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RANK TWO STABLE ULRICH BUNDLES ON ANTICANONICALLY EMBEDDED SURFACES

GIANFRANCO CASNATI

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Abstract

Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface of degree $d \geq 3$. In this note, we classify stable Ulrich bundles on S of rank two. We also study their moduli spaces.

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1. Introduction and notation

Throughout the paper, $\mathbb{P}^N$ denotes the projective space of dimension N over an algebraically closed field k.

Let $S \subseteq \mathbb{P}^N$ be a surface. We assume that S is smooth integral and closed, without making explicit mention of these properties. The surface S is endowed with a polarisation $\mathcal{O}_S(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_S$. We are interested in the natural problem of studying vector bundles supported by S. We can obviously restrict our attention to *indecomposable* bundles, that is, bundles which do not split as a direct sum of bundles of lower rank. Moreover, at least from a cohomological viewpoint, the simplest vector bundles are the *arithmetically Cohen–Macaulay* (*aCM* for short) ones, that is, bundles $\mathcal{E}$ such that $h^1(S, \mathcal{E}(th)) = 0$ for $t \in \mathbb{Z}$. Such a property is trivially invariant up to shifting degrees, and thus we focus on *initialised* bundles, that is, bundles $\mathcal{E}$ such that $h^0(S, \mathcal{E}(-h)) = 0$ and $h^0(S, \mathcal{E}) \neq 0$.

Horrocks theorem (see [17] and the references therein) asserts that $\mathcal{O}_S$ is the unique initialised, indecomposable, aCM bundle when $S \subseteq \mathbb{P}^N$ is a plane. Let $S \subseteq \mathbb{P}^N$ be an aCM surface with ideal sheaf $\mathcal{I}_{S|\mathbb{P}^N}$ (that is, $h^1(\mathbb{P}^N, \mathcal{I}_{S|\mathbb{P}^N}(t)) = 0$, for $t \in \mathbb{Z}$, and $\mathcal{O}_S$ is aCM) supporting finitely many aCM bundles. A consequence of a general result of Eisenbud and Herzog (see [9]) then implies that S is necessarily either a plane, or a smooth quadric, or a rational scroll of degree up to four, or the Veronese surface.

Indeed, the general surface $S \subseteq \mathbb{P}^N$ is of *wild representation type*, that is, it actually supports families of dimension p of pairwise nonisomorphic initialised,

The author is a member of the GNSAGA group of INdAM.

indecomposable aCM bundles for arbitrarily large p (see the very recent paper [12]). One of the first examples of such behaviour is on cubic surfaces and the classification of aCM bundles of ranks one and two is due to Faenzi (see [11]).

Each smooth cubic surface in $\mathbb{P}^3$ is anticanonically embedded: that is, the embedding $S \subseteq \mathbb{P}^3$ satisfies $\omega_S^{-1} \cong \mathcal{O}_S(h)$. The anticanonically embedded surfaces $S \subseteq \mathbb{P}^d$ are exactly the *del Pezzo surfaces* S such that ω_S^{-1} is very ample. It is well known that the anticanonical map embeds S in $\mathbb{P}^d$ as a surface of degree d with $3 \leq d \leq 9$.

Pons-Llopis and Tonini studied aCM line bundles on every anticanonically embedded surface S in [18]. As an application, they prove that S is of wild representation type when $d \leq 6$ by constructing families of *Ulrich bundles*, that is, bundles $\mathcal{E}$ on S whose minimal free resolution as sheaves on $\mathbb{P}^d$ is linear (and thus initialised and aCM). For this reason, we say that anticanonically embedded surfaces of degree $d \leq 6$ are of *Ulrich-wild representation type*. Later, Miró-Roig and Pons-Llopis in [16] proved the Ulrich-wildness of anticanonically embedded surfaces $S \ncong \mathbb{P}^1 \times \mathbb{P}^1$ of degree $d \leq 8$ as a by-product of the construction of certain families of Ulrich bundles on them.

In [1] and [2], Casanellas and Hartshorne dealt with aCM bundles of arbitrary rank on smooth cubic surfaces, giving a complete description of the Ulrich ones. Some partial results on rank two Ulrich bundles on anticanonically embedded surfaces are also obtained by Coskun, Kulkarni and Mustopa in the context of the minimal resolution conjecture (see [7]).

The aim of this paper is to completely classify indecomposable rank two Ulrich bundles on anticanonically embedded surfaces. We recall some results on Ulrich bundles on a variety in Section 2. In Section 3, we list the results on anticanonically embedded surfaces that we need in the paper. In Section 4, we give the first results on simple Ulrich bundles on S. In Section 5, we find which are the admissible values for the first Chern class of an Ulrich bundle on S (see Proposition 5.1).

It is not difficult to construct Ulrich bundles of rank two on S with admissible class c_1 as direct sums of Ulrich line bundles. (A complete list can be found in [18, Theorem 4.2.2].) It is not evident that each admissible class can be obtained for an indecomposable Ulrich bundle. We show that this is the case in Theorem 6.7 (see Section 6), where the bundle can even be chosen to be stable.

In Section 7, we use this existence theorem and the results of Costa and Miró-Roig in [8] to describe the moduli spaces of rank two stable vector bundles with Chern classes c_1 and $c_2 = 2 - d + c_1^2/2$ for each admissible class c_1. As a by-product, we give two applications. First, we reprove the Ulrich-wildness of every anticanonically embedded surface (see Proposition 7.2). Then we prove the rationality of a moduli space of rank two Ulrich bundles on the Segre embedding $F \subseteq \mathbb{P}^7$ of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with some particular Chern classes (see Theorem 7.4), which improves the results of [5, Section 8].

For all the notation and standard results we refer to [13].

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2. General results on Ulrich bundles

In this section, we summarise some general results on Ulrich bundles on a smooth, irreducible, closed subscheme $X \subseteq \mathbb{P}^N$. In what follows, $\mathcal{O}_X(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$.

Definition 2.1. Let $X \subseteq \mathbb{P}^N$ be a smooth, irreducible, closed subscheme and let $\mathcal{F}$ be a vector bundle on X.

(a) $\mathcal{F}$ is initialised if $h^0(X, \mathcal{F}(-h)) = 0 \neq h^0(X, \mathcal{F})$.
(b) $\mathcal{F}$ is aCM if $h^i(X, \mathcal{F}(th)) = 0$ for each $t \in \mathbb{Z}$ and $i = 1, \dots, \dim(X) - 1$.
(c) $\mathcal{F}$ is Ulrich if $h^i(X, \mathcal{F}(-ih)) = h^j(X, \mathcal{F}(-(j+1)h)) = 0$ for each $i > 0$ and $j < \dim(X)$.

Ulrich bundles enjoy many important properties (see [10, Section 2]). They are initialised, aCM and globally generated; every direct summand of an Ulrich bundle is Ulrich, and $\mathcal{F}$ is Ulrich if and only if it has a linear minimal free resolution over $\mathbb{P}^N$.

Ulrich bundles also behave well with respect to the notions of (semi)stability and μ-(semi)stability. Recall that, for each bundle $\mathcal{F}$ on X, the slope $\mu(\mathcal{F})$ and the reduced Hilbert polynomial $p_{\mathcal{F}}(t)$ are defined as

$$\mu(\mathcal{F}) = c_1(\mathcal{F})h^{\dim(X)-1}/\mathrm{rk}(\mathcal{F}), \quad p_{\mathcal{F}}(t) = \chi(\mathcal{F}(th))/\mathrm{rk}(\mathcal{F}).$$

The bundle $\mathcal{F}$ is μ-semistable (respectively, μ-stable) if for all subsheaves $\mathcal{G}$ with $0 < \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$ we have $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$ (respectively, $\mu(\mathcal{G}) < \mu(\mathcal{F})$).

The bundle $\mathcal{F}$ is called semistable (respectively, stable) if for all $\mathcal{G}$, as above, $p_{\mathcal{G}}(t) \leq p_{\mathcal{F}}(t)$ (respectively, $p_{\mathcal{G}}(t) < p_{\mathcal{F}}(t)$) for $t \gg 0$. We recall that, in order to check the semistability and stability of a bundle, one can restrict attention only to subsheaves such that the quotient is torsion free.

The following chain of implications holds for $\mathcal{F}$.

$$\mathcal{F} \text{ is } \mu\text{-stable} \Rightarrow \mathcal{F} \text{ is stable} \Rightarrow \mathcal{F} \text{ is semistable} \Rightarrow \mathcal{F} \text{ is } \mu\text{-semistable}.$$

Theorem 2.2 [2, Theorem 2.9]. *Let $X \subseteq \mathbb{P}^N$ be a smooth, irreducible, closed subscheme. If $\mathcal{E}$ is an Ulrich bundle on X, then the following assertions hold.*

(a) *$\mathcal{E}$ is semistable and μ-semistable.*
(b) *$\mathcal{E}$ is stable if and only if it is μ-stable.*
(c) *If*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0$$

is an exact sequence of coherent sheaves with $\mathcal{M}$ torsion free and $\mu(\mathcal{L}) = \mu(\mathcal{E})$, then both $\mathcal{L}$ and $\mathcal{M}$ are Ulrich bundles.

Lemma 2.3 [6, Lemma 2.3]. *Let $X \subseteq \mathbb{P}^N$ be a smooth, irreducible, closed subscheme with $h^1(X, \mathcal{O}_X) = 0$. If $\mathcal{E}$ is an Ulrich bundle of rank two on X, then $\mathcal{E}$ is simple if and only if it is indecomposable.*

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3. General results on anticanonically embedded surfaces

We recall some facts on anticanonically embedded surfaces $S \subseteq \mathbb{P}^d$ of degree $d \geq 3$. We will denote by $\mathcal{O}_S(h)$ the hyperplane line bundle of the surface S.

By definition, $\omega_S \cong \mathcal{O}_S(-h)$, and hence $q(S) = p_a(S) = p_g(S) = 0$. The first important fact is that the Serre duality for each locally free sheaf $\mathcal{F}$ on S becomes

$$h^i(S, \mathcal{F}) = h^{2-i}(S, \mathcal{F}^\vee(-h)), \quad i = 0, 1, 2.$$

Moreover, the Riemann–Roch theorem on S is

$$h^0(S, \mathcal{F}) + h^2(S, \mathcal{F}) = h^1(S, \mathcal{F}) + \mathrm{rk}(\mathcal{F}) + \frac{c_1(\mathcal{F})^2}{2} + \frac{c_1(\mathcal{F})h}{2} - c_2(\mathcal{F}). \tag{3.1}$$

The following result summarises the needed characterisations for anticanonically embedded surfaces $S \subseteq \mathbb{P}^d$. Recall that up to six points in $\mathbb{P}^2$ are in general position if no three of them are collinear and six of them do not lie on the same conic.

LEMMA 3.1 [15, Theorems IV.24.3, IV.24.4, IV.24.5]. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface. Then the degree of S is d and S is either isomorphic to the blow up of $\mathbb{P}^2$ at $9-d$ points in general position embedded in $\mathbb{P}^d$ via the linear system of cubics through such points, or it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^8$ via $\mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(2)$. Conversely, each surface in $\mathbb{P}^d$ obtained in the above way is an anticanonically embedded surface of degree d.*

LEMMA 3.2 [18, Theorem 2.16]. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface. Then $\mathcal{O}_S(h)$ is aCM.*

The following results are classical. For their proofs when $d = 3$ see [13, Section IV.4]. The proofs given there can be easily generalised to cover all the cases $3 \leq d \leq 9$.

LEMMA 3.3. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface. If $S \subseteq \mathbb{P}^d$ is the blow up of $\mathbb{P}^2$ at $9-d$ points in general position and $\pi\colon S \to \mathbb{P}^2$ is the corresponding morphism, then the following assertions hold.*

(a) *$\mathrm{Pic}(S)$ is freely generated by the class ℓ of $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and by the classes $e_1, \dots, e_{9-d}$ of the exceptional divisors of π, that is, the inverse images of the blown-up points via π.*
(b) *$\ell^2 = 1$, $\ell e_i = 0$ and $e_i e_j = -\delta_{i,j}$ (the Kronecker symbol) for $i, j = 1, \dots, 9-d$.*
(c) *The hyperplane class h is $3\ell - \sum_{i=1}^{9-d} e_i$.*

If $S \subseteq \mathbb{P}^8$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then the following assertions hold.

(d) *$\mathrm{Pic}(S)$ is freely generated by two effective classes ℓ_1, ℓ_2 such that $\ell_1^2 = \ell_2^2 = 0$ and $\ell_1\ell_2 = 1$.*
(e) *The hyperplane class h is $2(\ell_1 + \ell_2)$.*

It is important to have information on the lines contained in S. For the proofs of the following results, see [18, Propositions 3.8 and 3.9] and the references quoted therein.

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Lemma 3.4. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface.*

(a) *If $S \subseteq \mathbb{P}^d$ is the blow up of $\mathbb{P}^2$ at $9-d$ points in general position, then the lines on S are the following: $e_1, \dots, e_{9-d}$, $\ell - e_i - e_j$ for $1 \le i < j \le 9-d$, $2l - \sum_{i=1}^5 e_i$ (if $d = 3, 4$) and $2l + e_j - \sum_{i=1}^6 e_i$ for $2 \le j \le 6$ (if $d = 3$).*

(b) *If $S \subseteq \mathbb{P}^8$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then it does not contain lines.*

In particular, each line $L \subseteq S$ (if any) satisfies $L^2 = -1$, and thus $h^0(S, \mathcal{O}_S(L)) = 1$. Hence S contains finitely many lines.

Lemma 3.5. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface. If $L_1, \dots, L_{9-d}$ are pairwise mutually disjoint lines, then there is a blow up morphism $\pi_{L_\bullet} : S \to \mathbb{P}^2$ of $9-d$ points in general position such that $L_1, \dots, L_{9-d}$ are the exceptional divisors of $\pi_{L_\bullet}$.*

4. Ulrich bundles on anticanonically embedded surfaces

In this section, we prove some general preliminary results about Ulrich bundles on anticanonically embedded surfaces $S \subseteq \mathbb{P}^d$ of degree $d \ge 3$, collected from [1, 2, 7]. The proof of the following result is completely analogous to the proof of [6, Lemma 4.1]. It slightly generalises [7, Propositions 2.10 and 2.11].

Lemma 4.1. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface. The following assertions are equivalent for a vector bundle $\mathcal{E}$ of rank r on S.*

(a) *$\mathcal{E}$ is Ulrich.*

(b) *$\mathcal{E}^\vee(2h)$ is Ulrich.*

(c) *$\mathcal{E}$ is aCM and*

$$c_1(\mathcal{E})h = dr, \quad c_2(\mathcal{E}) = \frac{c_1(\mathcal{E})^2 - (d-2)r}{2}. \tag{4.1}$$

(d) *$h^0(S, \mathcal{E}(-h)) = h^0(S, \mathcal{E}^\vee(h)) = 0$ and the equalities (4.1) hold.*

Lemma 4.2 [2, Corollary 2.13]. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Ulrich bundles on an anticanonically embedded surface $S \subseteq \mathbb{P}^d$. Then*

$$\chi(\mathcal{E}_1 \otimes \mathcal{E}_2^\vee) = (d-1)\mathrm{rk}(\mathcal{E}_1)\mathrm{rk}(\mathcal{E}_2) - c_1(\mathcal{E}_1)c_1(\mathcal{E}_2). \tag{4.2}$$

Let $\mathcal{E}$ be a simple vector bundle of rank r on S. Then $h^0(S, \mathcal{E} \otimes \mathcal{E}^\vee) = 1$ and $h^2(S, \mathcal{E} \otimes \mathcal{E}^\vee) = h^0(S, \mathcal{E} \otimes \mathcal{E}^\vee(-h))$. If H is any hyperplane section of S, then it is clear that the natural restriction map $H^0(S, \mathcal{E} \otimes \mathcal{E}^\vee) \to H^0(H, \mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{O}_H)$ is injective. Thus

$$h^2(S, \mathcal{E} \otimes \mathcal{E}^\vee) = h^0(S, \mathcal{E} \otimes \mathcal{E}^\vee(-h)) = 0.$$

Taking $\mathcal{H} = \mathcal{E}$ in (4.2) gives

$$h^1(S, \mathcal{E} \otimes \mathcal{E}^\vee) = 1 - \chi(\mathcal{E}) = c_1(\mathcal{E})^2 - (d-1)r^2 + 1. \tag{4.3}$$

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Proposition 4.3. *Let $\mathcal{E}$ be an Ulrich bundle of rank r on an anticanonically embedded surface $S \subseteq \mathbb{P}^d$. Then $c_1(\mathcal{E})^2$ is an even integer satisfying*

$$(d-2)r^2 \leq c_1(\mathcal{E})^2 \leq dr^2,$$

where $c_1(\mathcal{E})^2 = dr^2$ if and only if $c_1(\mathcal{E}) = rh$. If $\mathcal{E}$ is simple, then $(d-1)r^2 - 1 \leq c_1(\mathcal{E})^2$.

Proof. Trivially, $c_1(\mathcal{E})^2$ must be even because (see Lemma 4.1)

$$c_2(\mathcal{E}) = \frac{c_1(\mathcal{E})^2 - (d-2)r}{2}$$

is an integer. For the first chain of inequalities, see [7, Proposition 2.21].

If $c_1(\mathcal{E})^2 = dr^2$, the Hodge index theorem implies that the divisors h and $c_1(\mathcal{E})$ are numerically dependent in $\mathrm{Pic}(S)$. Thus they are linearly dependent and $c_1(\mathcal{E}) = \pm rh$. Since $\mathcal{E}$ is globally generated, the same is true for $\det(\mathcal{E})$, and so $c_1(\mathcal{E}) = \pm rh$. The converse is also trivial. Finally, if $\mathcal{E}$ is simple, it follows from (4.3) that

$$c_1(\mathcal{E})^2 - (d-1)r^2 + 1 = h^1(S, \mathcal{E} \otimes \mathcal{E}^\vee) \geq 0,$$

that is, $(d-1)r^2 - 1 \leq c_1(\mathcal{E})^2$. □

5. Admissible Chern classes for indecomposable Ulrich bundles of rank two

Let $\mathcal{E}$ be an Ulrich bundle on S of rank two. If $\mathcal{E}$ is indecomposable, it is simple (see Lemma 2.3), and thus Proposition 4.3 implies that $c_1(\mathcal{E})^2 \geq 4(d-1)$. In particular, if $c_1(\mathcal{E})^2 \leq 4(d-1) - 1$, then $\mathcal{E} \cong \mathcal{O}_S(A) \oplus \mathcal{O}_S(B)$, and hence $\mathcal{O}_S(A)$ and $\mathcal{O}_S(B)$ are both Ulrich. We will now deal with indecomposable Ulrich bundles of rank two on S.

Proposition 5.1. *Let $\mathcal{E}$ be an indecomposable Ulrich bundle of rank two on an anticanonically embedded surface $S \subseteq \mathbb{P}^d$, $S \not\cong \mathbb{P}^1 \times \mathbb{P}^1$. Then, for each d, its first Chern class $c_1 := c_1(\mathcal{E})$ is either $2h = 6\ell - \sum_{i=1}^{9-d} e_i$ (if $c_1^2 = 4d$) or, up to an automorphism of S, it is in the following list (if $4d - 4 \leq c_1^2 \leq 4d - 2$).*

$d = 3$: *$5\ell - 2\sum_{i=1}^{3} e_i - \sum_{i=4}^{6} e_i$ if $c_1^2 = 10$; $5\ell - 2\sum_{i=1}^{4} e_i - e_5$ if $c_1^2 = 8$.*
$d = 4$: *$5\ell - 2\sum_{i=1}^{2} e_i - \sum_{i=3}^{5} e_i$ if $c_1^2 = 14$; $5\ell - 2\sum_{i=1}^{3} e_i - e_4$ if $c_1^2 = 12$.*
$d = 5$: *$5\ell - 2e_1 - \sum_{i=2}^{4} e_i$ if $c_1^2 = 18$; $5\ell - 2\sum_{i=1}^{2} e_i - e_3$ if $c_1^2 = 16$.*
$d = 6$: *$5\ell - \sum_{i=1}^{3} e_i$ or $6\ell - 3e_1 - 2e_2 - e_3$ if $c_1^2 = 22$; $5\ell - 2e_1 - e_2$ if $c_1^2 = 20$.*
$d = 7$: *$6\ell - 3e_1 - e_2$ if $c_1^2 = 26$: $5\ell - e_1$ if $c_1^2 = 24$.*

In all the above cases

$$c_2(\mathcal{E}) = 2 - d + \tfrac{1}{2}c_1(\mathcal{E})^2.$$

Proof. The equality $c_2(\mathcal{E}) = 2 - d + c_1^2/2$ follows from (4.1). We give the details of the proof in the case $d = 3$, suggesting the necessary changes for the cases $d \geq 4$.

Let $c_1 = \alpha\ell - \sum_{i=1}^{9-d} \beta_i e_i$. Since $\mathcal{E}$ is globally generated, the same is true for $\mathcal{O}_S(c_1)$, and hence $\beta_i = c_1 e_i \geq 0$, $i = 1, \dots, 9-d$. Thus we can assume that $(\beta_1, \dots, \beta_{9-d})$ is a

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nonincreasing sequence of positive integers. Moreover,

$$c_1(\mathcal{E}^\vee(2h)) = (12-\alpha)\ell - \sum_{i=1}^{9-d}(4-\beta_i)e_i,$$

and thus bundles with $\alpha \geq 7$ are in one-to-one correspondence with bundles with $\alpha \leq 5$. Hence we can also restrict our attention to $\alpha \leq 6$. The Cauchy–Schwartz inequality for the two vectors $(\beta_1, \ldots, \beta_{9-d})$ and $(1, \ldots, 1)$ gives

$$\Big(\sum_{i=1}^{9-d}\beta_i\Big)^2 \leq (9-d)\Big(\sum_{i=1}^{9-d}\beta_i^2\Big). \tag{5.1}$$

From (4.1) and Proposition 4.3,

$$3\alpha - \sum_{i=1}^{9-d}\beta_i = 2d, \quad 4(d-1) \leq c_1^2 = \alpha^2 - \sum_{i=1}^{9-d}\beta_i^2 \leq 4d. \tag{5.2}$$

Thus $\sum_{i=1}^{9-d}\beta_i = 3\alpha - 2d$ and $\sum_{i=1}^{9-d}\beta_i^2 = \alpha^2 - c_1^2$. When $d=3$, $4(d-1)=8$ and $4d=12$, and so substituting the above identities in (5.1) yields

$$3(\alpha-6)^2 - 72 + 6c_1^2 \leq 0. \tag{5.3}$$

If $c_1^2 = 12$, then $\alpha = 6$, and thus (5.2) becomes $\sum_{i=1}^{6}\beta_i = 12$ and $\sum_{i=1}^{6}\beta_i^2 = 24$. In particular, the second equality implies that $\beta_1 \leq 4$. By writing all the possible sequences, one checks that $(\alpha, \beta_1, \ldots, \beta_6) = A := (6, 2, \ldots, 2)$. Thus $c_1 = 2h$ in this case.

If $c_1^2 = 10$, then (5.3) becomes $(\alpha-6)^2 - 4 \leq 0$, and thus $4 \leq \alpha \leq 6$. If $\alpha = 4$, then $\sum_{i=1}^{6}\beta_i = 6$ and $\sum_{i=1}^{6}\beta_i^2 = 6$, and thus $\beta_1 \leq 2$. The unique solution is $B := (4, 1, \ldots, 1)$. If $\alpha = 5$, then $\sum_{i=1}^{6}\beta_i = 9$ and $\sum_{i=1}^{6}\beta_i^2 = 15$, and thus $\beta_1 \leq 3$. The solution is $B' := (5, 2, 2, 2, 1, 1, 1)$. Finally, if $\alpha = 6$, then $\sum_{i=1}^{6}\beta_i = 12$ and $\sum_{i=1}^{6}\beta_i^2 = 26$. Arguing as before, $(\alpha, \beta_1, \ldots, \beta_6) = B'' := (6, 3, 2, 2, 2, 2, 1)$.

If $c_1^2 = 8$, then (5.3) becomes $(\alpha-6)^2 - 8 \leq 0$, and thus again $4 \leq \alpha \leq 6$. If $\alpha = 4$, then $\sum_{i=1}^{6}\beta_i = 6$ and $\sum_{i=1}^{6}\beta_i^2 = 8$, and the unique solution is $C := (4, 2, 1, 1, 1, 1, 0)$. If $\alpha = 5$, then $\sum_{i=1}^{6}\beta_i = 9$ and $\sum_{i=1}^{6}\beta_i^2 = 17$: in this case, we have the two solutions $C' := (5, 3, 2, 1, 1, 1, 1)$ and $C'' := (5, 2, 2, 2, 2, 1, 0)$. If $\alpha = 6$, then $\sum_{i=1}^{6}\beta_i = 12$ and $\sum_{i=1}^{6}\beta_i^2 = 28$, and we have the solution $C''' := (6, 3, 3, 2, 2, 1, 1)$.

We now choose the pairwise mutually disjoint lines

$$\overline{e}_1 := \ell - e_2 - e_3, \quad \overline{e}_2 := \ell - e_1 - e_3, \quad \overline{e}_3 := \ell - e_1 - e_2,$$
$$\overline{e}_4 := e_4, \quad \overline{e}_5 := e_5, \quad \overline{e}_6 := e_6.$$

Lemma 3.5 guarantees the existence of a blow up $S \to \mathbb{P}^2$ such that the above lines are its exceptional divisors. Thus $\mathrm{Pic}(S)$ is freely generated by the classes of the above lines and by a further divisor $\overline{\ell}$ which must satisfy the equality

$$3\overline{\ell} - \sum_{i=1}^{6}\overline{e}_i = h = 3\ell - \sum_{i=1}^{6}e_i$$

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(see Lemma 3.3). A substitution yields $\overline{\ell} = 2\ell - e_1 - e_2 - e_3$. The idempotent matrix

$$M := \begin{pmatrix} 2 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is the basis change matrix from $\overline{\mathcal{B}} := (\overline{\ell}, -\overline{e_1}, \dots, -\overline{e_6})$ to $\mathcal{B} := (\ell, -e_1, \dots, -e_6)$: that is, if X are the components of c_1 with respect to $\overline{\mathcal{B}}$, then the components of c_1 with respect to $\mathcal{B}$ are $X^t M$.

It is straightforward to check that $B'^t M$ and $B''^t M$ are, up to permutations of the last six components, B and B'. Thus, up to a proper choice of the blow up base, we can always assume that $c_1 = 5\ell - 2\sum_{i=1}^{3} e_i - \sum_{i=4}^{6} e_i$ if $c_1^2 = 10$. Similarly, $C'^t M$, $C''^t M$ and $C'''^t M$ are C up to permutations of the last six components. As in the previous case, we can thus assume that $c_1 = 5\ell - 2\sum_{i=1}^{4} e_i - e_5$ if $c_1^2 = 8$.

If $d = 4$, then the same argument as above using the basis

$$\overline{e}_1 := \ell - e_2 - e_3, \quad \overline{e}_2 := \ell - e_1 - e_3, \quad \overline{e}_3 := \ell - e_1 - e_2, \quad \overline{e}_4 := e_4, \quad \overline{e}_5 := e_5,$$

leads to $c_1 = 6\ell - 2\sum_{i=1}^{5} e_i = 2h$ if $c_1^2 = 16$, $c_1 = 5\ell - 2e_1 - 2e_2 - \sum_{i=3}^{5} e_i$ if $c_1^2 = 14$, and $c_1 = 5\ell - 2\sum_{i=1}^{3} e_i - e_4$ if $c_1^2 = 12$.

If $d = 5$, then using the basis

$$\overline{e}_1 := \ell - e_2 - e_3, \quad \overline{e}_2 := \ell - e_1 - e_3, \quad \overline{e}_3 := \ell - e_1 - e_2, \quad \overline{e}_4 := e_4,$$

we can restrict to the cases $c_1 = 6\ell - 2\sum_{i=1}^{5} e_i = 2h$ if $c_1^2 = 20$, $c_1 = 5\ell - 2e_1 - \sum_{i=2}^{4} e_i$ if $c_1^2 = 18$, and $c_1 = 5\ell - 2\sum_{i=1}^{2} e_i - e_3$ if $c_1^2 = 16$.

When $d \geq 6$, the change of basis does not shorten the list of possible c_1. Indeed, if $d = 6$, $c_1 = 6\ell - 2\sum_{i=1}^{3} e_i = 2h$ if $c_1^2 = 24$, c_1 is either $5\ell - \sum_{i=1}^{3} e_i$ or $6\ell - 3e_1 - 2e_2 - e_3$ if $c_1^2 = 22$, and $c_1 = 5\ell - 2e_1 - e_2$ if $c_1^2 = 20$. If $d = 7$, then $c_1 = 6\ell - 2e_1 - 2e_2 = 2h$ if $c_1^2 = 28$, $c_1 = 6\ell - 3e_1 - e_2$ if $c_1^2 = 26$, and $c_1 = 5\ell - e_1$ if $c_1^2 = 24$. If $d = 8$, then only the case $c_1 = 2h$ is admissible and $c_1^2 = 32$. Finally, when $d = 9$, $\mathrm{Pic}(S) \cong \mathbb{Z}$ is generated by h, and thus $c_1 = \alpha h$. Now $c_1 h = 18$ trivially implies $c_1 = 2h$, and hence $c_1^2 = 36$. □

Now we turn our attention to the remaining anticanonically embedded surface, that is, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ which has degree eight.

Proposition 5.2. *Let $\mathcal{E}$ be an indecomposable Ulrich bundle of rank two on the anticanonically embedded surface $S \subseteq \mathbb{P}^8$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Up to an automorphism of S, $c_1 := c_1(\mathcal{E})$ is either $3\ell_1 + 5\ell_2$ or $2h$.*

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Proof. We know that $c_1 = \alpha_1\ell_1 + \alpha_2\ell_2$ for suitable integers α_i, $i = 1, 2$, which must be nonnegative because $\mathcal{E}$ is globally generated, and hence the same is true for $\mathcal{O}_S(c_1)$. Moreover, $h = 2(\ell_1 + \ell_2)$ and $c_1 h = 16$, and thus $\alpha_1 + \alpha_2 = 8$. By computing c_1^2, Proposition 4.3 yields the statement. □

Each Chern class listed in the above statements will be called an *admissible class*.

Remark 5.3. The existence of rank two Ulrich bundles with $c_1 = 2h$ is already known (see [10, Corollary 6.5]). In [7, Proposition 3.7], the authors prove the existence on each anticanonically embedded surface $S \subseteq \mathbb{P}^d$ with $d \le 7$ of Ulrich bundles of rank two whose first Chern class is $h + C$, where C is a rational normal curve of degree d. The adjunction formula on S implies that $C^2 = d - 2$: thus $c_1^2 = 4d - 2$ for such bundles.

In the next section we will reprove these facts when $d \le 8$. We also prove the existence of Ulrich bundles of rank two such that $c_1^2 = 4d - 4$.

6. Existence of indecomposable Ulrich bundles of rank two

In this section, we construct stable Ulrich bundles of rank two on an anticanonically embedded surface $S \subseteq \mathbb{P}^d$ of degree $d \le 8$ whose first Chern class is admissible.

We will make use of the Hartshorne–Serre correspondence on surfaces.

Theorem 6.1 [14, Theorem 5.1.1]. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface and let $Z \subseteq S$ be a locally complete intersection subscheme of dimension zero. If $\mathcal{O}_S(A)$ is initialised, then there exists a vector bundle $\mathcal{F}$ of rank two on S with* $\det(\mathcal{F}) = \mathcal{O}_S(A)$ *and with a section s such that $Z = (s)_0$.*

Remark 6.2. Let Z and S be as in Theorem 6.1. The theorem implies the existence of an exact Koszul complex

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Z(A) \longrightarrow 0, \tag{6.1}$$

where $\mathcal{F}$ is a vector bundle and $\mathcal{I}_Z$ denotes the ideal sheaf of Z inside S. The proof in [14] shows that each general choice of $\xi \in \operatorname{Ext}^1_S(\mathcal{I}_E(A), \mathcal{O}_S) \cong H^1(S, \mathcal{I}_E(A - h))^\vee$ returns a bundle in the above way: thus the extensions are parameterised by an open subset of $\mathbb{P}(H^1(S, \mathcal{I}_E(A - h)))$.

For each divisor A on S, the adjunction formula implies that $A^2 - Ah$ is an even integer.

Proposition 6.3. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface and let $\mathcal{O}_S(A)$, $\mathcal{O}_S(D)$ be initialised line bundles on S with $h^1(S, \mathcal{O}_S(D - h)) = 0$, $Ah = 2a$ and $Dh = d - a$. Let $Z \subseteq S$ be a locally complete intersection subscheme of dimension zero and degree $z := AD + (A^2 - Ah)/2$ that is not contained in any curve of degree a on S. Then there is an Ulrich bundle $\mathcal{E}$ of rank two on S with $c_1(\mathcal{E}) = A + 2D$. Further, $\mathcal{E}(-D)$ has a section with Z as its zero-locus. The bundle $\mathcal{E}$ is stable if and only if $a > 0$.*

PROOF. The line bundle $\mathcal{O}_S(D)$ is initialised, so it is effective, and $h^2(S, \mathcal{O}_S(D-h)) = h^0(S, \mathcal{O}_S(-D)) = 0$. Since $h^0(S, \mathcal{O}_S(D-h)) = h^1(S, \mathcal{O}_S(D-h)) = 0$ (by hypothesis), it follows that $D^2 = d - a - 2$ from (3.1). By hypothesis $h^0(S, \mathcal{O}_S(A-h)) = 0$, thus there is a rank two bundle $\mathcal{F}$ on S that fits into the sequence (6.1). Trivially, $c_1(\mathcal{F}) = A$ and $c_2(\mathcal{F}) = z$. The bundle $\mathcal{E} := \mathcal{F}(D)$ fits into the sequence

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z(D+A) \longrightarrow 0. \tag{6.2}$$

We have $c_1(\mathcal{E}) = c_1(\mathcal{F}) + 2D = A + 2D$, and hence $c_1(\mathcal{E})h = Ah + 2Dh = 2d$. Thus

$$\begin{aligned} c_2(\mathcal{E}) &= c_2(\mathcal{F}) + c_1(\mathcal{F})D + D^2 = AD + \frac{A^2 - Ah}{2} + AD + D^2 \\ &= \frac{(A+2D)^2}{2} + 2 - d = \frac{c_1(\mathcal{E})^2}{2} + 2 - d. \end{aligned}$$

We deduce that both the equalities (4.1) hold.

Since $(D + A - h)h = a$, the hypotheses on Z imply that $h^0(S, \mathcal{I}_Z(D + A - h)) = 0$. The cohomology of sequence (6.2) tensored by $\mathcal{O}_S(-h)$ implies that $h^0(S, \mathcal{E}(-h)) = 0$. Tensoring sequence (6.2) by $\mathcal{O}_S(h - A - 2D)$ and noting $\mathcal{E}(-A - 2D) \cong \mathcal{E}^\vee$ gives

$$0 \longrightarrow \mathcal{O}_S(h - A - D) \longrightarrow \mathcal{E}^\vee(h) \longrightarrow \mathcal{I}_Z(h - D) \longrightarrow 0.$$

We have $(h - D - A)h = -a$, and hence $h^0(S, \mathcal{O}_S(h - D - A)) = 0$. Moreover, $(h - D)h = a$. The hypotheses on Z again imply that $h^0(S, \mathcal{I}_Z(h - D)) = 0$, and hence $h^0(S, \mathcal{E}^\vee(h)) = 0$. We conclude that the above construction yields an Ulrich bundle $\mathcal{E}$ of rank two on S with $c_1(\mathcal{E}) = A + 2D$.

We have to prove that it is stable or, equivalently (see Theorem 2.2), μ-stable if and only if $a > 0$. If $a = 0$, then $\mu(D) = d = \mu(\mathcal{E})$, and hence $\mathcal{E}$ is not μ-stable because $\mathcal{O}_S(D)$ is a subbundle of $\mathcal{E}$ (see sequence (6.2)).

Assume that $a > 0$. If $\mathcal{E}$ is not μ-stable, by Theorem 2.2(a), it is strictly μ-semistable, and thus it fits into a sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0,$$

where $\mu(\mathcal{L}) = \mu(\mathcal{E}) = d$ and $\mathcal{M}$ is nonzero and torsion free. Thus Theorem 2.2(c) implies that $\mathcal{L} \cong \mathcal{O}_S(B)$ is an Ulrich line bundle on S. Suppose the injection $\mathcal{O}_S(B) \to \mathcal{E}$ induces a nonzero morphism $\mathcal{O}_S(B) \to \mathcal{I}_Z(D + A)$. Then $h^0(S, \mathcal{I}_Z(D + A - B)) \neq 0$, which contradicts the choice of Z because $(D + A - B)h = a$.

Thus $\mathcal{O}_S(B) \to \mathcal{I}_Z(D + A)$ must be zero. It follows that we can lift the injection $\mathcal{O}_S(B) \to \mathcal{E}$ to a nonzero map $\mathcal{O}_S(B) \to \mathcal{O}_S(A)$. In particular, we should have $h^0(S, \mathcal{O}_S(A - B)) \neq 0$, which is not possible because $Bh = d$ and $Ah = d - a < d$. It follows that there are no nonzero morphisms $\mathcal{O}_S(B) \to \mathcal{E}$, and hence $\mathcal{E}$ is μ-stable. □

EXAMPLE 6.4. We prove the existence of stable rank two Ulrich bundles $\mathcal{E}$ with $c_1(\mathcal{E}) = 2h$ on each anticanonically embedded surface $S \not\cong \mathbb{P}^1 \times \mathbb{P}^1$ based on Proposition 6.3.

Assume first that $d \leq 8$ and let

$$D := 3\ell - 2e_1 - \sum_{i=2}^{9-d} e_i, \quad A := 2e_1.$$

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Then $a = 1, A^2 - Ah + 2AD = 2$ and the divisor A is trivially initialised. Also, $D = h - e_1$, and thus D is initialised and (3.1) for $D - h$ yields $h^1(S, \mathcal{O}_S(D - h)) = 0$. The complement $S_0 \subseteq S$ of the union of lines is dense in S (see Lemma 3.4). Thus, for each point $P \in S_0$, the choice of the scheme $Z := \{P\}$ allows us to apply Proposition 6.3, showing the existence of stable Ulrich bundles $\mathcal{E}$ on S such that

$$c_1(\mathcal{E}) = 6\ell - 2\sum_{i=1}^{9-d} e_i, \quad c_2(\mathcal{E}) = d + 2.$$

Now let $d = 9$ and choose $A = D = 2\ell$. Trivially, $\mathcal{O}_S(A) \cong \mathcal{O}_S(D)$ is initialised and, from (3.1), $h^1(S, \mathcal{O}_S(A - h)) = 0$. In this case, $a = 3$ and $A^2 - Ah + 2AD = 6$. Since each curve of degree three in S is necessarily in the linear system $|\ell|$, which has dimension two, a general choice of a reduced scheme $Z \subseteq S$ of degree six and dimension zero leads again to a rank two stable Ulrich bundle $\mathcal{E}$ on S such that $c_1(\mathcal{E}) = 2h$ and $c_2(\mathcal{E}) = 11$.

When $d = 9$, each Ulrich bundle $\mathcal{E}$ of rank two on S with $c_1(\mathcal{E}) = 2h$ satisfies $c_2(\mathcal{E}) = 11$. By Lemma 4.1, these bundles are exactly aCM bundles. It follows from [3, Example 5.6] that $\mathcal{E} \cong \mathcal{F}(h)$, where $\mathcal{F}$ is a stable bundle of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = 2$.

Example 6.5. We will now complete our list of examples of stable Ulrich bundles on each anticanonically embedded surface $S \ncong \mathbb{P}^1 \times \mathbb{P}^1$ for the remaining admissible classes. The second table of [18, Theorem 4.2.2] gives the complete list of initialised aCM bundles on S. In particular, the line bundles $\mathcal{O}_S(D)$ listed in the Table 1 below are initialised and aCM of degree $Dh = d - 1$ on each anticanonically embedded surface S of degree d. It is easy to check that the line bundles $\mathcal{O}_S(A)$ listed below are initialised of degree $Ah = 2$. Finally $A^2 - Ah + 2AD = 2$. Choosing $Z := \{ P \}$, as in Example 6.4, one can immediately check that all the hypotheses of Proposition 6.3 are satisfied. As in Example 6.4, let $P \in S$ be a point not lying on any line in S.We again obtain a stable Ulrich bundle $\mathcal{E}$ of rank two such that $c_1(\mathcal{E}) = c_1$.

Example 6.6. Now consider the anticanonically embedded surface $S \subseteq \mathbb{P}^8$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. By Proposition 5.2, the admissible classes are $3\ell_1 + 5\ell_2$ and $4(\ell_1 + \ell_2)$.

If $D := \ell_1 + 2\ell_2$, then the line bundle $\mathcal{O}_S(D)$ is trivially initialised with respect to $\mathcal{O}_S(h) \cong \mathcal{O}_S(2\ell_1 + 2\ell_2)$. The surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ can be embedded in $\mathbb{P}^3$ via $\mathcal{O}_Q(\ell_1 + \ell_2)$ as a smooth quadric. Via such an embedding, the curve D is a rational cubic. It is well known that $h^1(\mathbb{P}^3, \mathcal{I}_{D|\mathbb{P}^3}(t)) = 0$ for $t \in \mathbb{Z}$ for such a curve. Since $\mathcal{I}_{S|\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-2)$, it follows from the exact sequence

$$0 \longrightarrow \mathcal{I}_{S|\mathbb{P}^3}(t) \longrightarrow \mathcal{I}_{D|\mathbb{P}^3}(t) \longrightarrow \mathcal{I}_{D|S}(th) \longrightarrow 0$$

that $h^1(S, \mathcal{I}_{D|S}(th)) = 0$ for $t \in \mathbb{Z}$. In particular, $\mathcal{O}_S(-D) \cong \mathcal{I}_{D|S}$ is aCM with respect to $\mathcal{O}_S(\ell_1 + \ell_2)$, and thus it is trivially aCM also with respect to $\mathcal{O}_S(h) \cong \mathcal{O}_S(2\ell_1 + 2\ell_2)$. Serre duality finally implies that $\mathcal{O}_S(D)$ is aCM as a line bundle on $S \subseteq \mathbb{P}^8$.

Set $A := \ell_1 + \ell_2$ (respectively, $2\ell_1$) if $c_1 = 3\ell_1 + 5\ell_2$ (respectively, $c_1 = 4(\ell_1 + \ell_2)$), so that $a = 2$. It is clear that $\mathcal{O}_S(A)$ is initialised, and $A^2 - Ah + 2AD = 4$. Notice that

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TABLE 1. Line bundles in Example 6.5.

d	A	D	$c_1 = A + 2D$
3	$\ell + e_4 - e_5 - e_6$	$2\ell - e_1 - e_2 - e_3 - e_4$	$5\ell - 2\sum_{i=1}^{3} e_i - \sum_{i=4}^{6} e_i$
3	$\ell - e_5$	$2\ell - e_1 - e_2 - e_3 - e_4$	$5\ell - 2\sum_{i=1}^{4} e_1 - e_5$
4	$\ell + e_3 - e_4 - e_5$	$2\ell - e_1 - e_2 - e_3$	$5\ell - 2\sum_{i=1}^{2} e_i - \sum_{i=3}^{5} e_i$
4	$\ell - e_4$	$2\ell - e_1 - e_2 - e_3$	$5\ell - 2\sum_{i=1}^{3} e_i - e_4$
5	$\ell + e_2 - e_3 - e_4$	$2\ell - e_1 - e_2$	$5\ell - 2e_1 - \sum_{i=2}^{4} e_i$
5	$\ell - e_3$	$2\ell - e_1 - e_2$	$5\ell - 2\sum_{i=1}^{2} e_i - e_3$
6	$\ell + e_1 - e_2 - e_3$	$2\ell - e_1$	$5\ell - \sum_{i=1}^{3} e_i$
6	$2\ell - e_1 - 2e_2 - e_3$	$2\ell - e_1$	$6\ell - 3e_1 - 2e_2 - e_3$
6	$\ell - e_2$	$2\ell - e_1$	$5\ell - 2e_1 - e_2$
7	$e_1 + e_2$	$3\ell - 2e_1 - e_2$	$6\ell - 3e_1 - e_2$
7	$\ell - e_1$	2ℓ	$5\ell - e_1$

the curves of degree two on $S \subseteq \mathbb{P}^8$ are exactly the ones in $|\ell_1| \cup |\ell_2|$, and thus each point on S is contained in exactly two curves of degree two. Thus there is a subscheme Z of degree two not contained in any curve of degree two on S. Again, by Proposition 6.3, we obtain the existence of rank two stable Ulrich bundles $\mathcal{E}$ on S such that $c_1(\mathcal{E}) = 3\ell_1 + 5\ell_2$ (respectively, $4(\ell_1 + \ell_2)$) and $c_2 = 9$ (respectively, $c_2 = 10$).

Examples 6.4–6.6 provide a proof of the following result.

THEOREM 6.7. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface. For each admissible class c_1, there exists a stable Ulrich bundle $\mathcal{E}$ of rank two such that $c_1(\mathcal{E}) = c_1$.*

7. Moduli spaces

In this section, we will deal with the moduli spaces of the bundles constructed in Theorem 6.7. We start with the following proposition.

PROPOSITION 7.1. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface. For each admissible class c_1, the moduli space $\mathcal{M}_S^s(2; c_1, c_2)$ of rank two vector bundles $\mathcal{E}$ on S with Chern classes*

$$c_1(\mathcal{E}) = c_1, \quad c_2(\mathcal{E}) = \frac{c_1^2}{2} + 2 - d,$$

which are stable with respect to ω_S^{-1}, is nonempty, rational and irreducible of dimension $c_1^2 + 5 - 4d$. The general point in $\mathcal{M}_S^s(2; c_1, c_2)$ is Ulrich.

PROOF. The moduli space $\mathcal{M}_S^s(2; c_1, c_2)$ is nonempty and it contains at least one Ulrich bundle by Theorem 6.7. By [8, Theorems 4.2.4 and 4.3.8], it is rational and irreducible of dimension $c_1^2 + 5 - 4d$. (The hypothesis $c_2 \gg 0$ in [8, Theorem 4.3.8] is used only at the end of the proof of Proposition 4.3.6. Nevertheless, it is actually unnecessary

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for proving that statement and hence also [8, Theorem 4.3.8].) Finally, the property of a bundle to be Ulrich is open in the moduli space by semicontinuity, and thus the general point in $\mathcal{M}_S^s(2; c_1, c_2)$ is Ulrich. □

As a first application of the above proposition we deal with the Ulrich-wildness of anticanonically embedded surfaces. More precisely, we give the following very short proof of a result proved in [18] when $d \le 6$ and in [16] when $d \le 8$.

Proposition 7.2. *Let $S \subseteq \mathbb{P}^d$ be an anticanonically embedded surface. Then S is of Ulrich-wild representation type.*

Proof. We take two nonisomorphic stable Ulrich bundles of rank two, say, $\mathcal{E}_1$ and $\mathcal{E}_2$, on S with Chern classes $c_1 = 2h$ and $c_2 = d + 2$. Since they are nonisomorphic and stable, it follows from [14, Proposition 1.2.7] that

$$h^0(S, \mathcal{E}_1 \otimes \mathcal{E}_2^\vee) = h^0(S, \mathcal{E}_1^\vee \otimes \mathcal{E}_2) = 0,$$

and hence

$$h^2(S, \mathcal{E}_1 \otimes \mathcal{E}_2^\vee) = h^0(S, \mathcal{E}_1^\vee \otimes \mathcal{E}_2(-h)) \le h^0(S, \mathcal{E}_1^\vee \otimes \mathcal{E}_2) = 0.$$

Thus (4.2) implies that $h^1(S, \mathcal{E}_1 \otimes \mathcal{E}_2^\vee) = -\chi(\mathcal{E}_1 \otimes \mathcal{E}_2^\vee) = 4$. We conclude that S is Ulrich-wild, by [12, Theorem 1 and Corollary 1]. □

As a second application of Proposition 7.1, we deal with certain rank two Ulrich bundles on the image $F \subseteq \mathbb{P}^7$ of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In what follows, we set $\mathcal{O}_F(H) := \mathcal{O}_F \otimes \mathcal{O}_{\mathbb{P}^7}(1)$. The group $\mathrm{Pic}(F)$ is freely generated by the classes of the fibres h_1, h_2 and h_3 of the three projections $F \to \mathbb{P}^1$. Moreover, $\mathcal{O}_F(h_1 + h_2 + h_3) \cong \mathcal{O}_F(H)$ and $\omega_F \cong \mathcal{O}_F(-2h_1 - 2h_2 - 2h_3) \cong \mathcal{O}_F(-2H)$.

From now on, we will assume that the characteristic of k is zero in order to make use of the results proved in [4] and [5]. In those papers, both the complete description of aCM vector bundles of rank two on F and the construction of their moduli spaces are given. In particular, it is shown that if $\mathcal{E}$ is an Ulrich bundle on F, then the pair $(c_1(\mathcal{E}), c_2(\mathcal{E}))$ is necessarily one of the following (up to permutations of the h_i).

$$\begin{aligned} &(2H, 2h_2h_3 + 2h_1h_3 + 4h_1h_2), && (2H, 2h_2h_3 + 3h_1h_3 + 3h_1h_2),\\ &(h_1 + 2h_2 + 3h_3, 3h_2h_3 + 3h_1h_3 + h_1h_2), && (h_1 + 2h_2 + 3h_3, 4h_2h_3 + h_1h_3 + 2h_1h_2). \end{aligned}$$

In the first three cases, the moduli space $\mathcal{M}_F^{s,U}(2; c_1(\mathcal{E}), c_2(\mathcal{E}))$ of stable Ulrich bundles with respect to H is nonempty, unirational and irreducible and it has dimension $(4c_2(\mathcal{E}) - c_1^2(\mathcal{E}))h - 3$ (see [5, Propositions 6.4 and 6.5]). In the fourth case, all the rank two Ulrich bundles $\mathcal{E}$ are strictly semistable and form a family isomorphic to $\mathbb{P}^3$ (see [5, Proposition 6.2]). Moreover, all such bundles are S-equivalent (see [14] for the definition of S-equivalence). In [5, Section 8], the rationality of the moduli space $\mathcal{M}_F^{s,U}(2; 2H, 2h_2h_3 + 2h_1h_3 + 4h_1h_2)$ is proved by constructing a rational map to a suitable moduli space of bundles of rank two on a quadric surface which is known to be nonempty and rational due to an old result (see [19]).

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We reproduce such an argument for $\mathcal{M}_F^{s,U}(2; h_1 + 2h_2 + 3h_3, 3h_2h_3 + 3h_1h_3 + h_1h_2)$, starting from the analogous properties of $\mathcal{M}_S^s(2; 3\ell_1 + 5\ell_2, 9)$ listed in Proposition 7.1. Let $\varphi\colon F \to \mathbb{P}^1$ be the projection onto the second factor and $\psi\colon F \to S := \mathbb{P}^1 \times \mathbb{P}^1$ be the projection on the remaining two factors. If ℓ_1 and ℓ_2 are the two standard generators of $\mathrm{Pic}(S)$, then $\psi^*\mathcal{O}_S(\ell_1) = \mathcal{O}_F(h_1)$, $\psi^*\mathcal{O}_S(\ell_2) = \mathcal{O}_F(h_3)$. Moreover, $\varphi^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_F(h_2)$.

For each general $\mathcal{E} \in \mathcal{M}_S^s(2; 3\ell_1 + 5\ell_2, 9)$, we set

$$e(\mathcal{E}) := \psi^*\mathcal{E}(-\ell_1 - \ell_2) \otimes \mathcal{O}_F(h_2) \cong \psi^*\mathcal{E} \otimes \mathcal{O}_F(-h_1 + h_2 - h_3).$$

Lemma 7.3. *If $\mathcal{E} \in \mathcal{M}_S^s(2; 3\ell_1 + 5\ell_2, 9)$ is general, then $e(\mathcal{E})$ is in the moduli space $\mathcal{M}_F^{s,U}(2; h_1 + 2h_2 + 3h_3, 3h_2h_3 + 3h_1h_3 + h_1h_2)$.*

Proof. By definition, $\mathcal{E}$ is stable with respect to the natural polarisation on S, namely, $\omega_S^{-1} \cong \mathcal{O}_S(-2\ell_1 - 2\ell_2)$. It is also Ulrich, and hence aCM, with respect to the same polarisation. In particular,

$$\begin{aligned} h^0(S, \mathcal{E}(-2\ell_1 - 2\ell_2)) &= h^1(S, \mathcal{E}(-2\ell_1 - 2\ell_2)) \\ &= h^1(S, \mathcal{E}(-4\ell_1 - 4\ell_2)) = h^2(S, \mathcal{E}(-4\ell_1 - 4\ell_2)) = 0. \end{aligned} \tag{7.1}$$

Notice that

$$\begin{aligned} c_2(e(\mathcal{E})) &= \psi^*c_2(\mathcal{E}) + (3h_1 + 5h_3)(-h_1 + h_2 - h_3) + (-h_1 + h_2 - h_3)^2 \\ &= 3h_2h_3 + 3h_1h_3 + h_1h_2 \end{aligned}$$

because $\psi^*c_2(\mathcal{E}) = \psi^*(9\ell_1\ell_3) = 9h_1h_3$. The Künneth formulas imply that

$$\begin{aligned} h^p(F, e(\mathcal{E})(-tH)) &= h^p(F, \psi^*\mathcal{E}(-(t+1)\ell_1 - (t+1)\ell_2) \otimes \mathcal{O}_F((1-t)h_2)) \\ &= \sum_{i=0}^{p} h^i(S, \mathcal{E}(-(t+1)\ell_1 - (t+1)\ell_2))h^{p-i}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1-t)), \end{aligned}$$

because $\varphi^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_F(h_2)$. From $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ and (7.1),

$$\begin{aligned} h^0(F, e(\mathcal{E})(-H)) &= h^1(F, e(\mathcal{E})(-H)) = h^1(F, e(\mathcal{E})(-2H)) \\ &= h^2(F, e(\mathcal{E})(-2H)) = h^2(F, e(\mathcal{E})(-3H)) = h^3(F, e(\mathcal{E})(-3H)) = 0. \end{aligned}$$

Thus, by definition, $e(\mathcal{E})$ is Ulrich. Ulrich bundles are always semistable (see Theorem 2.2), and thus $e(\mathcal{E}) \in \mathcal{M}_F^{ss,U}(2; h_1 + 2h_2 + 3h_3, 3h_2h_3 + 3h_1h_3 + h_1h_2)$. But this last space coincides with $\mathcal{M}_F^{s,U}(2; h_1 + 2h_2 + 3h_3, 3h_2h_3 + 3h_1h_3 + h_1h_2)$, by [5, Proposition 6.4]. □

The above lemma implies that

$$e\colon \mathcal{M}_S^s(2; 3\ell_1 + 5\ell_2, 9) \dashrightarrow \mathcal{M}_F^{s,U}(2; h_1 + 2h_2 + 3h_3, 3h_2h_3 + 3h_1h_3 + 1h_1h_2)$$

is a well-defined rational map. The map $\psi\colon F \to S$ has a section $\sigma\colon S \to F$. Thus $\sigma^*\psi^* = (\psi\sigma)^*$ is the identity, and hence ψ^* is injective. From the definition of e, we deduce that e is injective as well. Since both the moduli spaces have dimension three, we deduce that e is also dominant, and hence birational.

Theorem 7.4. *Let $F \subseteq \mathbb{P}^7$ and h_1, h_2, h_3 be as specified above. Then the moduli space $\mathcal{M}_F^{s,U}(2; h_1 + 2h_2 + 3h_3, 3h_2h_3 + 3h_1h_3 + 1h_1h_2)$ is rational.*

Proof. The statement follows from the birationality of e and Proposition 7.1. □

Remark 7.5. For each d other than six, there is at most one admissible class c_1 with $c_1^2 = 4d - 2$. The existence of two different admissible classes c_1 with $c_1^2 = 22$ when $d = 6$ can be motivated as follows.

As pointed out above, on the image $F \subseteq \mathbb{P}^7$ of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, there exist two families of rank two Ulrich bundles $\mathcal{G}$ with $c_1(\mathcal{G}) = h_1 + 2h_2 + 3h_3$. For each $\mathcal{G}$ in these families, $c_2(\mathcal{G})$ represents a rational normal curve in $\mathbb{P}^7$. Such an F is the unique del Pezzo threefold supporting rank two Ulrich bundles $\mathcal{G}$ with $c_1(\mathcal{G}) \neq 2H$.

Let $S \in |\mathcal{O}_F(H)|$ be general. Then $S \subseteq \mathbb{P}^6$ is an anticanonically embedded surface. We can assume the classes of $\ell, e_1, e_2, e_3 \in \mathrm{Pic}(F)$ inside $A(F)$ are $h_2h_3 + h_1h_3 + h_1h_2$, h_2h_3, h_1h_3, h_1h_2, respectively (see the proof of [4, Lemma 6.4]). The restriction $\mathcal{E} := \mathcal{G} \otimes \mathcal{O}_S$ satisfies

$$c_1(\mathcal{E}) = c_1(\mathcal{G})H = 6\ell - 3e_1 - 2e_2 - e_3, \quad c_2(\mathcal{E}) = c_2(\mathcal{G})H = 7.$$

In particular, bundles on S with $c_1(\mathcal{E}) = 6\ell - 3e_1 - 2e_2 - e_3$ come by restriction from the aforementioned families on F.

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References

[1] M. Casanellas and R. Hartshorne, 'ACM bundles on cubic surfaces', *J. Eur. Math. Soc.* **13** (2011), 709–731.

[2] M. Casanellas, R. Hartshorne, F. Geiss and F. O. Schreyer, 'Stable Ulrich bundles', *Int. J. Math.* **23** 1250083 (2012), 50 pages.

[3] G. Casnati, 'On rank two aCM bundles', *Comm. Algebra* (2016). Doi: 10.1080/00927872.2016.1222397.

[4] G. Casnati, D. Faenzi and F. Malaspina, 'Rank two aCM bundles on the del Pezzo threefold with Picard number 3', *J. Algebra* **429** (2015), 413–446.

[5] G. Casnati, D. Faenzi and F. Malaspina, 'Moduli spaces of rank two aCM bundles on the Segre product of three projective lines', *J. Pure Appl. Algebra* **220** (2016), 1554–1575.

[6] G. Casnati and F. Galluzzi, 'Stability of rank 2 Ulrich bundles on projective $K3$ surfaces', *Math. Scand.*, to appear.

[7] E. Coskun, R. S. Kulkarni and Y. Mustopa, 'The geometry of Ulrich bundles on del Pezzo surfaces', *J. Algebra* **375** (2013), 280–301.

[8] L. Costa and M. R. Miró-Roig, 'On the rationality of moduli spaces of vector bundles on Fano surfaces', *J. Pure Appl. Algebra* **137** (1999), 199–220.

[9] D. Eisenbud and J. Herzog, 'The classification of homogeneous Cohen–Macaulay rings of finite representation type', *Math. Ann.* **280** (1988), 347–352.

[10] D. Eisenbud, F. O. Schreyer and J. Weyman, 'Resultants and Chow forms via exterior syzigies', *J. Amer. Math. Soc.* **16** (2003), 537–579.

[11] D. Faenzi, 'Rank 2 arithmetically Cohen–Macaulay bundles on a nonsingular cubic surface', *J. Algebra* **319** (2008), 143–186.
[12] D. Faenzi and J. Pons-Llopis, 'The CM representation type of projective varieties', Preprint, 2015, arXiv:1504.03819 [math.AG].
[13] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52 (Springer, New York, 1977).
[14] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, 2nd edn (Cambridge University Press, Cambridge, 2010).
[15] Y. Manin, *Cubic Forms* (North Holland, Amsterdam, 1986).
[16] R. M. Miró-Roig and J. Pons-Llopis, 'n-dimensional Fano varieties of wild representation type', *J. Pure Appl. Algebra* **218** (2014), 1867–1884.
[17] C. Okonek, M. Schneider and H. Spindler, *Vector Bundles on Complex Projective Spaces*, Progress in Mathematics, 3 (Springer, New York, 1980).
[18] J. Pons-Llopis and F. Tonini, 'ACM bundles on del Pezzo surfaces', *Matematiche (Catania)* **64** (2009), 177–211.
[19] S. Soberon-Chavez, 'Rank 2 vector bundles over a complex quadric surface', *Q. J. Math. Oxford* **36** (1985), 159–172.

GIANFRANCO CASNATI, Dipartimento di Scienze Matematiche,
Politecnico di Torino, c.so Duca degli Abruzzi 24, 10129 Torino, Italy
e-mail: gianfranco.casnati@polito.it