(t)^{\alpha+h}}{\Gamma(\alpha+h+1)}-\frac{P(t)^{\alpha}}{\Gamma(\alpha+1)}\right|.\end{aligned}$$

Let

$$E = \left|\frac{P(t)^{\alpha+h}}{\Gamma(\alpha+h+1)}-\frac{P(t)^{\alpha}}{\Gamma(\alpha+1)}\right|.$$

We want to show it converges uniformly to 0 for $t \in [0, 1]$ as $h \to 0$, $\forall \alpha > 0$.

$$\begin{aligned}E &= \left|\frac{P(t)^{\alpha}}{\Gamma(\alpha+h+1)}\right|\left|P(t)^{h}-\frac{\Gamma(\alpha+h+1)}{\Gamma(\alpha+1)}\right| \\ &\leq \left|\frac{P(t)^{\alpha}}{\Gamma(\alpha+1)}\right|\left|\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+h+1)}\right|\left\{|P(t)^{h}-1|+\left|1-\frac{\Gamma(\alpha+h+1)}{\Gamma(\alpha+1)}\right|\right\}.\end{aligned}$$

For small h,

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+h+1)} = e^{-h}\frac{\Gamma'(\alpha+1)}{\Gamma(\alpha+1)}+O(h^{3/2}), \quad |h| < 1.$$

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Choose h so small that $1-\varepsilon < [\Gamma(\alpha+1)]/[\Gamma(\alpha+h+1)] < 1+\varepsilon$.

$$|P(t)^h - 1| < \frac{\varepsilon}{2M}, \qquad \left|1-\frac{\Gamma(\alpha+h+1)}{\Gamma(\alpha+1)}\right| < \frac{\varepsilon}{2M}$$

where

$$M = \max_{0\le t\le 1} \frac{P(t)^\alpha}{\Gamma(\alpha+1)}.$$

Then

$$\|I(\alpha+h)g - I(\alpha)g\| < M(1+\varepsilon)\left[\frac{\varepsilon}{2M}+\frac{\varepsilon}{2M}\right]\|g\|$$

or

$$\|I(\alpha+h)g - I(\alpha)g\| < \varepsilon(1+\varepsilon)\|g\|$$

and this implies

$$\sup_{\|g\|\le 1} \|I(\alpha+h)g - I(\alpha)g\| = \|I(\alpha+h)-I(\alpha)\| < \varepsilon(1+\varepsilon)\to 0 \quad \text{as } \varepsilon\to 0.$$

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