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UPPER BOUNDS FOR THE SEPARATION OF REAL ZEROS OF POLYNOMIALS

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Let $f(x)=\prod_1^n(x-a_i)$ be a polynomial with distinct real zeros whose separation is defined by $\delta(f)=\min_{i\neq j}(a_i-a_j)$. We establish upper estimates for $\delta(f'-kf)$ in terms of n, k, and $\delta(f)$. The results give sufficient conditions for the inverse operator $(D-kI)^{-1}$ to preserve the reality of the zeros of a polynomial.

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1. Introduction

Let f be a polynomial $f(x)=\prod_1^n(x-a_i)$ with real zeros which we suppose are in increasing order: $a_1<a_2<\cdots<a_n$. Let D be the differentiation operator, $Df=f'$ and let I be the identity.

It is elementary that for real k the property of having only real zeros is preserved by $D-kI$. In [**2**] we proved more, namely that the separation of the zeros, defined by $\delta(f)=\min_i(a_{i+1}-a_i)$, is increased by $D-kI$ (see also [**1**] for another proof). In [**3**] we gave an explicit lower bound of the form $\delta(f'-kf)\geqq\delta(f)(1+O(1/nk^2))$.

In this paper we find upper bounds for $\delta(f'-kf)$. The problem is of interest because of its application to the inverse operator $(D-kI)^{-1}$. For $k\neq 0$ the inverse is well-defined on the class of polynomials; explicitly we have for a polynomial of degree n that

$$(D-kI)^{-1}=-\frac{1}{k}\left(I+\frac{1}{k}D+\cdots+\frac{1}{k^n}D^n\right)$$

and for $k>0$ also

$$(D-kI)^{-1}f(x)=-e^{kx}\int_x^\infty e^{-kt}f(t)\,dt.$$

However unlike $D-kI$, the inverse operator does not in general preserve the reality of the zeros. For instance if $f(x)=x^2-a^2$, $a>0$, then

$$(D-kI)^{-1}f(x)=-(x^2-a^2+2x/k+2/k^2)/k$$

has real zeros only when $|k|>1/a=2/\delta(f)$.

In general $(D-kI)^{-1}f$ will have only real zeros when $|k|$ is sufficiently large, and an upper bound for $\delta(f'-kf)$ leads to estimates for how large $|k|$ must be. For instance from one of the simpler inequalities of this type, $\delta^2(f'-kf)\leqq\delta^2(f)+n^2/k^2$, which was proved in [3], we have equivalently $\delta^2((D-kI)^{-1}f)\geqq\delta^2(f)-n^2/k^2$. Hence if f is a polynomial of degree n with only real zeros then $(D-kI)^{-1}f$ will have only real zeros if $|k|>n/\delta(f)$. The purpose of this paper is to give results of this type which more accurately reflect the dependence on n and $\delta(f)$. For instance Theorem 2 gives $\delta(f'-kf)\leqq\delta(f)+10h(n)h_2(n)/|k|$, where for $x>-1$, $h(x)=\sum_{k=1}^{\infty}(1/k-1/(k+x))$ (so that for integer n, $h(n)=1+1/2+\cdots+1/n$) and $h_2(n)=h(h(n))$. It follows that $(D-kI)^{-1}$ preserves the reality of the zeros of f if $|k|>10h(n)h_2(n)/\delta(f)$. The final section gives some calculations which show that these estimates are close to the correct orders of magnitude.

2. Notation

As in the introduction, let f be a polynomial $f(x)=\prod_1^n(x-a_i)$ with real zeros which are in increasing order: $a_1<a_2<\cdots<a_n$. Let $d_i=a_{i+1}-a_i$ and $d=\delta(f)=\min_i d_i$. Let $g(x)=f'(x)/f(x)=\sum_1^n 1/(x-a_i)$.

For $k\in\mathbf{R}$ and $1\leqq j\leqq n-1$, let b_j be the zero of $f'-kf$ which lies in the interval (a_j,a_{j+1}); additionally, for $k>0$ let b_n be the zero of $f'-kf$ with $b_n>a_n$, while for $k<0$ let b_0 be the zero of $f'-kf$ with $b_0<a_1$.

Write $b_j=a_j+u_j=a_{j+1}-v_{j+1}$ so that $d_j=u_j+v_{j+1}$. Let $r_j=b_{j+1}-b_j=u_{j+1}+v_{j+1}$ so that $\delta(f'-kf)=\min_j r_j$. Let $\sigma_j=\sum_{i\neq j}1/(a_j-a_i)$.

For $x>-1$ let $h(x)=\sum_{n=1}^{\infty}(1/n-1/(x+n))$ from which for $n\in\mathbf{N}$, $h(n)=1+1/2+\cdots+1/n$, and we have the asymptotic expression $h(n)=\log(n)+\gamma+O(1/n)$ as $n\to\infty$ and $1/x>h'(x)>1/(x+1)$ for $x>0$. Let $h_2(n)=h(h(n))$ for which $h_2(n)=\log\log(n)+\gamma+O(1/\log n)$ as $n\to\infty$. It is easy to show inductively that $\sum_{i=1}^n 1/(ih(i))\leqq 4h_2(n)$.

For $0<x<d$ let $P_d(x)=1/x+1/(x-d)$ so that P_d is a decreasing function from $(0,d)$ onto $\mathbf{R}$. Hence P_d^{-1} is a decreasing function from $\mathbf{R}$ onto $(0,d)$ with $P_d^{-1}(y)+P_d^{-1}(-y)=d$. Explicitly,

$$P_d^{-1}(y)=\frac{2d}{2+dy+\sqrt{4+d^2y^2}}$$

and for $y>0$ we have the estimates $d/(2+dy)<P_d^{-1}(y)<1/y$.

3. Upper bounds

We establish estimates for u_j and v_j in terms of the algebraic function P_d^{-1} defined above and use them to deduce results for r_j.

Lemma 1. *For $1\leqq j\leqq n$ and $k>\sigma_j$ we have $u_j<1/(k-\sigma_j)$.*

Proof. We know that

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$$k=g(b_j)=\sum_{i=1}^{N}\frac{1}{a_j+u_j-a_i}=\frac{1}{u_j}+\sum_{i\neq j}\frac{1}{a_j+u_j-a_i}$$

and the terms in the summation are decreasing functions of u_j, so $k<1/u_j+\sigma_j$ and the result follows.

Lemma 2. *For all $k\in\mathbf{R}$ and $1\leqq j\leqq n-1$ we have*

$$P_{d_j}^{-1}(k-\sigma_{j+1}+1/d_j))\leqq u_j\leqq P_{d_j}^{-1}(k-\sigma_j-1/d_j)$$

and

$$v_{j+1}\leqq d_j-P_{d_j}^{-1}(k-\sigma_{j+1}+1/d_j).$$

Proof. We treat k as a function of u_j so $k=\sum_{i=1}^{n}1/(a_j+u_j-a_i)$ and

$$-\frac{dk}{du_j}=\sum_{i=1}^{n}\frac{1}{(u_j+a_j-a_i)^2}>\frac{1}{u_j^2}+\frac{1}{(u_j-d_j)^2}.$$

Note that as $u_j\to 0_+$, $k\to\infty$ and $k-1/u_j-\sigma_j\to 0$. Hence we can integrate from 0 to u_j to get

$$\left[-k+\frac{1}{u_j}\right]_0^{u_j}\geqq\left[-\frac{1}{u_j-d_j}\right]_0^{u_j}$$

$$k-\sigma_j-\frac{1}{d_j}\leqq\frac{1}{u_j}+\frac{1}{u_j-d_j}=P_{d_j}(u_j)$$

$$u_j\leqq P_{d_j}^{-1}\left(k-\sigma_j-\frac{1}{d_j}\right)$$

as required for one half of the first inequality. Similarly for v_j we have $k=\sum_{i=1}^{n}1/(a_j-v_j-a_i)$ and so $dk/dv_j>1/v_j^2+1/(d_{j-1}-v_j)^2$. If we integrate this from 0 to v_j we obtain as above

$$v_j\leqq P_{d_{j-1}}^{-1}(\sigma_j-1/d_{j-1}-k)$$
$$=d_{j-1}-P_{d_{j-1}}^{-1}(k-\sigma_j+1/d_{j-1})$$

which is the required result for v_j when we put $j+1$ for j. Putting $u_j=d_j-v_{j+1}$ then gives the other half of the required inequality for u_j.

Our first theorem gives an estimate of $\delta(f'-kf)$ for large values of $|k|$.

Theorem 1. *For all $n\geqq 2$ and $|k|\geqq 2h(n-1)/\delta(f)$, we have*

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$$\delta(f'-kf)\leqq\delta(f)+\frac{12}{k^2\delta(f)}.$$

Proof. From Lemma 1 we have $u_j<1/(k-\sigma_j)$ and from Lemma 2,

$$v_j<d_{j-1}-P^{-1}_{d_{j-1}}(k-\sigma_j+1/d_{j-1})$$

$$<d_{j-1}-\frac{d_{j-1}}{2+d_{j-1}(k-\sigma_j+1/d_{j-1})}$$

using the estimate for $P^{-1}_{d_j}$. Hence

$$u_j+v_j<\frac{1}{k-\sigma_j}+d_{j-1}-\frac{d_{j-1}}{3+d_{j-1}(k-\sigma_j)}$$

$$=d_{j-1}+\frac{3}{(k-\sigma_j)(3+d_{j-1}(k-\sigma_j)}$$

$$<d_{j-1}+\frac{3}{d_{j-1}(k-\sigma_j)^2}.$$

Now choose $j=j_0$ say so that $d_{j_0-1}=d=\delta(f)$ and

$$\delta(f'-kf)\leqq u_{j_0}+v_{j_0}<d+\frac{3}{d(k-\sigma_{j_0})^2}.$$

But $|\sigma_j|\leqq h(n-1)/d$ for all j and so if $k>2h(n-1)/d\geqq 2|\sigma_{j_0}|$ then $k-\sigma_{j_0}\geqq k/2$ and $\delta(f'-kf)\leqq\delta(f)+12/(\delta(f)k^2)$ as required. The proof when $k<0$ is similar.

Note that we have proved $\delta(f'-kf)\leqq\delta(f)+O(k^{-2})$ as $|k|\to\infty$ with the implied constant independent of n. In the next section we show that the constant can be reduced from 12 to $2+o(1)$ as $k\to\infty$.

The next theorem gives a result which is valid for all $k\neq 0$.

Theorem 2. *For all* $n\geqq 2$ *and* $k\neq 0$,

$$\delta(f'-kf)\leqq\delta(f)+10h(n)h_2(n)/|k|.$$

Proof. We suppose throught the proof that j is chosen so that $a_j=0$, $a_{j+1}=d=\delta(f)$. We suppose $k>0$; the case $k<0$ is similar.

The proof divides into two cases, determined by the spacing of the points a_i with $i<j$.

When the points are widely separated (which includes the case $j=1$ when there are no points to the left of a_j) we use r_j as our upper estimate for $\delta(f'-kf)$. When the separation is less we transfer our attention to the smallest r_i with $i<j$.

To be more specific, suppose for the first case that for each $1 \leqq i \leqq j-1$ we have $a_{j-i} < -cih(i)$ where c is a parameter depending on n and k to be determined later. Then for the second case there is some $1 \leqq i \leqq j-1$ with $a_{j-i} \geqq -cih(i)$.

In the first case let F be defined by

$$F(t)=\frac{1}{t}+\sum_{i=0}^{j-1}\frac{1}{t+d+cih(i)}$$

and let u be the positive solution of the equation $F(t)=k$. Then $d+u=b_{j+1}$ in the case when all $a_{j-i}=-cih(i)$ and there are no points with $i>j+1$, and so in general $d+u$ is an upper bound for b_{j+1} and we have $r_j<b_{j+1}\leqq d+u$.

We shall show that $u \leqq 2ch(n)$ for which, since F is decreasing, it is sufficient to show that $F(2ch(n)) \leqq k$. But

$$F(2ch(n))=\frac{1}{2ch(n)}+\sum_{i=0}^{j-1}\frac{1}{2ch(n)+d+cih(i)}<\frac{1}{c}\left(\frac{1}{2h(n)}+\sum_{i=0}^{j-1}\frac{1}{2h(n)+ih(i)}\right)$$

$$<\frac{1}{c}\left(\frac{1}{h(n)}+\sum_{i=1}^{j-1}\frac{1}{ih(i)}\right)<\frac{1}{c}\left(\frac{1}{h(n)}+4h_2(j)\right)<\frac{5}{c}h_2(n)$$

which equals k if $c=5h_2(n)/k$. Hence in this case $\delta(f'-kf)\leqq d+u \leqq \delta(f)+2ch(n)\leqq \delta(f)+10h(n)h_2(n)/k$ as required.

In the second case choose $i \geqq 1$ with $a_{j-1} \geqq -cih(i)$. Then the sum of the lengths of the i pairs of consecutive intervals (d_j+d_{j-1}), $(d_{j-1}+d_{j-2})\ldots(d_{j-i+1}+d_{j-i})$ is at most $d+2cih(i)$ and so the length of the smallest pair, d_s+d_{s-1} say, is at most

$$(d+2cih(i))/i \leqq d+2ch(n).$$

Hence since the corresponding interval (b_{s-1}, b_s) of length r_{s-1} is contained in this pair of intervals we have $\delta(f'-kf)\leqq r_{s-1}\leqq \delta(f)+2ch(n)\leqq \delta(f)+10h(n)h_2(n)/k$ as before, and the proof is complete.

From Theorem 2 we have the following corollary.

Corollary. *For any polynomial f of degree n with distinct real zeros, $(D-kI)^{-1}f$ has also distinct real zeros if $|k|>10h(n)h_2(n)/\delta(f)$.*

Proof. A continuity argument shows that for sufficiently large $|k|$ the zeros of $(D-kI)^{-1}f$ are close to those of f and hence that all are real when those of f are real. Theorem 2 then gives $\delta((D-kI)^{-1}f)\geqq \delta(f)-10h(n)h_2(n)/|k|$ for these values of k. But the zeros depend continuously on k and so $\delta((D-kI)^{-1}f)$ will remain positive as long as $|k|>10h(n)h_2(n)/\delta(f)$ and so the zeros remain real for these values of k.

4. Asymptotic estimates

We begin with an asymptotic calculation concerning the best value of the constant in Theorem 1. As before we put $b_j=a_j+u_j$ so that $k=\sum_{i=1}^{n}1/(a_j+u_i-a_i)=1/u_j+\sigma_j+O(u_j)$ as $u_j\to 0_+$. Inverting this relation gives

$$u_j=\frac{1}{k}+\frac{\sigma_j}{k^2}+O(k^{-3})\quad\text{and so}\quad r_j=d_j+\frac{\sigma_{j+1}-\sigma_j}{k^2}+O(k^{-3})\text{ as }k\to\infty.$$

But for any j,

$$\begin{aligned}\sigma_{j+1}-\sigma_j&=\sum_{i\neq j+1}\frac{1}{a_{j+1}-a_i}-\sum_{i\neq j}\frac{1}{a_j-a_i}\\&=\frac{2}{d_j}-\sum_{i<j}\frac{a_{j+1}-a_j}{(a_{j+1}-a_i)(a_j-a_i)}-\sum_{i>j+1}\frac{a_{j+1}-a_j}{(a_i-a_{j+1})(a_i-a_j)}<\frac{2}{d_j}\end{aligned}$$

since both summations are positive. Hence taking $j=j_0$ so that $d_{j_0}=d=\delta(f)$ we have

$$\delta(f'-kf)\leqq r_{j_0}=d+\frac{2}{dk^2}+O(k^{-3})$$

as $k\to\infty$ and so the constant 12 in Theorem 1 can be replaced by $2+o(1)$ as $k\to\infty$.

To investigate the estimate in the corollary we argue as follows. Recall that for $k>0$ we have

$$F(x)=(D-kI)^{-1}f(x)=-e^{kx}\int_x^\infty e^{-kt}f(t)\,dt.$$

We consider the situation in which the zeros of f are equally spaced, say $f(x)=\prod_0^{n-1}(x+id)$, $d>0$ and estimate how large k must be to ensure that F has a sign change between $x=0$ and $x=-d$. The conjecture that equal spacing gives the extremal configuration is not proved but is supported by numerical evidence, as was the case for the operator $D-kI$ in [3].

Suppose then that $f(x)=\prod_0^{n-1}(x+id)$, $d=\delta(f)>0$. Then $F(0)=-\int_0^\infty e^{-kt}f(t)\,dt$ is obviously negative and so we need an estimate for the size of k to ensure that $F(-d)=-e^{-kd}\int_{-d}^\infty e^{-kt}f(t)\,dt$ is positive. To obtain this we consider the two integrals $I_0=\int_{-d}^0 e^{-kt}f(t)\,dt<0$ and $I_1=\int_0^\infty e^{-kt}f(t)\,dt>0$ separately and find how large k must be to make $|I_0|>I_1$.

To estimate I_0 note that for $-d\leqq x\leqq 0$ we have $|f(x)|\leqq(n-2)!d^{n-2}|x(d+x)|$ and so

$$\begin{aligned}|I_0| &\leqq (n-2)!d^{n-2}\int_{-d}^{0}|t(d+t)|\,e^{-kt}\,dt\\ &= k^{-2}(n-2)!d^{n-2}[d(e^{kd}+1)-2(e^{kd}-1)/k]\\ &\leqq 2(n-2)!d^{n-1}\,e^{kd}/k^2\end{aligned}$$

after two integrations by parts.

To estimate I_1 note that for $x>0$ we have $f(x)\geqq(n-1)!d^{n-1}x$ and so

$$I_1 \geqq (n-1)!\,d^{n-1}\int_0^\infty te^{-kt}\,dt = k^{-2}(n-1)!d^{n-1}.$$

Comparing these two estimates shows that values of k with $2e^{kd}<n-1$, i.e. $k<\log((n-1)/2)/d$ will give $|I_0|<I_1$ and so will not produce the required sign change of F. Hence $k\geqq\log((n-1)/2/d$ is necessary for F to have all real zeros. Comparing this with the corollary shows that the estimates agree in order of magnitude except for the $h_2(n)$ term which is of order $\log\log n$.

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