FORD AND DIRICHELET REGIONS FOR FUCHSIAN GROUPS

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1. Introduction. There has recently been some interest in a class of limit points for Fuchsian groups now known as Garnett points [5], [8]. In this paper we show that such points are intimately connected with the structure of Dirichlet regions and the same ideas serve to show that the Ford and Dirichlet regions are merely examples of one single construction which also yields fundamental regions based at limit points (and which properly lies in the subject of inversive geometry). We examine in the general case how the region varies continuously with the construction. Finally, we consider the linear measure of the set of Garnett points.

2. Hyperbolic space. Let \( \Delta \) be any open disc (or half-plane) in the extended complex plane \( \mathbb{C}_\infty \): usually \( \Delta \) will be the unit disc or the upper half-plane. We may regard \( \Delta \) as the hyperbolic plane in the usual way and the conformal isometries of \( \Delta \) are simply the Moebius transformations of \( \Delta \) onto itself.

Each conformal isometry \( g \) of \( \Delta \) may be written as \( g = \sigma_2 \sigma_1 \) where \( \sigma_j \) is the reflection in a geodesic \( L_j \) in \( \Delta \). This decomposition is not unique; however there is a one-parameter family \( F \) of geodesics to which the \( L_j \) belong and either \( L_1 \) or \( L_2 \) may be chosen arbitrarily from \( F \); the other \( L_j \) then being uniquely determined. If \( g \) is elliptic or parabolic, then \( g \) has a unique fixed point in \( \overline{\Delta} \) (the closure of \( \Delta \) ) and \( F \) is the class of geodesics through that point (if \( g \) is elliptic) or having that point as an end-point (if \( g \) is parabolic). If \( g \) is hyperbolic, \( F \) is the set of geodesics orthogonal to the axis of \( g \).

In general, if \( \sigma \) is the reflection in \( L \), then \( L \) is called the axis of \( \sigma \).

3. A unique representation. We now select any \( \xi \) in \( \overline{\Delta} \). For each isometry \( g \) not fixing \( \xi \) we can choose \( L_2 \) so that \( \xi \in L_2 \). This determines \( L_2 \) (and so \( L_1 \)) uniquely and we now have a unique representation \( g = \sigma_2 \sigma_1 \). We shall denote these choices of \( L_1 \) and \( L_2 \) by \( L_g \) and \( L_g^* \) respectively and also we use \( \sigma_{g^*} \) for \( \sigma \), \( \sigma_3 \).

It is geometrically obvious that as \( g(\xi) \neq \xi \) we have \( \xi \notin L_g \). For an analytic proof we need only observe that if \( \xi \in L_g \) then
\[
g(\xi) = \sigma_{g^*}(\sigma_g(\xi)) = \sigma_{g^*}(\xi) = \xi
\]
which is false by assumption.

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As $\xi \notin L_\varphi$ we can now define the hyperbolic half-planes $H_\varphi, I_\varphi$ as those half-planes which are bounded by $L_\varphi$ and which contain and do not contain $\xi$ respectively: thus $\xi \in H_\varphi, \xi \notin I_\varphi$.

4. Invariance under conjugation. It is an elementary fact that if $h$ is any Moebius transformation mapping $\Delta$ onto $\Sigma$, then $h\phi h^{-1}$ is an isometry of $\Sigma$ whenever $\phi$ is an isometry of $\Delta$. In particular, if $\sigma$ is the reflection in a geodesic $L$ in $\Delta$ then $h\sigma h^{-1}$ is the reflection in the geodesic $h(L)$ in $\Sigma$. These facts lead immediately to the invariance of the above concepts under conjugation: explicitly

$$
\sigma_{h\varphi h^{-1}} = h\sigma_\varphi h^{-1},
$$

$$
L_{h\varphi h^{-1}} = h(L_\varphi),
$$

$$
H_{h\varphi h^{-1}} = h(H_\varphi)
$$

and similarly for $\sigma_\varphi^*, L_\varphi^*$ and $I_\varphi$, where, of course, $\sigma_{h\varphi h^{-1}}$ (for example) refers to the point $h(\xi)$ in the same way that $\sigma_\varphi$ refers to $\xi$.

5. Basic properties. The following result should seem familiar to any reader who has met the concept of isometric circles.

**Proposition 1.** *In the above notation,*

(i) $\sigma_{\varphi^{-1} \varphi} = \sigma_\varphi^*$;

(ii) $g(L_\varphi) = L_{\varphi^{-1}}$;

(iii) $g(H_\varphi) = I_{\varphi^{-1}}, g(I_\varphi) = H_{\varphi^{-1}}$.

*Proof.* As every reflection is its own inverse we have

$$
\sigma_{\varphi^{-1} \varphi}^* g^{-1} = \sigma_\varphi^* \sigma_\varphi^* = \sigma_\varphi^*(\sigma_\varphi^* \sigma_\varphi^*) = \sigma_\varphi^* \sigma_\varphi^*.
$$

By the uniqueness of the decomposition this yields (i) together with the identity

$$
\sigma_{\varphi^{-1}} = \sigma_\varphi^* \sigma_\varphi^*.
$$

This identity shows that the axis of $\sigma_{\varphi^{-1}}$ is precisely $\sigma_\varphi^*(L_\varphi)$ and this proves (ii) as

$$
g(L_\varphi) = \sigma_\varphi^* \sigma_\varphi(L_\varphi) = \sigma_\varphi^*(L_\varphi).
$$

Finally, (ii) shows that the pair of half-planes $g(H_\varphi), g(I_\varphi)$ is the same as the pair of half-planes $H_{\varphi^{-1}}, I_{\varphi^{-1}}$. As

$$
\xi = \sigma_\varphi^*(\xi) \notin \sigma_\varphi^* \sigma_\varphi(H_\varphi) = g(H_\varphi)
$$

we see that (iii) follows.

In order to relate these ideas to the more familiar ideas concerning Ford and Dirichlet regions, we now identify $L_\varphi$ in certain cases.
Proposition 2. (i) If \( \xi \in \Delta \), then \( L_\sigma \) is the hyperbolic bisector of \( \xi \) and \( g^{-1}(\xi) \).

(ii) If \( \xi = \infty \), then \( L_\sigma \) is part of the isometric circle of \( g \).

Proof. To prove (i) observe first that \( \sigma_\phi^*(\xi) = \xi \). Next, if \( z \in L_\sigma \) then \( \sigma_\phi(z) = z \) and so (using \( \rho \) for the hyperbolic metric)

\[ \rho(z, g^{-1}(\xi)) = \rho(z, \sigma_\phi \sigma_\phi^*(\xi)) = \rho(z, \sigma_\phi(\xi)) = \rho(\sigma_\phi(z), \xi) = \rho(z, \xi). \]

This proves (i).

To prove (ii), observe first that when \( \xi = \infty \), \( \sigma_\phi^* \) is a Euclidean reflection, hence a Euclidean isometry. As \( \sigma_\phi \) acts as a Euclidean isometry (indeed, it is the identity) on \( L_\sigma \), we see that \( g \) acts as a Euclidean isometry on \( L_\sigma \). Then \( |g'(z)| = 1 \) on \( L_\sigma \) and (ii) follows.

Remark 1. If \( \Delta \) is the unit disc, and \( \xi = 0 \) then every extended \( L_\sigma^* \) also goes through \( \infty \) and so the construction of the \( L_\sigma^* \) for \( \xi = 0 \) and \( \xi = \infty \) yield the same line. This shows that the isometric circle of \( g \) coincides with the bisector of \( 0 \) and \( g^{-1}(0) \) and so gives a geometric insight into this well known fact [4, p. 151].

Remark 2. It is a simple matter to compute the equation of \( L_\sigma \) when \( \Delta \) is the unit disc. If \( \xi \in \Delta \), the equation is

\[ |z - g(\xi)|^2 |z - \xi|^2 = |g'(\xi)| \]

and (by continuity considerations or by taking conjugates) this remains true when \( |\xi| = 1 \). We shall not need this formula and we omit the proof.

In view of Proposition 2, it is natural to consider a Fuchsian group \( \Gamma \) acting on \( \Delta \) and to write

\[ D_\xi = \bigcap_{g \in \Gamma, g \neq I} H_\sigma \]

under the assumption that no non-trivial element of \( \Gamma \) fixes \( \xi \). Indeed, from the standard theory of Fuchsian groups, we know that \( D_\xi \) is a fundamental polygon for \( \Gamma \) when \( \xi \in \Delta \) (\( D_\xi \) is then the Dirichlet region with center \( \xi \)) or when \( \xi \in \partial \Delta \) and is an ordinary point of \( \Gamma \) (\( D_\xi \) is then the image of the Ford polygon for a conjugate group). We now make an observation which will be used repeatedly throughout the paper. If \( \Gamma \) acts in the upper half-plane \( \Delta \) and \( w \in \Delta \), we say that \( w \) is of strictly maximal height if for every \( g \in \Gamma, g \neq I \) we have \( \text{Im} \ g(w) < \text{Im} \ (w) \). We say that \( w \) is of maximal height if for every \( g \in \Gamma \) we have \( \text{Im} \ g(w) \leq \text{Im} \ (w) \).

Proposition 3. If \( \Gamma \) is a Fuchsian group in the upper half-plane \( \Delta \) then

(i) \( w \in \Delta \) is of strictly maximal height if and only if \( w \in D_\infty \);

(ii) \( w \in \Delta \) is of maximal height if and only if \( w \in \bar{D}_\infty \).
Proof. Part (i) follows easily from Proposition 2 and the fact that for a Moebius transform \( g \) preserving \( \Delta \)
\[
\text{Im } g(z) = |g'(z)|^2 \text{ Im } z.
\]
To prove part (ii) suppose first that \( w \in \Delta \) is of maximal height and let \( L \) denote the open half-geodesic joining \( w \) to \( \infty \). No isometric circle of a transform in \( \Gamma \) can meet \( L \) (otherwise \( w \) is interior to an isometric circle and so has an image of greater height) thus \( L \subset D_\infty \) and \( w \in \tilde{D}_\infty \). Conversely, if \( w \) is not of maximal height then for some \( g \in \Gamma \), \( \text{Im } g(w) > \text{Im } (w) \) and from our comments above we see that \(|g'(w)| > 1\). From Proposition 2 we see that \( w \in I_{g} \). As \( I_{g} \) is open and \( \tilde{D}_\infty \subset \tilde{H}_g = H_g \cup I_{g} \) we see that \( w \notin D_\infty \).

We aim to consider the continuity of \( D_\xi \) as a function of \( \xi \) and also to characterize those limit points \( \xi \) for which \( D_\xi \) is a fundamental region.

6. Continuity. We recall the definition of convergence of a sequence of sets \( \{A_i\} \). Defining
\[
\lim \inf A_i = \bigcup_{i=1}^{\infty} \bigcap_{k \geq i} A_k \quad \text{and} \quad \lim \sup A_i = \bigcap_{i=1}^{\infty} \bigcup_{k \geq i} A_k
\]
we say that \( \{A_i\} \) converges to \( A \) if \( \lim \inf A_i = \lim \sup A_i = A \). Given a sequence \( \xi_n \) in \( \Delta \) which converges to \( \xi \) in \( \bar{\Delta} \) we ask whether the regions \( D_{\xi_n} \) converge to \( D_\xi \). If \( \xi \in \Delta \) this result is almost trivial to prove. In the case where \( \xi \notin \partial \Delta \) we have the following result.

Theorem 1. Let \( \Gamma \) be a Fuchsian group acting in \( \Delta \) and \( \xi \in \partial \Delta \). If \( \xi \) is not fixed by any element of \( \Gamma \) (except the identity) and if \( \xi_n \) is a sequence of points converging to \( \xi \) in a cusp then
\[
D_\xi \subset \lim \inf D_{\xi_n} \subset \lim \sup D_{\xi_n} \subset \bar{D}_\xi.
\]

Proof. By conjugation we assume \( \Delta \) is the upper half-plane and \( \xi = \infty \). Thus \( \xi_n \) converge to \( \infty \) in some strip \( \{z: a < \text{Re } z < b\} \). Now choose \( w \) in \( \Delta \) and define
\[
A_n(w) = \{z \in \Delta: \rho(z, \xi_n) < \rho(w, \xi_n)\}
\]
where \( \rho \) is the non-euclidean metric in \( \Delta \); note that \( A_n(w) \) is a disc in \( \Delta \). We note that
\[
A_n(w) = \left\{ z \in \Delta: \frac{|z - \xi_n|^2}{\text{Im } z} < \frac{|w - \xi_n|^2}{\text{Im } w} \right\}.
\]
If \( z \in \lim \sup A_n(w) \) then, for infinitely many \( n \),
\[
\frac{|z - \xi_n|^2}{\text{Im } z} < \frac{|w - \xi_n|^2}{\text{Im } w}
\]
and, since \( \xi_n \) converges to \( \infty \) in a cusp, we see that \( \text{Im } w \leq \text{Im } z \). Similarly, if \( \text{Im } w < \text{Im } z \), then for all \( n \) large enough

\[
\frac{|z - \xi_n|^2}{\text{Im } z} < \frac{|w - \xi_n|^2}{\text{Im } w}
\]

and so \( z \in \lim \inf A_n(w) \). We have proved

(1) \( \{ z : \text{Im } z > \text{Im } w \} \subseteq \lim \inf A_n(w) \subseteq \lim \sup A_n(w) \subseteq \{ z : \text{Im } z \geq \text{Im } w \} \).

Suppose that \( w \notin \bar{D}_\infty \). Then for some \( g \in \Gamma \), \( \text{Im } g(w) > \text{Im } w \) and so from (1),

\( g(w) \in \lim \inf A_n(w) \).

As \( D_{\xi_n} \) is the Dirichlet region centered at \( \xi_n \) we see that \( w \notin D_{\xi_n} \) for all \( n \geq n(w) \) say and hence that \( w \notin \lim \sup D_{\xi_n} \). This shows that

\[
\lim \sup D_{\xi_n} \subset \bar{D}_\infty.
\]

To complete the proof we suppose \( w \in D_\infty \) and so \( w \) is of strictly maximal height. If \( w \notin \lim \inf D_{\xi_n} \) then, for infinitely many \( n \), \( w \notin D_{\xi_n} \) and there exists a sequence \( g_n \) of elements of \( \Gamma \) with \( g_n(w) \in A_n(w) \). However, since

\[
\{ z : 0 < \text{Im } z < \text{Im } w \} \cap \left( \bigcup_{n=1}^{\infty} A_n(w) \right)
\]

is a compact subset of \( \Delta \) it can contain only finitely many \( \Gamma \) images of \( w \). Thus the sequence \( g_n \) contains only finitely many distinct transforms. We see then that for some \( g \in \Gamma \), \( g \neq I \), \( g(w) \in A_n(w) \) for infinitely many \( n \) and we obtain a contradiction with (1) since \( \text{Im } g(w) < \text{Im } w \). We have shown that

\( D_\infty \subset \lim \inf D_{\xi_n} \)

and this completes the proof of Theorem 1.

We conclude this section with an example to show that Theorem 1 is essentially best possible. For each positive integer \( n \) let \( r_n = (n!)^2 \) and set

\[
x_n = 2 \sum_{i=1}^{n-1} r_i + r_n.
\]

Let \( C_n \) be the circle of radius \( r_n \) centered at \( x_n \) on the real axis and let \( C_n' \) be the reflection of \( C_n \) across the imaginary axis. Let \( P \) be a parabolic transform preserving the upper half-plane with isometric circles \( C_1, C_1' \). For each \( n \geq 2 \) let \( H_n \) be a hyperbolic transform preserving the upper half-plane with isometric circles \( C_n, C_n' \). Now \( \Gamma \) given by

\[
\Gamma = \langle P, H_2, H_3, \ldots \rangle
\]
is a Fuchsian group preserving the upper half-plane. This group was considered in [5] where it was shown that \( i \) is the unique point of maximal height in its orbit and that, for \( n \geq 2 \),

\[
1 > \text{Im} \, H_n(i) > (1 + 2/n)^{-2}.
\]

Now \( D_\infty \) is the open region in the upper half-plane exterior to all the circles \( C_n, C'_n \) and clearly contains \( i \). We set, for \( n \geq 2 \),

\[
w_n = \text{Re} \, H_n(i) + iy_n
\]

where \( y_n \) is chosen so that \( y_n/\text{Re} \, H_n(i) \to +\infty \). Thus the points \( w_n \) approach \( \infty \) in arbitrarily thin Stolz angles. An easy calculation shows that for \( n \) large enough

\[
\rho(w_n, i) > \rho(w_n, V_n(i))
\]

so \( i \) is exterior to the Dirichlet region \( D_{w_n} \) for \( n \) large enough. Thus, in this example, \( D_\infty \) is not the limit of \( D_{w_n} \).

7. Garnett points. The open hyperbolic disc with center \( w \) in \( \Delta \) and radius \( r \) is

\[
D(w, r) = \{ z \in \Delta : \rho(z, w) < r \}
\]

We shall now extend this terminology to include cases where \( w \in \partial \Delta \). A horocyclic region \( Q \) in \( \Delta \) is an open Euclidean disc contained in \( \Delta \) and internally tangent to \( \partial \Delta \) at some point \( \eta \) in \( \partial \Delta \); we say \( Q \) is based at \( \eta \). The Euclidean circle \( \partial Q \) is called a horocycle at \( \eta \) and it has equation

\[
P(z, \eta) = k
\]

where \( P(z, \eta) \) is the Poisson kernel for \( \Delta \) and \( k \) is some positive constant.

When \( \eta \in \partial \Delta \), we use \( D(\eta, r) \) to denote the set \( Q \cup \{ \eta \} \) where \( Q \) is the horocyclic region based at \( \eta \) and given by \( P(z, \eta) > r^{-1} \).

For the remainder of this section we will assume that \( \Gamma \) is a Fuchsian group acting on \( \Delta \), that \( \xi \in \overline{\Delta} \) and that \( g(\xi) \neq \xi \) whenever \( g \in \Gamma \) and \( g \neq I \). We denote the set of elliptic fixed points of \( \Gamma \) by \( E \) and the set of parabolic fixed points of \( \Gamma \) by \( P \): thus

\[
\xi \notin P \cup E.
\]

We now distinguish certain types of limit points of \( \Gamma \).

Definition. (i) We say that \( w \) in \( \partial \Delta \) is a horocyclic limit point for \( \Gamma \) if for some (and hence all) \( a \) in \( \Delta \), the orbit \( \Gamma(a) \) of \( a \) meets every horocyclic region based at \( w \).

(ii) We say that \( w \) in \( \partial \Delta \) is a Garnett point for \( \Gamma \) if there is some \( a \in \Delta \) and some positive \( r \) such that \( D(w, r) \) does not meet \( \Gamma(a) \) whereas for every positive \( \epsilon \), \( D(w, r + \epsilon) \) does meet \( \Gamma(a) \).
We denote the set of horocyclic limit points by $H$ and the set of Garnett points by $G$. The existence of Garnett points has recently been established by Nicholls [5]. If $\Gamma$ is a finitely generated group then $G = \emptyset$ and the limit set is the disjoint union of $P$ and $H$ [2].

8. Fundamental polygons. In this section we characterize those points $\xi$ for which $D_\xi$ is a fundamental region. Throughout we assume that $\Gamma$ is a Fuchsian group acting on $\Delta$ and $\xi \in \bar{\Delta}$. To begin with we record the following trivial result.

**Proposition 4.** Let $\Gamma$ be a Fuchsian group acting in $\Delta$. Then $D_\xi \cap g(D_\xi) = \emptyset$ whenever $g \in \Gamma$, $g(\xi) \neq \xi$. Further, $D_\xi$ is convex.

**Proof.** As $D_\xi$ is an intersection of half-planes, it is convex. If $g \in \Gamma$ and $g \neq I$ then, by Proposition 1,

$$g(D_\xi) \cap D_\xi \subseteq g(H_\phi) \cap H_{\phi^{-1}} = I_{\phi^{-1}} \cap H_{\phi^{-1}} = \emptyset.$$  

With the notation of the previous section we state our theorem.

**Theorem 2.** The interior of $D_\xi$ is a convex fundamental region for $\Gamma$ if and only if $\xi \notin H \cup G \cup P \cup E$.

**Proof.** Suppose first that $\xi \in H \cup G \cup P \cup E$. If $\xi \in P \cup E$ then $D_\xi$ is not defined and so cannot be a fundamental region. In the other cases, $\xi \in \partial \Delta$ and by conjugation we may assume that $\Delta$ is the upper half-plane and that $\xi = \infty$. Thus we assume that $\infty \in H \cup G$ and that no element of $\Gamma$ (except the identity) fixes $\infty$. Suppose next that $D_\infty \neq \emptyset$; any $w \in D_\infty$ is of strictly maximal height and so $\infty \notin H$. Thus if $D_\infty \neq \emptyset$ then $\infty \notin H$ and we have only to consider the case $\infty \in G$. In this case there is a point $a$ in $\Delta$ and a positive real $r$ such that

$$\{z: \text{Im} \ z \geq r\} \cap \Gamma(a) = \emptyset$$

and, for any $\delta > 0$,

$$\{z: \text{Im} \ z > r - \delta\} \cap \Gamma(a) \neq \emptyset.$$

This implies that no point in $\Gamma(a)$ is of maximal height, hence $\Gamma(a) \cap D_\infty$ is empty and so $\Gamma(a)$ does not meet $\text{Int} \ D_\infty$. Thus in all cases $\text{Int} \ D_\infty$ is not a fundamental region.

Now suppose that $\xi \notin H \cup P \cup G \cup E$. The set $D_\xi$ is defined and, in view of Proposition 4 we have only to show that each $z$ in $\Delta$ has an image in $\text{Int} \ D_\xi$. The argument is well known when $\xi \in \Delta$ (this is simply the case of the Dirichlet region) and we omit the details: even so, the following argument for the case $\xi \in \partial \Delta$ can easily be modified to cover the case $\xi \in \Delta$ (independently of the well known method).

We suppose then that $\xi \in \partial \Delta$ and, as before, we may assume that $\Delta$ is the upper half-plane and that $\xi = \infty$. The assumption that $\xi \notin P \cup H$
implies that no element of $\Gamma$ fixes $\infty$ (except the identity) so every non-trivial $g$ in $\Gamma$ possesses a corresponding geodesic $L_g$ which is a Euclidean semi-circle with center on the real axis (the isometric circle of $g$). As $\xi \notin H \cup G$, for each $z$ in $\Delta$ there is a maximal closed disc $\bar{D}(\infty, r)$ such that $z$ has an equivalent point $z'$ satisfying

$$z' \in \partial \bar{D}(\infty, r), \Gamma(z) \cap D(\infty, r) = \emptyset.$$ 

Thus $z'$ is of maximal height.

Now select two distinct orbits and select from them representatives, say $z_1, z_2$ of maximal height. We see, as in the proof of Proposition 3 that no $L_g$ can meet the half-geodesic joining $\infty$ to $z_1$ or the half-geodesic joining $\infty$ to $z_2$. We deduce from elementary geometry that no $L_g$ can penetrate the interior of the triangle with vertices $\infty, z_1, z_2$. This interior is therefore a subset of $\text{Int} \, D_\infty$ and, in particular, $z_1 \in \text{Int} \, D_\infty$. As this is true for any $z_1$ we have shown that any $z$ in $\Delta$ has a representative in the closure of the interior of $D_\xi$ and the proof is complete.

We conclude this section with several remarks.

Remark 1. If $\xi \notin H \cup G \cup P \cup E$ then the interior of $D_\xi$ is a convex fundamental region and its boundary is polygonal. This is well known when $\xi \in \Delta$ and in the case when $\xi \in \partial \Delta$ it may be deduced from the methods developed by Nicholls and Zarrow [6, Lemma 1 and Theorem 1].

Remark 2. The set $D_\xi$ may be empty and we can say precisely when this will happen. Recalling the definition of a Garnett point for a Fuchsian group $\Gamma$, we say that $w$ in $\partial \Delta$ is a universal Garnett point (written $w \in G^*$) if for every $a \in \Delta$ there exists a positive $r$ such that $\bar{D}(w, r)$ does not meet $\Gamma(a)$ whereas for every positive $\epsilon$, $D(w, r + \epsilon)$ does meet $\Gamma(a)$. The set $D_\xi$ is empty if and only if $\xi \in H \cup G^* \cup P \cup E$. The sufficiency of this condition follows immediately from the definition of $D_\xi$. Now suppose

$$\xi \notin H \cup G^* \cup P \cup E.$$ 

If $\xi \in \Delta$ then $D_\xi$ is a Dirichlet region and consequently non-empty. Thus we suppose that $\xi \in \partial \Delta$ and we conjugate $\xi$ to infinity. Since $\infty \notin H \cup G^* \cup P \cup E$ we see that there exists a point of $\Delta$, say $a$, which is of maximal height. It follows immediately from Proposition 3 that $D_\infty$ is not empty.

Remark 3. We observe that our construction yields new examples of convex fundamental regions (apart from the Dirichlet and Ford regions). See, for example, the region $D_\infty$ for the group $\Gamma$ mentioned at the end of Section 6.

Remark 4. The region $D_\xi$ need not be open or locally finite (group images of $D_\xi$ may accumulate in $\Delta$). To see this consider the group $\Gamma'$
defined at the end of Section 6. We may find a proper subgroup \( \Gamma_1 \) of \( \Gamma \) for which \( \infty \in G \) and the region \( D_\infty \) for \( \Gamma_1 \) is the closure of the region \( D_\infty \) for \( \Gamma \). The \( \Gamma_1 \) images of \( D_\infty \) accumulate in \( \Delta \) (see [5] for more details).

Note that in this example the interior of \( D_\infty \) is not a fundamental region for \( \Gamma_1 \). It may be the case that if the interior of \( D_\infty \) is a fundamental region for a group \( \Gamma \) then \( D_\infty \) will always be open and locally finite. However we are unable to prove this.

**Remark 5.** Observe that the family

\[ \{D(w, r) : w \in \Delta, r \geq 0\} \]

is a base for a Hausdorff topology on \( \Delta \) and the induced topology on \( \Delta \) is the Euclidean (and hyperbolic) topology. With this in mind we see that for \( \xi \in \partial \Delta \) the region \( D_\xi \) has a natural interpretation as a Dirichlet region centered at \( \xi \).

9. The measure of \( G \). It is clearly desirable to have some information on the size of the set \( G \) and we conclude that paper by showing that \( G \) has zero linear measure. This result is implicit both in the work of Pommerenke [7] and of Sullivan [8]. We will follow Pommerenke's approach.

**Theorem 3.** Let \( \Gamma \) be a Fuchsian group acting in \( \Delta \) then \( m(G) = 0 \).

**Proof.** Following Pommerenke [7] we define \( Q \), the set of oricyclic points for \( \Gamma \), to be the set of points of \( \partial \Delta \) with the property that every horocyclic region at the point contains only finitely many \( \Gamma \) images of some point (and hence all points) of \( \Delta \). We see from this definition that

\[ (\partial \Delta \setminus G) \supset H \cup Q. \]

If \( w \in \Delta \) we define

\[ E_w = \{z \in \partial \Delta : V(z) \in \partial D_w \text{ for some } V \in \Gamma\}. \]

We need one more definition. If \( \xi \in \partial \Delta \) and \( w \in \Delta \) we say that \( \xi \) is a Garnett point with respect to the orbit of \( w \) if there is a positive \( r \) so that \( \bar{D}(\xi, r) \) does not meet \( \Gamma(w) \) whereas for every positive \( \epsilon \), \( D(\xi, r + \epsilon) \) does meet \( \Gamma(w) \). We denote the set of such points \( \xi \) by \( g_w \) and note that

\[ G = \bigcup_{w \in \Delta} g_w. \]

Nicholls [5] has shown that, for any \( w \in \Delta \),

\[ \partial \Delta = H \cup E_w \cup g_w \]

and Sullivan [8, Section 4] has shown that, for any \( w \in \Delta \)

\[ m(g_w) = 0. \]
Thus, for any $w$ in $\Delta$, $H \cup E_w$ is a set of full measure on $\partial \Delta$. We now use Pommerenke's result [7, Theorem 1] that $m(Q) = m(E_w)$ for any $w$ in $\Delta$ to deduce that $H \cup Q$ is a set of full measure on $\partial \Delta$. In view of (2) the proof of Theorem 3 is complete.

References


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