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ON SOME CONJECTURES ON NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In [2], Ladas and Sficas made two conjectures about the asymptotic behavior of solutions of some neutral differential equations. In this paper we confirm that these conjectures are indeed correct.

1. **Introduction.** In [2], Ladas and Sficas dealt with the asymptotic and oscillatory behavior of solutions of the NDDE of order $n \geqq 1$,

$$\frac{d^n}{dt^n}[y(t)+py(t-\tau)]+qy(t-\sigma)=0, \quad t \geqq t_0 \tag{1}$$

under the following hypothesis:

(H) q is a positive constant, the delays τ and σ are nonnegative real numbers and the coefficient p is a real parameter.

They proved the following results:

THEOREM A([2]). *Assume that n is odd, $p = -1$ and the hypothesis (H) is satisfied. Then every solution of Eq. (1) oscillates.*

THEOREM B([2]). *Consider the NDDE(1) and assume that the hypothesis (H) is satisfied. Then the following statements are true.*

(i) Assume that n is odd and that $p < -1$. Then every nonoscillatory solution of Eq. (1) tends to $+\infty$ or $-\infty$ as $t \longrightarrow \infty$.

(ii) Assume that n is odd or even and that $p \geqq -1$. Then every nonoscillatory solution of Eq. (1) tends to zero as $t \longrightarrow \infty$.

The conclusion of Theorem A is not true for n even. For example, the NDDE

$$\frac{d^2}{dt^2}[y(t)-y(t-\log 2)]+\left(\frac{1}{e}\right)y(t-1)=0$$

has the nonoscillatory solution $y(t) = e^{-t}$.

Also when n is even and $p < -1$, it is not true that the nonoscillatory solutions of Eq. (1) tend to $+\infty$ or $-\infty$ as $t \longrightarrow \infty$, as in case (i) of Theorem B, and it is not true that they tend to zero either. This can be seen from the example,

$$\frac{d^2}{dt^2}[y(t)-2y(t-\log 2)]+\left(\frac{3}{e}\right)y(t-1)=0$$

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which has the nonoscillatory solution $y(t) = e^{-t}$ and from

$$\frac{d^2}{dt^2}\,[y(t) - 4y(t - \log 2)] + ey(t - 1) = 0$$

which has the nonoscillatory solution $y(t) = e^t$.

Motivated by the above examples, Ladas and Sficas made the following two conjectures in [2].

CONJECTURE 1([2]). *Assume that n is even and $p = -1$. Then every nonoscillatory solution of Eq. (1) tends to zero as $t \longrightarrow \infty$.*

CONJECTURE 2([2]). *Assume that n is even and $p < -1$. Then every nonoscillatory solution of Eq. (1) either tends to ∞ or tends to zero as $t \longrightarrow \infty$.*

In this paper, we confirm that the above two conjectures are true for the nonoscillatory solutions of Eq. (1) with continuously differentiable initial functions. In addition, when n is even and $p < -1$ we will prove that all bounded nonoscillatory solutions of Eq. (1) tend to zero as $t \longrightarrow \infty$.

Set $\rho = \max\{\tau, \sigma\}$. Let $\phi \in C[[t_0 - \rho, t_0], \mathbf{R}]$ be a given initial function and let $z_k, k = 0, 1, \ldots, n-1$ be given initial constants. Using the method of steps it follows that Eq. (1) has a unique solution $y \in C[[t_0 - \rho, \infty), \mathbf{R}]$ in the sense that

$$y(t) = \phi(t) \text{ for } t \in [t_0 - \rho,\ t_0],$$

$$\frac{d^k}{dt^k}\,[y(t) + p\phi(t - \tau)]_{t=t_0} = z_k \text{ for } k = 0,\ 1, \ldots, n-1,$$

$y(t) + py(t - \tau)$ is n-times continuously differentiable on $[t_0, \infty)$, and $y(t)$ satisfies Eq. (1) for all $t \geqq t_0$.

2. **Some Lemmas.** First we prove the following lemma.

LEMMA 1. *Assume that $f \in C[[t_0, \infty), \mathbf{R}^+]$ and suppose that there exists an interval $[a, b] \subset [t_0, \infty)$ and constants $c > 0$ and $r \geqq 1$ such that on the intervals $[a + nc, b + nc]$ for $n = 0, 1, 2, \ldots, f(t)$ is continuously differentiable and satisfies*

$$f'(t) \geqq rf'(t - c) > 0. \tag{3}$$

Then

$$\int_{t_0}^{\infty} f(t)dt = \infty.$$

PROOF. If $b - a \geqq c$, the conclusion of the Theorem is obvious. Assume that

$$0 < b - a < c.$$

In view of (3) and the continuity of $f'(t)$ on $[a, b]$, there exists a positive constant m such that on the intervals $[a + nc, b + nc]$ $n = 0, 1, 2, \ldots$

$$f'(t) \geqq m > 0.$$

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Without loss of generality, we assume that $a \geqq 0$. By the mean value theorem, for each $t \in [a+nc, b+nc]$ there is a ξ_n such that $a+nc \leqq \xi_n \leqq t$ and

$$f(t) = f(a+nc) + f'(\xi_n)(t-(a+nc)).$$

Thus

$$f(t) \geqq m(t-(a+nc))$$

and

$$\int_{a+nc}^{b+nc} f(t)dt \geqq m \int_{a+nc}^{b+nc} (t-(a+nc))dt = m \int_0^{b-a} s ds = \frac{1}{2} m(b-a)^2.$$

Let $t^* = \max\{t_0, 0\}$. It is easy to see that

$$\int_{t^*}^{\infty} f(t)dt \geqq \sum_{n=1}^{\infty} \int_{a+nc}^{b+nc} f(t)dt = \lim_{k\to\infty} \sum_{n=1}^{k} \int_{a+nc}^{b+nc} f(t)dt \geqq \lim_{k\to\infty} \frac{m}{2}(b-a)^2 k = \infty.$$

Then

$$\int_{t_0}^{\infty} f(t)dt = \infty$$

and the proof is complete. ∎

The following two lemmas which will also be used in the proofs of our main results, have been extracted from [1] and [2], respectively.

LEMMA 2([1]). *Let* $f, g \in C[[t_0,\infty), (0,\infty)]$ *satisfy* $f(t) = g(t) + pg(t-c)$, $t \geqq t_0 + \max\{0, c\}$ *where* $p, c \in \mathbf{R}$ *and* $p \neq \pm 1$. *Assume that* g *is bounded on* $[t_0,\infty)$ *and that* $\lim_{t\to\infty} f(t) = \alpha$, *exists. Then* $\lim_{t\to\infty} g(t)$ *exists.*

LEMMA 3([2]). *Consider the NDDE(1). Assume that the hypothesis* (H) *is satisfied and that* n *is even. Let* $y(t)$ *be an eventually positive solution of Eq. (1) and set* $z(t) = y(t) + py(t-\tau)$. *Then the following statements are true.*

(i) Assume that $0 > p \geqq -1$, *then*

$$z(t) < 0,\ z'(t) > 0,\ z''(t) < 0, \ldots, z^{(n-1)}(t) > 0. \tag{4}$$

(ii) Assume that $p < -1$. *Then either* (4) *holds or*

$$\lim_{t\to\infty} z^{(i)}(t) = -\infty \quad \text{for} \quad i = 0, 1, \ldots, n-1. \tag{5}$$

3. **Main Results.**

THEOREM 1. *Consider the NDDE(1). Assume that the hypothesis* (H) *is satisfied and that* n *is even and* $p = -1$. *Then every nonoscillatory solution* $y(t)$ *of Eq. (1) with* *continuously differentiable initial function* $\phi(t)$ *converges monotonically to zero as* $t \longrightarrow \infty$.

PROOF. The proof is trivial for $\tau = 0$. Assume that $\tau > 0$. As the negative of a solution of Eq. (1) is also a solution of the same equation, it suffices to prove the theorem for an eventually positive solution $y(t)$ of Eq. (1). Set

$$z(t) = y(t) - y(t-\tau). \tag{6}$$

Since $z(t)$ is continuously differentiable on $[t_0, \infty)$ and $y(t) = \phi(t)$ is continuously differentiable on $[t_0-\rho, t_0]$ where $\rho = \max\{\tau, \sigma\}$, it follows from (6) that $y(t)$ is continuously differentiable on $[t_0, \infty)$ except perhaps at the points $t_0+n\tau$ for $n = 0, 1, 2, \ldots$. We claim that $y'(t) \leqq 0$.

Assume, for the sake of contradiction, that there exists a $t^* > t_0+\rho$ such that $y'(t^*) > 0$. Then by the continuity of $y'(t)$ there exists an interval $[a, b]$ which contains t^* such that for $t \in [a, b]$, $y'(t)$ exists and satisfies $y'(t) \geqq r$ where r is a positive constant. From Lemma 3(*i*), $z'(t) > 0$. Then on the intervals $[a+n\tau, b+n\tau]$ for $n = 1, 2, \ldots$, $y'(t) > y'(t-\tau) \geqq r > 0$. Hence by lemma 1,

$$\int_{t_0}^{\infty} y(t)dt = \infty. \tag{7}$$

By integrating Eq. (1), we have

$$z^{(n-1)}(t) - z^{(n-1)}(t_0) + q\int_{t_0}^{t} y(s-\sigma)ds = 0 \tag{8}$$

which, in view of (7), implies $\lim_{t\to\infty} z^{(n-1)}(t) = -\infty$. This contradicts (4). So, we see that $y'(t) \leqq 0$ which implies that $y(t)$ is monotonically decreasing and $\lim_{t\to\infty} y(t) = \beta \in \mathbf{R}$ exists. By (8), $\beta = 0$ and the proof is complete. ∎

THEOREM 2. *Consider the NDDE(1). Assume that the hypothesis (H) is satisfied with* n *even and* $p < -1$. *Then every nonoscillatory solution of Eq. (1) with continuously differentiable initial function either tends to infinity or tends monotonically to zero as* $t \longrightarrow \infty$.

PROOF. It suffices to prove the theorem for an eventually positive solution $y(t)$ of Eq. (1). Set $z(t) = y(t) + py(t-\tau)$. By Lemma 2(*ii*) wither (5) holds or (4) holds. When (5) holds, $py(t-\tau) < z(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$ and so $\lim_{t\to\infty} y(t) = \infty$. When (4) holds, we see $y'(t)$ is continuously differentiable except perhaps at the points $t_0+n\tau$ for $n = 0, 1, 2, \ldots$ and $y'(t) > -py'(t-\tau)$. Then by Lemma 1 and 3 and an argument similar to that in Theorem 1 we see that $y(t)$ tends monotonically to zero as $t \longrightarrow \infty$. The proof is complete. ∎

REMARK 1. From the proof of Theorems 1 and 2 we see, in fact, that it is enough to require only that the initial function $\phi(t)$ of $y(t)$ is continuously differentiable on $(t_0 - \tau, t_0)$.

In the following Theorem we do not require that the initial function is continuously differentiable.

THEOREM 3. *Consider the NDDE(1). Assume that the hypothesis (H) is satisfied, n is even and $p < -1$. Then every bounded nonoscillatory solution $y(t)$ of Eq. (1) tends to zero.*

PROOF. Assume $y(t)$ is an eventually positive and bounded solution. Set

$$z(t) = y(t) + py(t - \tau).$$

Then $z(t)$ is also bounded. By Lemma 3, we see that

$$\lim_{t\to\infty} z(t) = \gamma \in \mathbf{R}, \text{ exists.} \tag{9}$$

Then by Lemma 2, we find

$$\lim_{t\to\infty} y(t) = \beta \in \mathbf{R}, \text{ exists.}$$

We claim $\beta = 0$. Otherwise $\beta > 0$, and so

$$\int_{t_0}^{\infty} y(t)dt = \infty. \tag{10}$$

By integrating Eq. (1) we find

$$z^{(n-1)}(t) - z^{(n-1)}(t_0) + q\int_{t_0}^{t} y(s-\sigma)ds = 0$$

and, in view of (10), $\lim_{t\to\infty} z^{(n-1)}(t) = -\infty$. Then $\lim_{t\to\infty} z(t) = -\infty$ which contradicts (9) and completes the proof. ∎

REMARK 2. By an argument similar to that in the proof of Theorem 3, we can extend the conclusion of Theorem 3 to the neutral delay differential equation with variable coefficient

$$\frac{d^n}{dt^n}[y(t) + py(t-\tau)] + Q(t)y(t-\sigma) = 0, \quad t \geqq t_0 \tag{11}$$

where $\tau, \sigma \in [0, \infty)$, $p \in (-\infty, -1)$ and $Q \in C[[t_0, \infty), \mathbf{R}^+]$. Assume that n is even and

$$\int_{t_0}^{\infty} Q(t)dt = \infty.$$

Then every bounded nonoscillatory solution of (11) tends to zero as $t \longrightarrow \infty$.

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