Forum of Mathematics, Pi (2016), Vol. 4, e5, 28 pages
doi:10.1017/fmp.2016.3

THE α-INVARIANT AND THOMPSON'S CONJECTURE

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Received 8 January 2016; accepted 13 June 2016

Abstract

In 1981, Thompson proved that, if $n \geqslant 1$ is any integer and G is any finite subgroup of $\mathrm{GL}_n(\mathbb{C})$, then G has a semi-invariant of degree at most $4n^2$. He conjectured that, in fact, there is a universal constant C such that for any $n \in \mathbb{N}$ and any finite subgroup $G < \mathrm{GL}_n(\mathbb{C})$, G has a semi-invariant of degree at most Cn. This conjecture would imply that the α-invariant $\alpha_G(\mathbb{P}^{n-1})$, as introduced by Tian in 1987, is at most C. We prove Thompson's conjecture in this paper.

2010 Mathematics Subject Classification: 20C15 (primary); 13A50, 14B05, 20C33 (secondary)

1. Introduction

Invariants, and more generally, semi-invariants, have been studied since the very beginning of finite group theory. Let $V = \mathbb{C}^n$ and let $G < \mathrm{GL}(V)$ be a finite group. Then G acts on the dual space V^*, and a nonzero element $f \in \mathrm{Sym}^k(V^*)$ is said to be an *invariant*, respectively a *semi-invariant*, of degree k for G if G fixes f, respectively if G fixes the 1-dimensional space $\langle f \rangle_{\mathbb{C}}$. In spite of many classical results in invariant theory and representation theory of finite groups, very little is known about the smallest degree of invariants and semi-invariants for arbitrary finite subgroup of $\mathrm{GL}(V)$. Let

$$d(G) := \min\{k \in \mathbb{N} \mid G \text{ has a semi-invariant of degree } k\}.$$

Since $\mathrm{Sym}^k(V)^* \cong \mathrm{Sym}^k(V^*)$, G has a semi-invariant of degree k if and only if the G-module $\mathrm{Sym}^k(V)$ contains a 1-dimensional submodule.

In 1981, Thompson proved the following theorem.

THEOREM 1.1 [Th]. *Let $n \in \mathbb{N}$ be any integer and $G < \mathrm{GL}_n(\mathbb{C})$ be any finite subgroup. Then $d(G) \leqslant 4n^2$.*

It turns out that this result also has interesting implications in algebraic geometry, in particular, in regard to the α-invariant $\alpha_G(\mathbb{P}^{n-1})$ when $G < \mathrm{GL}(V)$ acts on the projective space $\mathbb{P}V = \mathbb{P}^{n-1}$.

The *α-invariant* $\alpha_G(X)$ for a compact group G of automorphisms of a Kähler manifold X was introduced by Tian in 1987 [Ti, TY]. This invariant is of importance in differential geometry and algebraic geometry. In the case X is a Fano variety, Tian proved in [Ti] that X admits a G-invariant Kähler–Einstein metric if

$$\alpha_G(X) > \frac{\dim(X)}{\dim(X)+1}.$$

Moreover, it was proved by Demailly and Kollár [DK], see also [CS1, Appendix A], that in this case the α-invariant coincides with the *log-canonical threshold*

$$\begin{aligned}&\mathrm{lct}(X, G)\\ &\quad = \sup\left\{\lambda \in \mathbb{Q} \,\middle|\, \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ has log-canonical singularities}\\ \text{for every } G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\end{array}\right\},\end{aligned}$$

cf. [CS1, page 863].

An important example of Fano varieties is the projective space $\mathbb{P}V = \mathbb{P}^{n-1}$, where $V = \mathbb{C}^n$. Consider the natural action of any finite subgroup $G < \mathrm{GL}(V)$ on $\mathbb{P}^{n-1}$. Then Thompson's Theorem 1.1 implies that $\alpha_G(\mathbb{P}^{n-1})$ can be bounded linearly in terms of n.

THEOREM 1.2 [Th]. *Let $n \in \mathbb{N}$ be any integer and $G < \mathrm{GL}_n(\mathbb{C})$ be any finite subgroup. Then $\alpha_G(\mathbb{P}^{n-1}) \leqslant 4n$.*

The connection between the α-invariant and semi-invariants follows from the following inequality

$$\alpha_G(\mathbb{P}V) \leqslant \frac{d(G)}{\dim(V)}, \tag{1.1}$$

see [CS2, Section 1]. Under the additional assumption that G contains no complex reflection, [CS2, Theorems 1.17 and 3.16] shows that the quotient singularity $\mathbb{C}^n/G$ is *exceptional* if $\alpha_G(\mathbb{P}^{n-1}) > 1$, not exceptional if $\alpha_G(\mathbb{P}^{n-1}) < 1$ or $d(G) \leqslant n$, and *weakly exceptional* if and only if $\alpha_G(\mathbb{P}^{n-1}) \geqslant 1$. Furthermore, for $n \geqslant 24$ the upper bound $\alpha_G(\mathbb{P}^{n-1}) \leqslant 4n$ can be improved to $\alpha_G(\mathbb{P}^{n-1}) \leqslant 12n/5$, see [CS2, Theorem 1.24].

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In the same paper [Th], Thompson raised (the first part of) the following conjecture.

THOMPSON'S CONJECTURE [Th]. *There is a positive constant C such that for any $n \in \mathbb{N}$ and for any finite subgroup $G < \mathrm{GL}_n(\mathbb{C})$, the following statements hold:*

(i) $d(G) \leqslant Cn$; *and*

(ii) $\alpha_G(\mathbb{P}^{n-1}) \leqslant C$.

The goal of the paper is to prove the following result.

MAIN THEOREM. *Thompson's Conjecture is true, with $C = 1184036$.*

Note that, with substantially more efforts, the upper bound 1184036 in Main Theorem for the constant C in Thompson's conjecture can be driven down a bit, but we did not try to do it. On the other hand, this constant C cannot be smaller than 1, see Example 3.6(i). (In fact, for some specific n there are examples showing that $d(G)$ can be larger than n for $G < \mathrm{GL}_n(\mathbb{C})$, see Example 3.6(ii), (iii).) We also note that Corollary 3.8 gives an example of a weakly exceptional, but nonexceptional, quotient singularity $\mathbb{C}^p/G$ with $\alpha_G(\mathbb{P}^{p-1}) = 1$, for any prime $p > 2$. It is an open question, see [CS2, Question 1.9] and [CS3], whether there exists a finite subgroup $G < \mathrm{GL}_n(\mathbb{C})$ such that $\alpha_G(\mathbb{P}^{n-1}) > 1$ for $n \geqslant 8$.

The paper is organized as follows. In Section 2 we prove some key estimates on the character values of symmetric powers, which allow us to give in Proposition 2.6 a criterion for the existence of semi-invariants of certain degrees. Analogues of these results for all symmetrizations of a given complex representation are also established. In Section 3, the classification of finite simple groups, and their representation theory and structure theory are used to study some base cases of Thompson's Conjecture. Relying on these results and Aschbacher's theorem on finite subgroups of classical groups [A], we then prove Thompson's Conjecture in Section 4. We also give an upper bound on the smallest degree of a (nonzero) polynomial invariant for any finite linear group, cf. Corollary 4.4.

2. Character estimates

Our approach relies on bounding character values of symmetric powers $\mathrm{Sym}^k(\chi)$ of a given complex character χ. The symmetric power $\mathrm{Sym}^k(\chi)$ is just a partial case of λ-symmetrizations $\mathrm{Sym}^\lambda(\chi)$ for any $\lambda \in \mathrm{Irr}(\mathsf{S}_k)$, which can be defined as follows. Let the complex character χ of a finite group G be

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afforded by a $\mathbb{C}G$-module V. Then we can extend the action of G to the tensor product $W = V^{\otimes k}$ of k copies of V. We can also define the action of S_k on W via naturally permuting these k copies, and this action is commuting with the action of G on W. Thus we obtain an action of $\mathsf{S}_k \times G$ on W, with character say ω. Then we can decompose

$$\omega(sg) = \sum_{\lambda \in \mathrm{Irr}(\mathsf{S}_k)} \lambda(s) \cdot \mathrm{Sym}^{\lambda}(\chi)(g),$$

where $s \in \mathsf{S}_k$, $g \in G$, and

$$\mathrm{Sym}^{\lambda}(\chi)(g) = \frac{1}{k!} \sum_{\pi \in \mathsf{S}_k} \lambda(\pi)\omega(\pi g), \tag{2.1}$$

cf. [**LBST**, Lemma 5.5] (note that $\lambda = \bar{\lambda}$). Thus, $\lambda \otimes \mathrm{Sym}^{\lambda}(\chi)$ is the $\mathsf{S}_k \times G$-character afforded by the λ-isotypic component of the S_k-module W. In particular, $\mathrm{Sym}^k(\chi) = \mathrm{Sym}^{\lambda}(\chi)$ if $\lambda = 1_{\mathsf{S}_k}$ is the trivial character of S_k.

For any $\pi \in \mathsf{S}_k$, we write π as a product of disjoint cycles and let $a_j(\pi)$ denote the number of cycles of length j in this decomposition, where $1 \leqslant j \leqslant k$. In particular,

$$\sum_{j=1}^{k} j a_j(\pi) = k. \tag{2.2}$$

With this notation, one has the following lemma.

LEMMA 2.1. *For any finite group G with a complex character χ, any $g \in G$, any positive integer k, and any $\lambda \in \mathrm{Irr}(\mathsf{S}_k)$, one has*

$$\mathrm{Sym}^{\lambda}(\chi)(g) = \frac{1}{k!} \sum_{\pi \in \mathsf{S}_k} \lambda(\pi) \prod_{j=1}^{k} \chi(g^j)^{a_j(\pi)}.$$

Proof. The formula is well known, but we give a proof for the reader's convenience. According to (2.1), it suffices to prove that

$$\omega(\pi g) = \prod_{j=1}^{k} \chi(g^j)^{a_j(\pi)} \tag{2.3}$$

for all $g \in G$ and $\pi \in \mathsf{S}_k$. First we consider the case $\pi = (1, 2, \ldots, k)$ is a k-cycle. We can find a matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ and a basis $(e_j \mid 1 \leqslant j \leqslant n := \dim(V))$ of V, such that

$$g(e_j) = \sum_{l=1}^{n} a_{lj} e_l, \quad \pi : v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_k \mapsto v_k \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{k-1}$$

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for all $v_1, \ldots, v_k \in V$. Now $(e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k} \mid 1 \leqslant j_1, \ldots, j_k \leqslant n)$ is a basis for W, and it is straightforward to check that

$$\omega(\pi g) = \sum_{j_1, j_2, \ldots, j_k = 1}^{n} a_{j_2 j_1} a_{j_3 j_2} \ldots a_{j_k j_{k-1}} a_{j_1 j_k} = \mathrm{Tr}(A^k) = \chi(g^k),$$

establishing (2.3) in this special case. The general case of (2.3) reduces to this special case by decomposing π into a product of disjoint cycles. □

PROPOSITION 2.2. *Let G be a finite group and $0 < \gamma \leqslant 1$ be such that $|\chi(x)|/\chi(1) \leqslant \gamma$ for some $\chi \in \mathrm{Irr}(G)$ of degree n and for all $x \in G \smallsetminus \mathbf{Z}(G)$. Then for any $g \in G \smallsetminus \mathbf{Z}(G)$ and any $k \in \mathbb{N}$ we have*

$$|\mathrm{Sym}^k(\chi)(g)| \leqslant \frac{\prod_{j=0}^{k-1}(\gamma n + j/\gamma)}{k!}.$$

In particular, if k is such that $n \geqslant 3k/\gamma^{3/2}$, then

$$\frac{|\mathrm{Sym}^k(\chi)(g)|}{\mathrm{Sym}^k(\chi)(1)} \leqslant \gamma^{k/2}. \tag{2.4}$$

Proof. Applying Lemma 2.1 to the case $\lambda = 1_{\mathsf{S}_k}$ we get:

$$\mathrm{Sym}^k(\chi)(g) = \frac{1}{k!} \sum_{\pi \in \mathsf{S}_k} \prod_{j=1}^{k} \chi(g^j)^{a_j(\pi)}.$$

Setting

$$f(t) := \frac{1}{k!} \prod_{j=0}^{k-1} (t + j)$$

for any $t \in \mathbb{R}$, and evaluating the above formula (for characters ϑ of varying degree n) at $g = 1$, we see that

$$f(t) = \frac{1}{k!} \sum_{\pi \in \mathsf{S}_k} \prod_{j=1}^{k} t^{a_j(\pi)}. \tag{2.5}$$

Now, by substituting $t = \gamma^2 n$ in (2.5) and using (2.2) as well as the bounds

$$|\chi(g)| \leqslant \gamma n, \quad |\chi(g^j)| \leqslant n, \ \forall j \geqslant 2,$$

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we obtain

$$\begin{aligned}\frac{\prod_{j=0}^{k-1}(\gamma n + j/\gamma)}{k!} &= \frac{\prod_{j=0}^{k-1}(\gamma^2 n + j)}{\gamma^k k!} = \frac{f(\gamma^2 n)}{\gamma^k} = \frac{1}{k!}\sum_{\pi\in\mathsf{S}_k}\gamma^{-k}\prod_{j=1}^{k}(\gamma^2 n)^{a_j(\pi)}\\ &= \frac{1}{k!}\sum_{\pi\in\mathsf{S}_k}\prod_{j=1}^{k}\left(\frac{\gamma^2 n}{\gamma^j}\right)^{a_j(\pi)} = \frac{1}{k!}\sum_{\pi\in\mathsf{S}_k}(\gamma n)^{a_1(\pi)}\prod_{j=2}^{k}(\gamma^{2-j}n)^{a_j(\pi)}\\ &\geqslant \frac{1}{k!}\sum_{\pi\in\mathsf{S}_k}(\gamma n)^{a_1(\pi)}\prod_{j=2}^{k}n^{a_j(\pi)} \geqslant \frac{1}{k!}\sum_{\pi\in\mathsf{S}_k}\prod_{j=1}^{k}|\chi(g^j)|^{a_j(\pi)}\\ &\geqslant |\mathrm{Sym}^k(\chi)(g)|.\end{aligned}$$

Next, assume that k is such that $n \geqslant 3k/\gamma^{3/2}$. Then for any $0 \leqslant j \leqslant k-1$ we have

$$\gamma n + j/\gamma \leqslant \gamma^{1/2}(n + j),$$

whence

$$\begin{aligned}|\mathrm{Sym}^k(\chi)(g)| &\leqslant \frac{\prod_{j=0}^{k-1}(\gamma n + j/\gamma)}{k!} \leqslant \gamma^{k/2}\frac{\prod_{j=0}^{k-1}(n+j)}{k!}\\ &= \gamma^{k/2}\,\mathrm{Sym}^k(\chi)(1).\end{aligned}$$ □

Note that

$$\left|\frac{1}{k!}\sum_{\pi\in\mathsf{S}_k}\lambda(\pi)\prod_{j=1}^{k}\chi(g^j)^{a_j(\pi)}\right| \leqslant \lambda(1)\cdot\frac{1}{k!}\sum_{\pi\in\mathsf{S}_k}\prod_{j=1}^{k}|\chi(g^j)|^{a_j(\pi)}.$$

Hence, the proof of Proposition 2.2 also yields the following corollary.

COROLLARY 2.3. *Let G be a finite group and $0 < \gamma \leqslant 1$ be such that $|\chi(x)|/\chi(1) \leqslant \gamma$ for some $\chi \in \mathrm{Irr}(G)$ of degree n and all $x \in G \smallsetminus \mathbf{Z}(G)$. Then for any $g \in G \smallsetminus \mathbf{Z}(G)$, $k \in \mathbb{N}$, and for all $\lambda \in \mathrm{Irr}(\mathsf{S}_k)$, we have*

$$|\mathrm{Sym}^\lambda(\chi)(g)| \leqslant \lambda(1)\cdot\frac{\prod_{j=0}^{k-1}(\gamma n + j/\gamma)}{k!}.$$

For a normal subgroup N of a finite group G, $\chi \in \mathrm{Irr}(G)$, and $\lambda \in \mathrm{Irr}(N)$, if λ is an irreducible constituent of the restriction χ_N, we will say that χ *lies above* λ and that λ *lies under* χ.

The next statement shows that the symmetrizations $\mathrm{Sym}^\lambda(\chi)$ are equidistributed in the following sense.

COROLLARY 2.4. *Let G be a finite group and $0 < \gamma \leqslant 1$ be such that $|\chi(x)|/\chi(1) \leqslant \gamma$ for some $\chi \in \mathrm{Irr}(G)$ and all $x \in G \smallsetminus \mathbf{Z}(G)$. Let χ be of degree*

$$\chi(1) = n \geqslant \max(k^2, 3k/\gamma^{3/2})$$

for some $k \in \mathbb{N}$, and let ν denote the character of $\mathbf{Z}(G)$ lying under χ. Also, assume that

$$|G/\mathbf{Z}(G)| \leqslant (2e^{-1/2} - 1)\gamma^{-k/2}.$$

Then for any $\lambda \in \mathrm{Irr}(\mathsf{S}_k)$ and any $\rho \in \mathrm{Irr}(G)$ lying above ν^k, ρ is an irreducible constituent of $\mathrm{Sym}^\lambda(\chi)$.

Proof. The assumption $n \geqslant k^2$ implies that

$$\frac{n^k}{k!} \leqslant \mathrm{Sym}^k(\chi)(1) = \frac{n^k}{k!}\prod_{j=1}^{k-1}(1 + j/n) \leqslant \frac{n^k}{k!} e^{\sum_{j=1}^{k-1} j/n} e^{1/2}\frac{n^k}{k!}. \tag{2.6}$$

On the other hand, Corollary 2.3 and the proof of Proposition 2.2 show under the assumption $n \geqslant 3k/\gamma^{3/2}$ that

$$|\mathrm{Sym}^\lambda(g)| \leqslant \lambda(1) \cdot \frac{\prod_{j=0}^{k-1}(\gamma n + j/\gamma)}{k!} \leqslant \lambda(1)\gamma^{k/2}\,\mathrm{Sym}^k(\chi)(1)$$

for all $g \in G \setminus \mathbf{Z}(G)$. It follows from (2.6) that

$$|\mathrm{Sym}^\lambda(g)| \leqslant \lambda(1) \cdot e^{1/2}\gamma^{k/2}n^k/k!.$$

Also, using (2.1) and (2.6) we have

$$\begin{aligned}\mathrm{Sym}^\lambda(\chi)(1) &= \frac{1}{k!}\sum_{\pi \in \mathsf{S}_k}\lambda(\pi)\prod_{j=1}^{k} n^{a_j(\pi)} \geqslant \frac{1}{k!}\left(\lambda(1)n^k - \lambda(1)\sum_{1 \neq \pi \in \mathsf{S}_k}\prod_{j=1}^{k} n^{a_j(\pi)}\right) \\ &= \frac{\lambda(1)}{k!}\left(2n^k - \sum_{\pi \in \mathsf{S}_k}\prod_{j=1}^{k} n^{a_j(\pi)}\right) = \lambda(1)\left(\frac{2n^k}{k!} - \mathrm{Sym}^k(\chi)(1)\right) \\ &> (2 - e^{1/2})\lambda(1)\frac{n^k}{k!}.\end{aligned}$$

Thus we have shown that

$$|\mathrm{Sym}^\lambda(\chi)(g)| \leqslant \frac{e^{1/2}}{2 - e^{1/2}}\gamma^{k/2}\,\mathrm{Sym}^\lambda(\chi)(1)$$

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for all $g \in G \smallsetminus \mathbf{Z}(G)$. Clearly, $\chi(z) = \nu(z)\chi(1)$ and $\mathrm{Sym}^{\lambda}(\chi)(z) = \nu^k(z)\,\mathrm{Sym}^{\lambda}(\chi)(1)$ for all $z \in \mathbf{Z}(G)$. Now we can estimate the scalar product $[\mathrm{Sym}^{\lambda}(\chi), \rho]_G$ from below:

$$
\begin{aligned}
&|G| \cdot [\mathrm{Sym}^{\lambda}(\chi), \rho]_G \\
&\quad = \left| \sum_{z \in \mathbf{Z}(G)} \mathrm{Sym}^{\lambda}(\chi)(z)\bar{\rho}(z) + \sum_{g \in G \smallsetminus \mathbf{Z}(G)} \mathrm{Sym}^{\lambda}(\chi)(g)\bar{\rho}(g) \right| \\
&\quad \geqslant |\mathbf{Z}(G)| \cdot \mathrm{Sym}^{\lambda}(\chi)(1)\rho(1) - |G \smallsetminus \mathbf{Z}(G)| \cdot \frac{e^{1/2}}{2-e^{1/2}} \gamma^{k/2}\,\mathrm{Sym}^{\lambda}(\chi)(1)\rho(1) \\
&\quad > |\mathbf{Z}(G)| \cdot \mathrm{Sym}^{\lambda}(\chi)(1)\rho(1) \cdot \left(1 - |G/\mathbf{Z}(G)| \cdot \frac{e^{1/2}}{2-e^{1/2}} \gamma^{k/2}\right) \geqslant 0,
\end{aligned}
$$

whence ρ is an irreducible constituent of $\mathrm{Sym}^k(\chi)$. □

Whereas the bound (2.4) applies to many finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ (particularly the ones 'close' to be quasisimple), it does not apply to the groups with a normal irreducible extraspecial subgroup of order p^3 (cf. Proposition 3.7 below). To handle the latter, we will need the following consequence of Proposition 2.2.

PROPOSITION 2.5. *Let G be a finite group, and let $\chi \in \mathrm{Irr}(G)$ of degree $n \geqslant 80$ be such that $|\chi(x)| \leqslant \chi(1)/\sqrt{2}$ for all $x \in G \smallsetminus \mathbf{Z}(G)$. Then for any $g \in G \smallsetminus \mathbf{Z}(G)$ we have*

$$
\frac{|\mathrm{Sym}^n(\chi)(g)|}{\mathrm{Sym}^n(\chi)(1)} \leqslant \frac{8}{7} \cdot \left(\frac{27}{32}\right)^{n/2}.
$$

Proof. (i) According to [AS, 6.1.38], for any $n \in \mathbb{N}$ we have

$$
n! = \sqrt{2\pi n} \cdot n^n e^{-n+\theta/12n}, \tag{2.7}
$$

where $0 < \theta < 1$. This version of Stirling's formula implies that

$$
\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! \leqslant e\sqrt{n}\left(\frac{n}{e}\right)^n. \tag{2.8}
$$

Indeed, the first inequality in (2.8) follows from (2.7). The second inequality in (2.8) is an equality for $n = 1$, and follows from (2.8) when $n \geqslant 2$, since

$$
\sqrt{2\pi}\, e^{1/24} < e.
$$

(ii) Note that

$$
\mathrm{Sym}^n(\chi)(1) = A/2, \tag{2.9}
$$

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where

$$A := \frac{1}{n!}\prod_{j=1}^{n}(n+j) = \frac{(2n)!}{(n!)^2}.$$

Applying (2.8), we obtain

$$\frac{2^{2n}}{\sqrt{n}}\cdot\frac{e}{\pi\sqrt{2}} \geqslant A \geqslant \frac{2^{2n}}{\sqrt{n}}\cdot\frac{2\sqrt{\pi}}{e^2}. \tag{2.10}$$

Next we choose $\gamma = 1/\sqrt{2}$ and apply (the first statement of) Proposition 2.2 to get

$$|\mathrm{Sym}^n(\chi)(g)| \leqslant \frac{B}{3\gamma^n}, \tag{2.11}$$

where

$$B := \frac{1}{n!}\prod_{j=1}^{n}(\gamma^2 n + j).$$

First suppose that $n = 2m$ is even. Then

$$B = \frac{1}{(2m)!}\prod_{j=1}^{2m}(m+j) = \frac{(3m)!}{m!(2m)!} \leqslant \frac{3^{3n/2}}{2^n\sqrt{n}}\cdot\frac{e\sqrt{3}}{2\pi} \tag{2.12}$$

by an application of (2.8).

If $n = 2m+1$ is odd, then

$$B = \frac{1}{2^n n!}\cdot\frac{(2m+2)(2m+3)\dots(6m+3)}{(2m+2)(2m+4)\dots(6m+2)} = \frac{(3n)!m!}{2^{2n}(n!)^2(3m+1)!}.$$

Using (2.8) to bound $(3n)!$ and $m!$ from the above, and $n!$ and $(3m+1)!$ from below, we get

$$B \leqslant \frac{3^{3n}}{2^{2n}}\cdot\frac{e^2\sqrt{3m}}{2\pi\sqrt{2\pi n(3m+1)}}\cdot C, \tag{2.13}$$

where

$$C := \frac{n^n m^m}{(3m+1)^{3m+1}} = \left(\frac{2m+1}{3m+1}\right)^{2m+1}\cdot\left(\frac{m}{3m+1}\right)^m = \frac{2^n}{3^{n+m}}\cdot xy. \tag{2.14}$$

Here,

$$x := \left(\frac{6m+3}{6m+2}\right)^{2m+1}, \quad y := \left(\frac{3m}{3m+1}\right)^m.$$

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Now,

$$e < x^3 = \left(\frac{6m+3}{6m+2}\right)^{6m+3} = \left(1 + \frac{1}{6m+2}\right)^{6m+2} \cdot \frac{6m+3}{6m+2} < e \cdot \frac{6m+3}{6m+2}.$$

Similarly,

$$e > 1/y^3 = \left(\frac{3m+1}{3m}\right)^{3m} = \left(1 + \frac{1}{3m}\right)^{3m+1} \cdot \frac{3m}{3m+1} > e \cdot \frac{3m}{3m+1}.$$

It follows that

$$1 < x^3 y^3 < \frac{6m+3}{6m+2} \cdot \frac{3m+1}{3m} = \frac{n}{n-1}$$

and so $xy < n/(n-1)$. Putting this bound in (2.14), we see that (2.13) yields

$$B \leqslant \frac{3^{3n/2}}{2^n \sqrt{n}} \cdot \frac{3e^2}{\sqrt{8\pi^3}} \cdot \sqrt{\frac{n^2}{(3n-1)(n-1)}}. \tag{2.15}$$

Note that the upper bound in (2.15) is larger than the one in (2.12). So for all $n \in \mathbb{N}$ we can use (2.15) to bound B from the above. Putting (2.9), (2.11), (2.10), and (2.15) together, we obtain for $n \geqslant 80$ that

$$\frac{|\mathrm{Sym}^n(\chi)(g)|}{\mathrm{Sym}^n(\chi)(1)} \leqslant \frac{2B}{3A\gamma^n} \leqslant \frac{3^{3n/2}}{2^{5n/2}} \cdot \frac{e^4}{\pi^2\sqrt{8}} \cdot \sqrt{\frac{n^2}{(n-1)(3n-1)}} \leqslant \frac{8}{7} \cdot \left(\frac{27}{32}\right)^{n/2},$$

and the statement follows. □

In the same fashion, one can also get an analogue of Proposition 2.5 for any γ between 0 and 1 (instead of $\gamma = 1/\sqrt{2}$), but we will not need it in the paper.

Now we can present a criterion for the existence of semi-invariants of certain degrees.

PROPOSITION 2.6. *Let G be a finite group, $\beta > 0$ a constant, $k \in \mathbb{N}$, and let $\chi \in \mathrm{Irr}(G)$ be such that*

(i) $|\mathrm{Sym}^k(\chi)(g)| \leqslant \beta \cdot \mathrm{Sym}^k(\chi)(1)$ *for all* $g \in G \smallsetminus \mathbf{Z}(G)$;

(ii) $|G/\mathbf{Z}(G)| \leqslant 1/\beta$; *and*

(iii) $|\mathbf{Z}(G) \cap G'|$ *divides* k.

Then $\mathrm{Sym}^k(\chi)$ *contains a constituent of degree* 1.

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Proof. Let $\lambda \in \mathrm{Irr}(\mathbf{Z}(G))$ be such that $\chi|_{\mathbf{Z}(G)} = \chi(1)\lambda$. The condition (iii) implies that λ^k is trivial on $\mathbf{Z}(G) \cap G'$, and so it can be viewed as a (linear) character of

$$X := \mathbf{Z}(G)G'/G' \cong \mathbf{Z}(G)/(\mathbf{Z}(G) \cap G').$$

As X is a subgroup of the abelian group G/G', it follows that we can find a linear character μ of G such that $\mu|_{\mathbf{Z}(G)} = \lambda^k$. Using the conditions (i) and (ii), we now obtain

$$\begin{aligned} |G| \cdot [\mathrm{Sym}^k(\chi), \mu] &= \left| \sum_{g \in \mathbf{Z}(G)} \mathrm{Sym}^k(\chi)(g)\bar{\mu}(g) + \sum_{g \in G \smallsetminus \mathbf{Z}(G)} \mathrm{Sym}^k(\chi)(g)\bar{\mu}(g) \right| \\ &\geqslant \mathrm{Sym}^k(\chi)(1) \cdot |\mathbf{Z}(G)| - \beta \cdot \mathrm{Sym}^k(\chi)(1) \cdot |G \smallsetminus \mathbf{Z}(G)| \\ &> \mathrm{Sym}^k(\chi)(1) \cdot |\mathbf{Z}(G)| \cdot (1 - \beta \cdot |G/\mathbf{Z}(G)|) \geqslant 0, \end{aligned}$$

that is, μ is a linear constituent of $\mathrm{Sym}^k(\chi)$. □

The following statement, pointed out to the author by N. Katz, is convenient in many situations.

LEMMA 2.7. *Let $G \leqslant \mathrm{GL}(V)$ be a finite group. Suppose that $\mathrm{Sym}^k(V)$ contains a 1-dimensional G-submodule. Then $\mathrm{Sym}^{km}(V)$ contains a 1-dimensional G-submodule for all $m \in \mathbb{N}$.*

Proof. Fixing a basis $(e_1, \ldots, e_n)$ of V, we can identify $\mathrm{Sym}^k(V)^* \cong \mathrm{Sym}^k(V^*)$ with the space of homogeneous polynomials of degree k in n variables $x_1, \ldots, x_n$. By assumption, G fixes $\langle f \rangle_{\mathbb{C}}$ for some nonzero $f \in \mathrm{Sym}^k(V^*)$. Hence G also fixes $\langle f^m \rangle_{\mathbb{C}}$, and $0 \neq f^m \in \mathrm{Sym}^{km}(V^*)$. □

We will also need the following consequence of [GT3, Corollary 2.14].

LEMMA 2.8. *Let G be a finite group with a perfect normal subgroup N. Let χ be a character of G and $0 < \beta \leqslant 1$ be a constant such that $|\chi(x)|/\chi(1) \leqslant \beta$ for all $x \in N \smallsetminus \mathbf{Z}(N)$. Then $|\chi(g)|/\chi(1) \leqslant (3+\beta)/4$ for all $g \in G \smallsetminus \mathbf{C}_G(N)$.*

Proof. Consider any $g \in G \smallsetminus \mathbf{C}_G(N)$. Note that there exists some $h \in N$ such that $[g, h] := ghg^{-1}h^{-1} \in N \smallsetminus \mathbf{Z}(N)$. (Indeed, otherwise we would have $[g, N] \subseteq \mathbf{Z}(N)$, and so

$$[g, N] = [g, [N, N]] \subseteq [[g, N], N] \cdot [[N, g], N] = 1,$$

contradicting the condition $g \notin \mathbf{C}_G(N)$.) Now $|\chi([g, h])| \leqslant \beta\chi(1)$, whence

$$4|\chi(g)| \leqslant 3\chi(1) + |\chi([g, h])| \leqslant (3+\beta)\chi(1)$$

by [GT3, Corollary 2.14(ii)]. □

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3. Base cases

LEMMA 3.1. *Let G be a finite group and let the $\mathbb{C}G$-module V be induced from a module W of a subgroup $H \leqslant G$. Suppose that $\mathrm{Sym}^k(W)$ contains a 1-dimensional H-submodule. Then $\mathrm{Sym}^{km}(V)$ contains a 1-dimensional G-submodule, where $m := [G : H]$.*

Proof. By hypothesis, V admits a G-invariant decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$, where $H = \mathrm{Stab}_G(V_1)$ and $V_1 \cong W$. Furthermore, $\mathrm{Sym}^k(V_1)$ contains a 1-dimensional H-submodule A_1. Write $G = \bigcup_{i=1}^m g_i H$ with $g_1 = 1$, and note that $A_i = g_i(A_1)$ is a 1-dimensional submodule for $g_i H g_i^{-1} = \mathrm{Stab}_G(V_i)$. Now, for any $g \in G$ and any $1 \leqslant i \leqslant n$, there are some $1 \leqslant j \leqslant n$ and some $h \in H$ such that $gg_i = g_j h$ and so $g(A_i) = g_j(A_1) = A_j$. Thus G permutes the m subspaces $A_1, \ldots, A_m$ and so acts on the 1-dimensional space $A_1 \otimes A_2 \otimes \cdots \otimes A_m$. Hence the statement follows by noting that $\mathrm{Sym}^{km}(V_1 \oplus V_2 \oplus \cdots \oplus V_m)$ contains

$$\mathrm{Sym}^k(V_1) \otimes \mathrm{Sym}^k(V_2) \otimes \cdots \otimes \mathrm{Sym}^k(V_m) \supseteq A_1 \otimes A_2 \otimes \cdots \otimes A_m,$$

see [**FH**, page 80]. $\square$

Recall that a finite group G is said to be *almost simple* if $S \lhd G \leqslant \mathrm{Aut}(S)$ for some nonabelian simple group S, in which case S is called the *socle of* G. Furthermore, G is said to be *almost quasisimple* if $S \lhd G/\mathbf{Z}(G) \leqslant \mathrm{Aut}(S)$ for some nonabelian simple group S, in which case $E(G) = G^{(\infty)}$ denotes the last term of the derived series of G. We also use the notation $\mathrm{Mult}(X)$ to denote the Schur multiplier $H^2(X, \mathbb{C}^\times)$ of a finite group X.

LEMMA 3.2. *Let G be a finite group. Then*

$$|\mathbf{Z}(G) \cap G'| \leqslant \mathrm{Mult}(H)$$

for $H := G/\mathbf{Z}(G)$. If in addition H is almost simple with socle S, then

$$|\mathrm{Mult}(H)| \leqslant |\mathrm{Mult}(S)| \cdot |\mathrm{Mult}(H/S)|.$$

Proof. Let $K \leqslant G$ be minimal subject to $KZ = G$, where $Z := \mathbf{Z}(G)$. Then

$$H = G/Z = KZ/Z \cong K/(K \cap Z),$$

and so $(K, K \cap Z)$ is a central extension of H. Next, if $L < K$ but $L(K \cap Z) = K$, then $LZ \geqslant KZ = G$, contradicting the minimality of K. Thus $L(K \cap Z) < K$ for all $L < K$, whence the central extension $(K, K \cap Z)$ of H is irreducible in the sense of [**Suz**, Ch. 2, (9.10)]. It then follows from [**Suz**, Ch. 2, (9.13)]

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that $|K' \cap Z| \leqslant |\mathrm{Mult}(H)$. Since $G' = (KZ)' = K'$, $G' \cap Z = K' \cap Z$, and so we arrive at the first statement.

For the second statement, observe that $H^1(S, \mathbb{C}^\times) = 0$ as S is perfect. Hence the sequence

$$0 \to H^2(H/S, \mathbb{C}^\times) \to H^2(H, \mathbb{C}^\times) \to H^2(S, \mathbb{C}^\times)$$

is exact, see [Suz, Ch. 2, (7.30)], whence we are done. □

LEMMA 3.3. *Let B be a normal subgroup of a finite group G.*

(i) *Assume G/B can be generated by s elements. Then*

$$|H^2(G, \mathbb{C}^\times)| \leqslant |H^2(B, \mathbb{C}^\times)| \cdot |H^2(G/B, \mathbb{C}^\times)| \cdot |B/B'|^s.$$

(ii) *If G/B is cyclic, then $|H^2(G, \mathbb{C}^\times)| \leqslant |H^2(B, \mathbb{C}^\times)| \cdot |B/B'|$.*

(iii) *If both B and G/B are cyclic, then $|H^2(G, \mathbb{C}^\times)| \leqslant |B/B'|$.*

Proof. By [Lyn, Theorem 4'], $H^2(G, \mathbb{C}^\times) = H^0 \geqslant H^1 \geqslant H^2$, where H^0/H^1 embeds in $H^0(G/B, H^2(B, \mathbb{C}^\times)) = H^2(B, \mathbb{C}^\times)^{G/B}$ and so has order at most $|H^2(B, \mathbb{C}^\times)|$. Next, H^2 is a factor group of $H^2(G/B, H^0(B, \mathbb{C}^\times)) = H^2(G/B, \mathbb{C}^\times)$ and so has order at most $|H^2(G/B, \mathbb{C}^\times)|$. Finally, H^1/H^2 is a subquotient of $H^1(G/B, H^1(B, \mathbb{C}^\times))$. Clearly,

$$X := H^1(B, \mathbb{C}^\times) = \mathrm{Hom}(B, \mathbb{C}^\times) = \mathrm{Hom}(B/B', \mathbb{C}^\times);$$

in particular, $|X| = |B/B'|$. Now, if G/B is generated by s elements, then

$$H^1(G/B, X) \leqslant |X|^s$$

(as each 1-cocycle on G/B is completely determined by its values at the s generators). Hence (i) follows. If G/B is cyclic, then we can take $s = 1$ and so (ii) follows, since $H^2(G/B, \mathbb{C}^\times) = 0$. If B is cyclic in addition, then $H^2(B, \mathbb{C}^\times) = 0$ and so (iii) follows. □

LEMMA 3.4. *Let $S \lhd H \leqslant \mathrm{Aut}(S)$ for some nonabelian finite simple group S. Then*

$$|\mathrm{Mult}(H/S)| \leqslant |\mathrm{Out}(S)|^2.$$

Proof. The statement is obvious if H/S is cyclic. If H/S is metacyclic, then we are done by Lemma 3.3(iii). It remains therefore to consider the case $O := \mathrm{Out}(S)$ is not metacyclic. According to [GLS, Theorem 2.5.12], in this

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case S is a simple group of Lie type over $\mathbb{F}_q$ with $q = p^f$, of type A_n, D_n, or E_6. If in addition S is not of type D_{2m}, then O has a metacyclic normal subgroup with cyclic quotient, whence H/S has a metacyclic normal subgroup B such that $(H/S)/B$ is cyclic. By Lemma 3.3(iii), $|H^2(B, \mathbb{C}^\times)| \leqslant |B|$, whence $|H^2(H/S, \mathbb{C}^\times)| \leqslant |B|^2 \leqslant |H/S|^2$ by Lemma 3.3(ii). The same argument applies if S is of type D_{2m} and $p = 2$.

In the final case, S is of type D_{2m} and $p > 2$. Let r be any prime, $R \in \mathrm{Syl}_r(H/S)$, and assume that r divides $|\mathrm{Mult}(H/S)|$. Then R cannot be cyclic, and so $r \leqslant 3$. If $r = 3$, then we must have that $m = 2$ and R is metacyclic; indeed, R is a subgroup of $C_f \rtimes C_3$. In this case,

$$|\mathrm{Mult}(H/S)|_3 \leqslant |\mathrm{Mult}(R)| \leqslant f_3$$

by Lemma 3.3(iii), where N_r denotes the r-part of any positive integer N for any prime r. Assume now that $r = 2$. Then R has a normal subgroup $B \leqslant C_2^2$, where R/B is metacyclic and so generated by two elements. Note that $|\mathrm{Mult}(B)| \leqslant 2$ and $|\mathrm{Mult}(R/B)| \leqslant f_2$ by Lemma 3.3(iii), since R/B has a cyclic normal subgroup of order at most f_2 with cyclic quotient. Also, $|B/B'| \leqslant 4$, whence

$$|\mathrm{Mult}(R)| \leqslant 2f_2 \cdot 4^2 = 32f_2$$

by Lemma 3.3(i). Thus

$$|\mathrm{Mult}(H/S)| \leqslant 32f_2f_3 \leqslant 32f < (8f)^2 \leqslant |\mathrm{Out}(S)|^2. \qquad \square$$

THEOREM 3.5 [**Th**]. *Let $G < \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup.*

(i) *Let p be any prime not dividing $|G|$. Then $\mathrm{Sym}^{(p-1)n}(V)$ contains a 1-dimensional G-submodule. The same conclusion holds if $p \nmid |G/\mathbf{Z}(G)|$ and the G-module V is irreducible.*

(ii) *Suppose that the G-module V is irreducible. If $p > 2n + 1$ is a prime, then $\mathrm{Sym}^{(p-1)n}(V)$ contains a 1-dimensional G-submodule. Furthermore, such p can be chosen to not exceed $4n + 1$.*

Proof. (i) The first statement is [**Th**, Lemma 2]. For the second statement, let $A := \mathbf{O}_p(\mathbf{Z}(G))$. By assumption, $p \nmid |G/A|$ and so $\gcd(|A|, |G/A|) = 1$. Hence, A has a complement B in G by the Schur–Zassenhaus theorem. In this case, $G = A \times B$, $p \nmid |B|$, and A acts on V as scalars. Now we can apply [**Th**, Lemma 2] to B.

(ii) Suppose first that the G-module V is primitive. Then the statement is established in the proof of [**Th**, Theorem 1]. In the general case, consider a

G-invariant decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_d$ with d largest possible, and let $H := \mathrm{Stab}_G(V_1)$. Then the H-module V_1 is irreducible and primitive. By the previous case, if $p > 2n/d + 1$ is a prime (and such prime can be chosen in the interval $[2n+2, 4n+1]$ by Bertrand's postulate), then $\mathrm{Sym}^{(p-1)n/d}(V)$ contains a 1-dimensional H-submodule. By Lemma 3.1, $\mathrm{Sym}^{(p-1)n}(V)$ contains a 1-dimensional G-submodule. □

EXAMPLE 3.6. (i) For any $n \in \mathbb{N}$, Thompson [**Th**] gave an example of a nilpotent group $G \in \mathrm{GL}_n(\mathbb{C})$ of order n^3 with $d(G) = n$.

(ii) Consider $G < \mathrm{GL}(V) = \mathrm{GL}_2(\mathbb{C})$ with $G \cong \mathrm{SL}_2(5)$. Then a computation with [**GAP**] shows that $d(G) = 12$. Thus the bound $(p-1)n$ in Theorem 3.5(i) is best possible: just take $(n, p) = (2, 7)$.

(iii) Let $G < \mathrm{GL}(V) = \mathrm{GL}_3(\mathbb{C})$ with $G \cong 3\mathsf{A}_6$, or $G < \mathrm{GL}(V) = \mathrm{GL}_6(\mathbb{C})$ with $G \cong 6\mathsf{A}_7, 2J_2$. Then using [**GAP**] one can show that $d(G) = 2\dim(V)$.

PROPOSITION 3.7. *Let $G < \mathcal{G} = \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ be a finite irreducible subgroup with $n \geqslant 8300$. Suppose that $n = p^m$ for some prime p, $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$ for some finite subgroup $H < \mathcal{G}$, $P \lhd H$ a p-group, and $P = \mathbf{Z}(P)E$ for some extraspecial p-group E of order p^{1+2m}. Then $\mathrm{Sym}^n(V)$ contains a 1-dimensional G-submodule.*

Proof. Since $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$, it suffices to show that $\mathrm{Sym}^n(V)$ contains a 1-dimensional H-submodule. So we may replace G by H and assume that $P \lhd G \leqslant \mathbf{N}_{\mathcal{G}}(P)$. Note that $|\mathbf{Z}(G) \cap G'|$ divides $n = \dim V$ by Schur's lemma, and that $V|_P$ is irreducible. Consider any element $g \in G \setminus \mathbf{Z}(G)$. Then $g \notin \mathbf{C}_G(P) = \mathbf{Z}(G)$ again by Schur's lemma, and so $\mathbf{C}_P(g) < P$. It follows that either $\mathbf{C}_{P/\mathbf{Z}(P)}(g) < P/\mathbf{Z}(P) \cong E/\mathbf{Z}(E)$, or $\mathbf{C}_{P/\mathbf{Z}(P)}(g) = P/\mathbf{Z}(P)$ but g does not act trivially on the complete inverse image P of $P/\mathbf{Z}(P)$ in P. Applying [**GT1**, Lemma 2.4], we get

$$|\chi(g)| \leqslant p^{m-1/2} \leqslant \chi(1)/\sqrt{2}, \tag{3.1}$$

where χ is the G-character afforded by V. Hence, by Proposition 2.5 we have

$$|\mathrm{Sym}^n(\chi)(g)| \leqslant \beta \cdot \mathrm{Sym}^n(\chi)(1),$$

with $\beta := (8/7) \cdot (27/32)^{n/2}$. Observe that

$$|G/\mathbf{Z}(G)| \leqslant |P/\mathbf{Z}(P)| \cdot |\mathrm{Out}(P)| \leqslant p^{2m} \cdot |Sp_{2m}(p)| < p^{2m^2+3m}.$$

To complete the proof, it therefore suffices by Proposition 2.6 to show that

$$\delta^{p^m/2} > p^{2m^2+3m+1}$$

for $\delta := 32/27$. Note that for $n = p^m \geqslant 8300$ we have

$$\begin{aligned} n > 48.96(\log_2 n)^2 &= 48.96m^2(\log_2 p)^2 \geqslant 48.96m^2 \cdot \log_2 p \\ &> 12m^2 \cdot \log_\delta p \geqslant 2(2m^2+3m+1)\log_\delta p, \end{aligned}$$

since $\log_\delta 2 < 4.08$. It follows that

$$p^m/2 > \log_\delta p^{2m^2+3m+1},$$

as required. □

COROLLARY 3.8. *Let $p > 2$ be any prime, $V = \mathbb{C}^p$, and let $G < \mathrm{SL}(V)$ be isomorphic to an extraspecial p-group of order p^3. Then*

(i) *V/G is weakly exceptional but not exceptional.*

(ii) $\alpha_G(\mathbb{P}^{p-1}) = 1$.

Proof. The weak exceptionality of V/G is [CS4, Theorem 1.15]. Let χ be the character of G afforded by V and $g \in G \setminus \mathbf{Z}(G)$. Then $|\chi(g)/\chi(1)| \leqslant 1/\sqrt{p} < 2/3$ by (3.1). On the other hand, if g were a complex reflection, then we would have that

$$\frac{|\chi(g)|}{\chi(1)} \geqslant \frac{\chi(1)-1}{\chi(1)} \geqslant 1 - 1/p \geqslant 2/3.$$

Also, any $z \in \mathbf{Z}(G)$ cannot be a complex reflection by irreducibility of V. Thus G contains no complex reflection. As $\mathbb{C}^p/G$ is weakly exceptional, [CS2, Theorem 3.16] now implies that $\alpha_G(\mathbb{P}^{p-1}) \geqslant 1$.

Next, observe that the G-module V can be induced from a 1-dimensional module of a subgroup of index p. It follows by Corollary 3.1 that $d(G) \leqslant p$. But $d(G) \geqslant p$ as $|\mathbf{Z}(G) \cap G'| = p$. Thus $d(G) = p$ and so $\alpha_G(\mathbb{P}^{p-1}) \leqslant 1$ by (1.1). Consequently, $\alpha_G(\mathbb{P}^{p-1}) = 1$. The equality $d(G) = p$ also implies by [CS2, Theorem 1.17] that V/G is not exceptional. □

PROPOSITION 3.9. *Let $G < \mathcal{G} = \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ be a finite group with a normal subgroup $L \cong \mathsf{A}_m$ or $2\mathsf{A}_m$. Suppose that $V|_L$ is irreducible and $m \geqslant 81$. Then $\mathrm{Sym}^k(V)$ contains a 1-dimensional G-submodule for $k := 2\lceil n/8 \rceil$.*

Proof. Let χ be the character of G afforded by V. First we consider the case where $L \cong \mathsf{A}_m$ and χ_L is extendible to, say, a character ρ of $\mathrm{Aut}(L) \cong \mathsf{S}_m$.

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It is well known, see for example [JK, Theorem 2.1.11], that ρ is afforded by a $\mathbb{Q}\mathsf{S}_m$-module W, and so W supports a nondegenerate S_m-invariant symmetric bilinear form. The same is certainly true for the L-module W_L. It follows that $[\mathrm{Sym}^2(\chi_L), 1_L] = 1$. Thus the subspace U of L-fixed points on $\mathrm{Sym}^2(V)$ is 1-dimensional, and U is certainly stabilized by $G \rhd L$. Hence we are done by Lemma 2.7.

Next we consider the case where $L \cong \mathsf{A}_m$ but χ_L is not S_m-invariant. It follows that the (unique) irreducible character ρ of S_m lying above χ_L is labeled by a self-conjugate partition $\lambda = (\lambda_1, \dots, \lambda_t) \vdash m$. In this case, $2\lambda_1 - 1 \leqslant m$, and so $\lambda_1 \leqslant (m+1)/2$. It follows by [GLT, Theorem 5.1] that

$$n = \chi(1) = \rho(1)/2 \geqslant 2^{(m-5)/4}. \tag{3.2}$$

Next, we observe that $\mathbf{C}_G(L) = \mathbf{Z}(G)$ by Schur's lemma, and $G/\mathbf{C}_G(L) \not\geqslant \mathsf{S}_m$ as χ_L is not S_m-invariant. Hence, $G = \mathbf{Z}(G) \times L$ and $\mathbf{Z}(G) \cap G' = 1$. Applying [GM, Theorem 1.6], we see that

$$|\chi(g)|/\chi(1) \leqslant \gamma := 1 - 1/2m \tag{3.3}$$

for all $g \in G \smallsetminus \mathbf{Z}(G)$. Choosing k as specified in the proposition, we also have that $k \geqslant n/4$ and $n > 3k/\gamma^{1.5}$. Hence, by Propositions 2.2 and 2.6, it suffices to prove the first inequality in the following chain

$$m! < \gamma^{-2^{(m-17)/4}} \leqslant \gamma^{-n/8} \leqslant \gamma^{-k/2}.$$

Note that $1/\gamma > e^{1/2m}$ and $m! < m^m = e^{m \log m}$. Hence, we are reduced to proving

$$2^{(m-21)/4} > m^2 \log m,$$

which indeed holds for all $m \geqslant 81$.

Finally, we consider the case $L \cong 2\mathsf{A}_m$. As $m \geqslant 81$ and V_L is irreducible and faithful, we see that (3.2) holds in this case. Furthermore, by [GT3, Lemma 2.23],

$$|\chi(x)|/\chi(1) \leqslant 7/8$$

for all $x \in L \smallsetminus \mathbf{Z}(L)$. Since $\mathbf{C}_G(L) = \mathbf{Z}(G)$ by Schur's lemma, Lemma 2.8 implies that

$$|\chi(g)|/\chi(1) \leqslant 31/32$$

for all $g \in G \smallsetminus \mathbf{Z}(G)$. As $m \geqslant 81$, it follows that (3.3) holds in this case as well. Now we can finish as in the previous case. $\square$

Now we can prove the main result of this section.

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THEOREM 3.10. *There is a constant D such that the following statement holds for any $n \in \mathbb{N}$ and any finite irreducible subgroup $G < \mathcal{G} := \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$. Suppose that the G-module V is irreducible, primitive, tensor indecomposable, and not tensor induced. Then one of the following holds.*

(i) *If $p \geqslant D$ is any prime then $\mathrm{Sym}^{(p-1)n}(V)$ contains a 1-dimensional G-submodule.*

(ii) *G is almost quasisimple and $V|_{E(G)}$ is irreducible. Furthermore, there is some $k \in \mathbb{N}$ depending on G such that either $k = 2\lceil n/8 \rceil$ or $k \leqslant n(10/31 + 1/174)$, and $\mathrm{Sym}^k(V)$ contains a 1-dimensional G-submodule.*

(iii) *$n = p^m$ for some prime p, $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$, H is finite, $P \lhd H \leqslant \mathbf{N}_{\mathcal{G}}(P)$, P a p-group, and $P = \mathbf{Z}(P)E$ for some extraspecial p-group E of order p^{1+2m}. Furthermore, $\mathrm{Sym}^n(V)$ contains a 1-dimensional G-submodule.*

In fact, D can be taken to be 592000.

Proof. (a) We will choose $D = 592000$ in this proof. Let $G < \mathrm{GL}(V)$ satisfy the hypothesis of the theorem. If all prime divisors of $|G/\mathbf{Z}(G)|$ do not exceed D, then conclusion (i) holds for G by Theorem 3.5(i). So in what follows we will assume that

($\star$): Some prime divisor of $|G/\mathbf{Z}(G)|$ exceeds D.

(b) Next, we apply [GT2, Proposition 2.8] (which is a simplified version of Aschbacher's theorem [A]) to G and let H be the finite subgroup of $\mathcal{G}$ obtained by this statement. In particular, $\mathbf{Z}(\mathcal{G})G = \mathbf{Z}(\mathcal{G})H$, and H is irreducible as so is G. It follows by Schur's lemma that $G \cap \mathbf{Z}(\mathcal{G}) = \mathbf{Z}(G)$ and $H \cap \mathbf{Z}(\mathcal{G}) = \mathbf{Z}(H)$. Hence,

$$G/\mathbf{Z}(G) \cong \mathbf{Z}(\mathcal{G})G/\mathbf{Z}(\mathcal{G}) = \mathbf{Z}(\mathcal{G})H/\mathbf{Z}(\mathcal{G}) \cong H/\mathbf{Z}(H).$$

If, furthermore, case (iii) of [GT2, Proposition 2.8] holds for H, then we are in case (iii) of the theorem with $p^m > D$ by virtue of ($\star$), whence we are done by Proposition 3.7. We may now assume that case (ii) of [GT2, Proposition 2.8] holds for H.

Clearly, $\mathrm{Sym}^k(V)$ contains a 1-dimensional G-module precisely when it does as an H-module, and $E(G) = E(H)$. So without loss we may replace G by H. Thus we are in case (ii) of the theorem, that is, G is almost quasisimple. Now, $L = E(G)$ is quasisimple, and $V|_L$ is irreducible by [GT2, Lemma 2.5]. Let S denote the simple quotient $L/\mathbf{Z}(L) \lhd G/\mathbf{Z}(G)$. Then the condition ($\star$) implies that S cannot be any of the 26 sporadic simple groups. If $L \cong \mathsf{A}_m$ or $2\mathsf{A}_m$, then again $m \geqslant D$ by virtue of ($\star$), and so we are done by Proposition 3.9.

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We have shown that L is a quasisimple group of Lie type, say in characteristic p. It follows by [GM, Theorem 2.4] that

$$|\chi(x)|/\chi(1) \leqslant 19/20$$

for all $x \in L \smallsetminus \mathbf{Z}(L)$, if χ denotes the character of G afforded by V. As $\mathbf{C}_G(L) = \mathbf{Z}(G)$ by Schur's lemma, we see by Lemma 2.8 that

$$|\chi(g)|/\chi(1) < \gamma := 0.99 \tag{3.4}$$

for all $g \in G \smallsetminus \mathbf{Z}(G)$. Furthermore, the condition ($\star$) implies that L is *generic*, in the broad sense that, first, $|\mathrm{Mult}(S)|$ is nonexceptional and so described in [KL, Theorem 5.1.4], and, second, the Landazuri–Seitz–Zalesskii bound $\mathfrak{d}(S)$, as given in the middle column of [KL, Table 5.3.A], applies to L. (Also see [T] for more recent improvements on the Landazuri–Seitz–Zalesskii bound.) As $V|_L$ is irreducible, we have

$$n \geqslant \mathfrak{d}(S). \tag{3.5}$$

We will assume that $|\mathbf{Z}(G) \cap G'| \leqslant \mathfrak{e}$ for some suitably chosen $\mathfrak{e} \in \mathbb{N}$. Lemmas 3.2 and 3.4 yield the universal choice $|\mathrm{Mult}(S)| \cdot |\mathrm{Out}(S)|^2$ for $\mathfrak{e}$, but in certain cases, depending on the structure of $\mathrm{Out}(S)$, which is described in [GLS, Theorem 2.5.12], we can make better choices for $\mathfrak{e}$. We will aim to show that

$$n \geqslant 174\mathfrak{e}. \tag{3.6}$$

If (3.6) holds, then we can choose an integer k,

$$\frac{10n}{31} \leqslant k \leqslant \frac{10n}{31} + \frac{n}{174} \tag{3.7}$$

such that $|\mathbf{Z}(G) \cap G'|$ divides k and $n > 3k/\gamma^{3/2}$, where γ is given in (3.4); in particular, all conclusions of Proposition 2.2 apply. At the same time, we will aim to show that

$$\frac{n}{430} \geqslant \log_2 |\mathrm{Aut}(S)|. \tag{3.8}$$

Claim that (3.8) implies $|G/\mathbf{Z}(G)| < \gamma^{-k/2}$. Indeed, $\bar{G} := G/\mathbf{Z}(G) \hookrightarrow \mathrm{Aut}(S)$ as G is almost quasisimple. Next, $k \geqslant 10n/31$ by (3.7), and $1/\gamma > 2^{1442/10^5}$ by (3.4). Hence,

$$\gamma^{-k/2} > 2^{(1442/10^5)\cdot(5/31)\cdot n} > 2^{n/430} \geqslant |\mathrm{Aut}(S)| \geqslant |\bar{G}|$$

as stated. Thus, all the hypotheses of Proposition 2.6 hold for $\beta := \gamma^{-k/2}$ and so $\mathrm{Sym}^k(V)$ contains a 1-dimensional G-submodule, provided that (3.6) and (3.8) hold for $n = \dim(V) = \chi(1)$.

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Now we proceed to establish (3.6) and (3.8). We will use the fact that

$$x > 1454(\log_2 x)^2 \quad \text{if } x \geqslant 526000, \quad \text{and} \quad x > 1610(\log_2 x)^2 \quad \text{if } x \geqslant 592000. \tag{3.9}$$

(Indeed, the function $f(x) := x/(\log_2 x)^2$ is strictly increasing on (e^2, ∞); furthermore, $f(526000) > 1454$ and $f(592000) > 1610$.) We will also assume that L is defined over $\mathbb{F}_q$ with $q = p^f$. Although the consideration of various series of finite groups of Lie type follows the same outline, we feel it would be best to give a treatment for each of these series.

(b) Consider the case $S = \mathrm{PSL}_2(q)$. Then $\mathrm{Out}(S)$ is abelian of order df with $d := \gcd(2, q-1)$; in fact, it has a cyclic subgroup B of order d such that $\mathrm{Out}(S)/B \cong C_f$. It follows by Lemma 3.3(iii) that $|\mathrm{Mult}(\bar{G}/S)| \leqslant d$, whence we can choose $\mathfrak{c} = d^2 \leqslant 4$. The condition $(\star)$ implies that $q \geqslant 592000 > 2^{19}$. Next, $n \geqslant \mathfrak{d}(S) = (q-1)/d \geqslant (q-1)/2$, and so (3.6) holds. Furthermore, $|\mathrm{Aut}(S)| = q(q^2-1)f < q^3 f < q^4$. Using (3.9) for $x := q$ we get

$$n \geqslant (q-1)/2 > 700(\log_2 q)^2 > (700 \cdot 19)\log_2 q > 430 \log_2 q^4,$$

and so (3.8) holds as well.

(c) Next, assume that $S = \mathrm{PSL}^{\epsilon}_m(q)$, where $m \geqslant 3$, and PSL^{ϵ} stands for PSL if $\epsilon = +$ and for PSU if $\epsilon = -$. Note that $\mathrm{Out}(S)$ has a metacyclic normal subgroup B such that $\mathrm{Out}(S)/B \cong C_f$. Here, B has a cyclic normal subgroup B_1 of order $d := \gcd(m, q - \epsilon 1)$ and $B/B_1 \cong C_2$. Certainly, $\bar{G}/S$ has a similar structure, with B replaced by $B \cap \bar{G}/S$. Now, Lemma 3.3(iii) applied to $B \cap \bar{G}/S$ yields $|\mathrm{Mult}(B \cap \bar{G}/S)| \leqslant |B|$. Applying Lemma 3.3(ii) to $\bar{G}/S$, we get $|\mathrm{Mult}(\bar{G}/S)| \leqslant |B|^2$. Hence by Lemma 3.2 we can choose $\mathfrak{c} = 4d^3$.

Recall by $(\star)$ that $|\mathrm{Aut}(S)|$ has a prime divisor r larger than 592000. Claim that

$$q^{m-1} > 526000 \tag{3.10}$$

in this case. Indeed, if r divides $|\mathrm{Out}(S)| = 2df$ then $r \leqslant q + 1 < q^{m-1}$. If $r \nmid |\mathrm{Out}(S)|$ and $\epsilon = -$, then $r \leqslant (q^m + 1)/(q + 1) < q^{m-1}$. Assume now that $r \nmid |\mathrm{Out}(S)|$ and $\epsilon = +$. Certainly, if $r | q$ or $r | (q^j - 1)$ for some $j \leqslant m - 1$, then $r \leqslant (q^{m-1} - 1)/(q - 1) < q^{m-1}$. In all these cases, we must have that $q^{m-1} > r > 592000$. It remains to consider the case where $r | (q^m - 1)$ but $r \nmid \prod_{j=1}^{m-1}(q^j - 1)$ and $q^{m-1} \leqslant 526000$. If m is not a prime, then we have $r \leqslant (q^m - 1)/(q^2 - 1) < q^{m-1}$, a contradiction. So we may assume m is a prime. If $q \geqslant 9$, then

$$r \leqslant \frac{q^m - 1}{q - 1} < \frac{q}{q - 1} \cdot q^{m-1} \leqslant \frac{9}{8} \cdot 526000 < 592000,$$

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again a contradiction. If $q = 2$, then $m \leqslant 20$, whence $m \leqslant 19$ and $r \leqslant 2^{19} - 1 < 592000$. A similar argument shows that $r < 592000$ if $3 \leqslant q \leqslant 8$. This contradiction completes the proof of (3.10).

Now we have that

$$n \geqslant \mathfrak{d}(S) \geqslant \frac{q^m - q}{q+1} = (1 - q^{-m}) \cdot \frac{q}{q+1} \cdot q^{m-1} > \frac{1.999}{3} q^{m-1}. \qquad (3.11)$$

In particular, $n > 350490$ by (3.10). Applying (3.9) and (3.10) to $x := q^{m-1}$ we get

$$n > \tfrac{1.999}{3} \cdot 1454 \cdot (m-1)^2 (\log_2 q)^2 \geqslant \tfrac{1.999}{3} \cdot 1454 \cdot \tfrac{4}{9} m^2 (\log_2 q)^2 > 430 m^2 \log_2 q.$$

Since $|\mathrm{Aut}(S)| < 2 f q^{m^2-1} \leqslant q^{m^2}$, (3.8) holds. Next, if $m \geqslant 9$, then $(m-4)/(m-1) \geqslant 5/8$, whence

$$n > \tfrac{1.999}{3} q^3 q^{m-4} \geqslant \tfrac{1.999}{3} q^3 \cdot (q^{m-1})^{5/8} > 2507 q^3.$$

As $\mathfrak{e} = 4d^3 \leqslant 4(q+1)^3 \leqslant 13.5 q^3$, (3.6) holds in this case. If $m \leqslant 7$, then

$$174\mathfrak{e} \leqslant 174 \cdot 4m^3 \leqslant 238728 < n.$$

If $m = 8$, then $q^7 > 526000$ implies that in fact $q \geqslant 7$, whence $n > 548750$ by (3.11), and so

$$174\mathfrak{e} \leqslant 174 \cdot 4m^3 \leqslant 356532 < n.$$

Thus we have verified (3.6) for all groups of type A.

(d) Now we consider the case $S = PSp_{2m}(q)$ with $m \geqslant 2$. It is easy to see that the condition ($\star$) now implies that $q^m \geqslant 592000$. Suppose first that q is odd. Then $n \geqslant \mathfrak{d}(S) \geqslant (q^m - 1)/2 > 295000$. Applying (3.9) to $x := q^m$ (and noting that $2m^2 + m + 1 \leqslant (11/4)m^2$), we get

$$\begin{aligned} n &> 804 m^2 (\log_2 q)^2 > (804 \cdot \log_2 3 \cdot 4/11)(2m^2 + m + 1) \log_2 q \\ &> 463(2m^2 + m + 1) \log_2 q. \end{aligned}$$

Next assume that $2|q$. Then $n \geqslant \mathfrak{d}(S) \geqslant q^m > 592000$. Applying (3.9) to $x := q^m$, we obtain

$$\begin{aligned} n &> 1610 m^2 (\log_2 q)^2 > (1608 \cdot 4/11)(2m^2 + m + 1) \log_2 q \\ &> 584(2m^2 + m + 1) \log_2 q. \end{aligned}$$

As $|\mathrm{Aut}(S)| \leqslant 2 f q^{2m^2+m} \leqslant q^{2m^2+m+1}$, (3.8) holds in both cases. Also, $\mathrm{Out}(S)$ has a normal subgroup B of order $\leqslant 2$ with $\mathrm{Out}(S)/B \cong C_f$. Hence by Lemma 3.3(iii) and Lemma 3.2 we can choose $\mathfrak{e} = 4$, and so (3.6) holds as well.

(e) Here we consider the orthogonal groups. Suppose first that $S = \Omega_{2m+1}(q)$ with q odd and $m \geqslant 3$. Again, the condition $(\star)$ implies that $q^m \geqslant 592000$. Also, $n \geqslant \mathfrak{d}(S) > q^m$, $|\mathrm{Aut}(S)| < q^{2m^2+m+1}$, and $\mathrm{Out}(S)$ has a similar structure as in the case of symplectic groups. So we can apply the same arguments as in (d).

Assume now that $S = P\Omega^{\epsilon}_{2m}(q)$ with $m \geqslant 4$, and note that $q^m \geqslant 592000$ by $(\star)$. Also, $|\mathrm{Aut}(S)| < 6fq^{m(2m-1)} < q^{2m^2}$, and $n \geqslant \mathfrak{d}(S) > q^{m+1}$. Applying (3.9) to $x := q^m$ we get $n > 1610m^2 \log_2 q$, and so (3.8) holds.

Suppose in addition that $2|q$ if $S = P\Omega^+_8(q)$. Then $|\mathrm{Out}(S)| \leqslant 8f$ and so we can take $\mathfrak{e} = 256f^2$ by Lemmas 3.2 and 3.4. Since $n \geqslant q^{m+1} \geqslant 1184000$, (3.6) holds if $f \leqslant 5$. On the other hand, if $f \geqslant 6$, then

$$174\mathfrak{e} = 44544f^2 < 2^{5f} \leqslant q^5 \leqslant n,$$

yielding (3.6) as well.

It remains to consider the case where $S = P\Omega^+_8(q)$ and $2 \nmid q$. Then $|\mathrm{Out}(S)| = 24f$ and so we can take $\mathfrak{e} = 2304f^2$ by Lemmas 3.2 and 3.4. Since $n \geqslant q^{m+1} \geqslant 1776000$, (3.6) holds if $f \leqslant 2$. On the other hand, if $f \geqslant 3$, then

$$174\mathfrak{e} = 400896f^2 < 3^{5f} \leqslant q^5 \leqslant n,$$

completing the proof of (3.6) for orthogonal groups.

(f) Finally, we handle the exceptional groups of Lie type, and we start with the smaller ones. Assume $S = {}^2B_2(q)$. The condition $(\star)$ implies that $q > 2^{19}$. Next, $n \geqslant (q-1)\sqrt{q/2}$, $|\mathrm{Aut}(S)| \leqslant fq^5 < q^6$, $\mathrm{Out}(S)$ is cyclic, and we can take $\mathfrak{e} = 1$. Hence, using (3.9) for $x := q$ we have

$$n > 500q > (1610 \cdot 500) \log_2 q > 430 \log_2 q^6,$$

yielding (3.6) and (3.8).

Assume $S = {}^2G_2(q)$. The condition $(\star)$ implies that $q > 3^{11}$. Next, $n \geqslant \mathfrak{d}(S) \geqslant q(q-1)$, $|\mathrm{Aut}(S)| \leqslant fq^7 < q^8$, $\mathrm{Out}(S)$ is cyclic, and we can take $\mathfrak{e} = 1$. Hence, using (3.9) for $x := q$ we have

$$n \geqslant 3^{11}q > (1610 \cdot 3^{11}) \log_2 q > 430 \log_2 q^8,$$

yielding (3.6) and (3.8).

Let $S = G_2(q)$. Then any prime divisor of $|\mathrm{Aut}(S)|$ is at most $q^2 + q + 1$, so $(\star)$ implies that $q \geqslant 769$. Next, $n \geqslant \mathfrak{d}(S) \geqslant q(q^2-1) > q^{5/2}$, $|\mathrm{Aut}(S)| < 2fq^{14} \leqslant q^{15}$, $\mathrm{Out}(S)$ is cyclic, and we can take $\mathfrak{e} = 1$. Hence, using (3.9) for $x := n$ we have

$$n > 1610(\log_2 n)^2 > 10^4(\log_2 q)^2 > 430 \log_2 q^{15},$$

yielding (3.6) and (3.8).

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Let $S = {}^3D_4(q)$. Then any prime divisor of $|\mathrm{Aut}(S)|$ is at most $q^4 - q^2 + 1$, so $(\star)$ implies that $q \geqslant 29$. Next, $n \geqslant \mathfrak{d}(S) \geqslant q^3(q^2-1) > q^4$, $|\mathrm{Aut}(S)| < 3fq^{28} < q^{30}$, $\mathrm{Out}(S)$ is cyclic, and we can take $\mathfrak{e} = 1$. Hence, using (3.9) for $x := n$ we have

$$n > 1610(\log_2 n)^2 > 25760(\log_2 q)^2 > 430 \log_2 q^{30},$$

yielding (3.6) and (3.8).

(g) For the remaining exceptional groups of Lie type, we prove the stronger result that (ii) holds even under the weaker assumption.

$(\star\star)$: Some prime divisor of $|G/\mathbf{Z}(G)|$ exceeds 73.

Suppose first $S = E_8(q)$. Then $n > \mathfrak{d}(S) > q^{28}$, $|\mathrm{Aut}(S)| < fq^{248} < q^{249}$, $\mathrm{Out}(S)$ is cyclic, and we can take $\mathfrak{e} = 1$. Hence, using (3.9) for $x := n$ we have

$$n > 1610(\log_2 n)^2 > (1610 \cdot 28^2)(\log_2 q)^2 > 430 \log_2 q^{249},$$

yielding both (3.6) and (3.8).

Next assume that $S = E_7(q)$. Then $n > \mathfrak{d}(S) > q^{16} \geqslant 2^{16}$, $|\mathrm{Aut}(S)| < fq^{133} < q^{134}$, $\mathrm{Out}(S)$ is metacyclic of order $\gcd(2, q-1)f$, and we can take $\mathfrak{e} = 4$. Hence,

$$n > 4096 \log_2 n > (4096 \cdot 16) \log_2 q > 430 \log_2 q^{134},$$

yielding (3.6) and (3.8).

Assume now that $S = E_6(q)$ or ${}^2E_6(q)$. The condition $(\star\star)$ implies that $q \geqslant 3$. Next, $n > \mathfrak{d}(S) > q^{10} \geqslant 3^{10}$, $|\mathrm{Aut}(S)| < 2fq^{78} \leqslant q^{79}$, and we can take $\mathfrak{e} = 108f^2$ by Lemmas 3.2 and 3.4. Hence,

$$n > 3725 \log_2 n > 37250 \log_2 q > 430 \log_2 q^{79},$$

yielding (3.8). Similarly,

$$n > 235(\log_2 n)^2 > 23500(\log_2 q)^2 \geqslant 23500f^2 > 174\mathfrak{e},$$

yielding (3.6).

Let $S = F_4(q)$. The condition $(\star\star)$ implies that $q \geqslant 4$. Next, $n > \mathfrak{d}(S) > q^6(q^2-1) \geqslant 5^7$ (note that $\mathfrak{d}(F_4(4)) > 5^7$), $|\mathrm{Aut}(S)| < 2fq^{52} \leqslant q^{53}$, and we can take $\mathfrak{e} = 1$ since $\mathrm{Out}(S)$ is cyclic. Hence,

$$n > 4806 \log_2 n > 33642 \log_2 q > 430 \log_2 q^{53},$$

yielding (3.6) and (3.8).

Finally, let $S = {}^2F_4(q)'$. The condition $(\star\star)$ implies that $q \geqslant 8$. Now, $n > \mathfrak{d}(S) > q^4(q-1)\sqrt{q/2} \geqslant 7 \cdot 2^{13}$, $|\mathrm{Aut}(S)| < fq^{26} < q^{27}$, and we can take $\mathfrak{e} = 1$ since $\mathrm{Out}(S)$ is cyclic. Hence,

$$n > 3627 \log_2 n > 18135 \log_2 q > 430 \log_2 q^{27},$$

yielding both (3.6) and (3.8). $\square$

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4. Proof of Main Theorem

First we begin with the tensor induced case.

PROPOSITION 4.1. *Let $G < \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ be a finite irreducible subgroup such that the G-module V is tensor induced: $V = V_1 \otimes V_2 \otimes \cdots \otimes V_m$ with $V_1, \ldots, V_m$ transitively permuted by G and $m \geqslant 2$. If $p > 2\dim(V_1) + 1$ is a prime, then $\mathrm{Sym}^k(V)$ contains a 1-dimensional G-submodule, with $k = (p-1)\dim(V_1)$. Furthermore, p can be chosen to not exceed $4\dim(V_1) + 1$, so that $k \leqslant 4n$.*

Proof. By hypothesis, G acts transitively on the set $\{V_1, V_2, \ldots, V_m\}$. Let $H := \mathrm{Stab}_G(V_1)$, and write $G = \bigcup_{i=1}^m g_i H$ with $g_1 = 1$.

(i) Suppose that $\mathrm{Sym}^a(V_1)$ contains a t-dimensional H-submodule A_1, for some $a, t \in \mathbb{N}$. Then $A_i = g_i(A_1)$ is a t-dimensional submodule for $g_i H g_i^{-1} = \mathrm{Stab}_G(V_i)$. Now, for any $g \in G$ and any $1 \leqslant i \leqslant n$, there are some $1 \leqslant j \leqslant n$ and some $h \in H$ such that $gg_i = g_j h$ and so $g(A_i) = g_j(A_1) = A_j$. Thus G permutes the m subspaces $A_1, \ldots, A_m$ and so acts on $A_1 \otimes A_2 \otimes \cdots \otimes A_m$.

(ii) Applying the observation in (i) with $a = 1$, we conclude by irreducibility of V that H is irreducible on V_1. By Theorem 3.5(ii), if $p > 2\dim(V_1) + 1$ is a prime then $\mathrm{Sym}^k(V_1)$ contains a 1-dimensional H-submodule A_1, with $k = (p-1)\dim(V_1)$. Again applying the observation in (i) with $(a, t) = (k, 1)$, we see that G acts on the 1-dimensional subspace $A_1 \otimes A_2 \otimes \cdots \otimes A_m$. Hence the statement follows by noting that $\mathrm{Sym}^k(V_1 \otimes V_2 \otimes \cdots \otimes V_m)$ contains

$$\mathrm{Sym}^k(V_1) \otimes \mathrm{Sym}^k(V_2) \otimes \cdots \otimes \mathrm{Sym}^k(V_m) \supseteq A_1 \otimes A_2 \otimes \cdots \otimes A_m,$$

see [**FH**, page 80]. □

PROPOSITION 4.2. *Let $G < \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ be a finite irreducible subgroup such that the G-module V is primitive, but tensor decomposable. Let $p \geqslant D$ be any prime, where D is the constant in Theorem 3.10. Then $\mathrm{Sym}^k(V)$ contains a 1-dimensional G-submodule for some $k \leqslant 2(p-1)n$.*

Proof. We decompose $V = A \otimes B \otimes C$, where the G-modules A, B, and C have the following properties. First,

$$A = A_1 \otimes A_2 \otimes \cdots \otimes A_l,$$

where each A_i is a tensor induced G-module. Next,

$$B = B_1 \otimes B_2 \otimes \cdots \otimes B_m, \quad C = C_1 \otimes C_2 \otimes \cdots \otimes C_q,$$

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where each B_i, respectively C_i, is a tensor indecomposable, not tensor induced, G-module. Furthermore, each B_i satisfies conclusion (i) of Theorem 3.10. Next, if K_i is the kernel of G acting on C_i, then the G/K_i-module C_i satisfies conclusion (ii) or (iii) of Theorem 3.10. Note that the irreducibility and primitivity of V imply that all A_i, B_i, and C_i are irreducible and primitive.

First we deal with A. Recall that $A_i = A_{i1} \otimes A_{i2} \otimes \cdots \otimes A_{il_i}$ is tensor induced, with $l_i \geqslant 2$. Let a_i be the common dimension of the A_{ij}, $1 \leqslant j \leqslant l_i$, and let

$$a := \max\{a_1, a_2, \ldots, a_l\}.$$

Then, by Proposition 4.1, there is a prime $r \in [2a + 2, 4a + 1]$ such that $\mathrm{Sym}^{(r-1)a_i}(A_i)$ contains a 1-dimensional G-module. By Lemma 2.7, $\mathrm{Sym}^{\mathfrak{l}}(A_i)$ contains a 1-dimensional G-module for

$$\mathfrak{l} := (r-1)\prod_{i=1}^{l} a_i \leqslant 4a \prod_{i=1}^{l} a_i \leqslant 4\dim(A).$$

As $\mathrm{Sym}^{\mathfrak{l}}(A)$ contains $\bigotimes_{i=1}^{l} \mathrm{Sym}^{\mathfrak{l}}(A_i)$, we see that $\mathrm{Sym}^{\mathfrak{l}}(A)$ contains a 1-dimensional G-module.

Next we fix a prime $p \geqslant D$. By Theorem 3.10(i), $\mathrm{Sym}^{(p-1)b_i}(B_i)$ contains a 1-dimensional G-module, if $b_i := \dim(B_i)$. By Lemma 2.7, $\mathrm{Sym}^{\mathfrak{m}}(B_i)$ contains a 1-dimensional G-module for

$$\mathfrak{m} := (p-1)\prod_{i=1}^{m} b_i \leqslant (p-1)\dim(B),$$

whence $\mathrm{Sym}^{\mathfrak{m}}(B)$ contains a 1-dimensional G-module.

Finally, by Theorem 3.10(ii), (iii) and Lemma 2.7, $\mathrm{Sym}^{q_i}(C_i)$ contains a 1-dimensional G-module for some $q_i \leqslant \dim(C_i)$. By Lemma 2.7, $\mathrm{Sym}^{\mathfrak{q}}(C_i)$ contains a 1-dimensional G-module for

$$\mathfrak{q} := \prod_{i=1}^{q} q_i \leqslant \dim(C),$$

whence $\mathrm{Sym}^{\mathfrak{q}}(C)$ contains a 1-dimensional G-module.

Now we can choose

$$\mathsf{k} = \mathrm{lcm}(\mathfrak{l}, \mathfrak{m}, \mathfrak{q}) \leqslant 2(p-1)\dim(A)\dim(B)\dim(C) = 2(p-1)n$$

and note by Lemma 2.7 that

$$\mathrm{Sym}^{\mathsf{k}}(V) \supseteq \mathrm{Sym}^{\mathsf{k}}(A) \otimes \mathrm{Sym}^{\mathsf{k}}(B) \otimes \mathrm{Sym}^{\mathsf{k}}(C)$$

contains a 1-dimensional G-submodule. □

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Now we can prove the main result of the paper.

THEOREM 4.3. *Let* $G < \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ *be any finite subgroup. Let* $p \geqslant D$ *be any prime, where* D *is the constant in Theorem 3.10. Then* $\mathrm{Sym}^k(V)$ *contains a* 1*-dimensional* G*-module for some* $k \leqslant 2(p-1)n$*. In particular, we can choose* $D = 592000$ *and* $p = 592019$.

Proof. Replacing $V = \mathbb{C}^n$ by any irreducible G-submodule U of V and G by the image of G acting on U, we may assume that the G-module V is irreducible.

(i) First we consider the case where the G-module V is primitive. If V is furthermore tensor induced, then we can apply Proposition 4.1. On the other hand, if V is tensor decomposable, then we are done by Proposition 4.2. Finally, if the G-module V is tensor indecomposable and not tensor induced, then the statement follows from Theorem 3.10.

(ii) In the general case, choose a subgroup $H \leqslant G$ of minimal order such that the G-module V is induced from an H-module W. Since V is irreducible, W is irreducible over H. The minimality of H implies that the H-module W is primitive. By the result of (i), $\mathrm{Sym}^k(W)$ contains a 1-dimensional H-module, with $k \leqslant 2(p-1)(\dim W)$ and $p \geqslant D$ any prime. It follows by Lemma 3.1 that $\mathrm{Sym}^{km}(V)$ contains a 1-dimensional G-module, with

$$m = [G : H] = (\dim V)/(\dim W),$$

and so we are done. □

COROLLARY 4.4. *Let* $G \leqslant \mathrm{GL}(V)$ *be a finite group. Then* G *has a nonzero polynomial invariant, of degree at most* $\min(1184036 \cdot \dim(V) \cdot \exp(G/G'), |G|)$.

Proof. There is no loss in assuming that G acts irreducibly on V. Now, by Main Theorem, G has a semi-invariant f of degree $k \leqslant 1184036 \cdot \dim(V)$. Arguing as in the proof of Lemma 2.7, we see that $f^{\exp(G/G')}$ is a nonzero polynomial invariant for G; in particular, $\mathbb{C}[V]^G \neq 0$. The existence of *nonzero* polynomial invariants of degree $\leqslant |G|$ now follows from the classical Noether bound [**N**]. Alternatively, the last statement also follows from the stronger result that, for any linear character λ of G, the G-submodule of $\mathbb{C}[V]$ consisting of all semi-invariants corresponding to λ is generated by homogeneous polynomials of degree at most $|G|$, see [**St**, Theorem 1.3]. □

As we mentioned in Section 1, the constant 1184036 in Corollary 4.4 can perhaps be improved. On the other hand, the next Example 4.5 shows that the term $1184036 \cdot \dim(V) \cdot \exp(G/G')$ cannot be replaced by $C \cdot \dim(V)$ or $C \cdot \exp(G/G')$ for any absolute constant C.

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EXAMPLE 4.5. Consider any fixed constant $C > 0$.

(i) For any integer $n \geqslant 4$ consider the irreducible subgroup $G = \mathsf{A}_{n+1} \times C_N < \mathrm{GL}_n(\mathbb{C})$, with $N > Cn$. As the central subgroup C_N acts faithfully as scalars on $V = \mathbb{C}^n$, any nonzero polynomial invariant f of G on V has degree divisible by N; in particular, $\deg(f) > C \cdot \dim(V)$.

(ii) For any prime $p > \max(C, 2)$, consider $q = 2^{p-1}$ and $H = \mathrm{SL}_p(q)$. It is well known, see for example [TZ, Theorem 3.1], that H has a faithful irreducible complex (Weil) representation of degree $n := (q^p - 1)/(q - 1)$, giving rise to an embedding $H < \mathrm{GL}(V)$ with $V = \mathbb{C}^n$. Note that $\mathbf{Z}(H) \cong C_p$ acts faithfully as scalars on V and $H = H'$. It follows that any nonzero semi-invariant f of H on V is also an invariant and has degree divisible by p. Thus $d(H) \geqslant p > C \cdot \exp(H/H')$. This example also shows that for quasisimple subgroups $G < \mathrm{GL}_n(\mathbb{C})$ with n arbitrarily large, $d(G)$ can exceed $\sqrt{\log_2 n}$ (indeed, taking $G = H$ as in this example, we get $2^{(p-1)^2} < q^{p-1} < n < q^p < 2^{p^2}$ and so $d(H) \geqslant p > \sqrt{\log_2 n}$).

Acknowledgments

The author gratefully acknowledges the support of the NSF (grant DMS-1201374) and the Simons Foundation Fellowship 305247. Parts of the paper were written while the author was visiting the Departments of Mathematics of Harvard University, Cambridge, MA, and Princeton University, Princeton, NJ. It is a pleasure to thank Harvard University and Princeton University for generous hospitality and stimulating environment. The author is grateful to I. Cheltsov, R. M. Guralnick, N. Katz, J. Kollár, and J. G. Thompson for many helpful discussions.

The author is also grateful to the referee for careful reading and helpful comments on the paper.

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