AN OPTIMAL PORTFOLIO PROBLEM IN A DEFAULTABLE MARKET

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Abstract

We consider a portfolio optimization problem in a defaultable market. The investor can dynamically choose a consumption rate and allocate his/her wealth among three financial securities: a defaultable perpetual bond, a default-free risky asset, and a money market account. Both the default risk premium and the default intensity of the defaultable bond are assumed to rely on some stochastic factor which is described by a diffusion process. The goal is to maximize the infinite-horizon expected discounted log utility of consumption. We apply the dynamic programming principle to deduce a Hamilton–Jacobi–Bellman equation. Then an optimal Markov control policy and the optimal value function is explicitly presented in a verification theorem. Finally, a numerical analysis is presented for illustration.

Keywords: Portfolio optimization; defaultable security; stochastic factor; Hamilton–Jacobi–Bellman equation; sub/super-solution

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1. Introduction

Stochastic portfolio optimization has been an attractive topic in the subject of mathematical finance. For the investment problem in a stock and a riskless bond, the pioneering works by Merton [19], [20], [21, pp. 95–212] first approached the strategy for maximizing the total expected discounted utility of consumption. From then on, various default-free optimal investment models have been proposed and investigated in the literature (see, e.g. [6], [11], [17], [22], and [23]). Among them, Fleming and Pang [11] discussed a classical Merton portfolio optimization problem, where the interest rate is assumed to fluctuate from time to time. The objective was to maximize the expected discounted HARA utility of consumption at the infinite horizon. In a subsequent paper, Pang [22] treated the analogue problem with log utility. Pham [23] studied an optimal investment problem, in which the instantaneous rate and the volatility are assumed to rely on a stochastic factor that is described by a Markov diffusion process, and the goal is to maximize the expected HARA utility of the terminal wealth.

Recently, the optimal investment and hedging with the defaultable claims have aroused much more attention (see, e.g. [1], [2], [3], [4], [14], [16], and [18]). Hou and Jin [16] studied an optimal investment problem with default risk under the conditional diversification assumption (which implies an asymptotic disappearance of the jump-risk premium), as in [15]. Dynamics

with constant default risk premium and constant default intensity for the price process of a defaultable bond was proposed in [14]. Bielecki and Jang [4] established an optimal asset allocation for maximizing the expected HARA utility of the terminal wealth.

In this paper we consider a stochastic portfolio optimization problem in a reduced-form defaultable market. The investor can dynamically choose a consumption rate and allocate his/her wealth among three financial securities: a defaultable perpetual bond, a default-free risky asset, and a money market account. Both the default risk premium and the default intensity of the defaultable bond are assumed to rely on some stochastic factor which is described by a diffusion process. (This assumption seems to be reasonable, and has been adopted in some known references; see, e.g. [23]. In fact, the stochastic differential equations in which the characteristic parameters depend on economic factors (state variables) are often used in finance to model different types of economic phenomenon, such as the random fluctuant interest rate and stochastic volatility; see, e.g. [10] and [11]. It is also natural to allow the default intensity and the default risk premium to depend on economic factors; see, e.g. Chapter 11 of [8] and the references therein.) Here we intend to maximize the infinite-horizon expected discounted log utility of consumption. For this purpose, we apply the dynamic programming principle to deduce a Hamilton–Jacobi–Bellman (HJB) equation, and then the sub/super-solution technique for partial differential equations (see, e.g. [11]) is adopted to study the solution of the HJB equation. The optimal Markov control policy and the optimal value function is explicitly presented in a verification theorem. Finally, we present a numerical analysis of the optimal control strategy and the value function.

The paper is organized as follows. In Section 2 we present the price dynamics of financial securities. The HJB equation for the optimal argument with a defaultable security is deduced and investigated in Section 3. Section 4 is devoted to proving a verification theorem. In Appendix A we present the derivation of the price dynamics for the perpetual defaultable bond with constant market parameters. In Appendix B we present a parametric sensitivity analysis on the optimal control strategy and the value function.

2. Price dynamics of financial securities

In this paper we consider an investor who dynamically allocates his/her wealth among a defaultable perpetual bond, a default-free risky asset, and a money market account. The main task in this section is to describe the price dynamics of the above three financial securities.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete real-world probability space, and let τ be a nonnegative random variable on this space. For $t \geq 0$, define a default indicator process $(z_t)_{t \geq 0}$ by

$$z_t = \mathbf{1}_{\{\tau \leq t\}}. \tag{2.1}$$

Suppose that $(W_t, \breve{W}_t)_{t \geq 0}$ is a two-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathrm{P})$, and let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the augmented natural filtration of $(W_t, \breve{W}_t)_{t \geq 0}$. Let $\mathcal{D}_t = \sigma(z_u; 0 \leq u \leq t)$ and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ with $t \geq 0$. Then τ is a $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ stopping time. We assume that all the filtrations satisfy the usual hypotheses of completeness and right continuity.

We suppose that τ has a positive $\mathbb{F}$-adapted intensity process $(\lambda_t)_{t \geq 0}$. (This implies that τ is a totally inaccessible $\mathbb{G}$-stopping time; see, e.g. Section VI.78 of [7]). Now

$$m_t := z_t - \int_{(0, t \wedge \tau]} \lambda_s \, \mathrm{d}s = z_t - \int_{(0, t]} (1 - z_s) \lambda_s \, \mathrm{d}s, \qquad t \geq 0, \tag{2.2}$$

is a $(P, \mathbb{G})$-martingale. Accordingly, for any $t > 0$, the survival probability

$$P(\tau > t) = E[E[1 - z_t \mid \mathcal{F}_t]] = E\left[\exp\left(-\int_0^t \lambda_s \, ds\right)\right]. \tag{2.3}$$

Herein, we suppose that the nonnegative intensity process $(\lambda_t)_{t \geq 0}$ and the default risk premium $(1/\eta_t)_{t \geq 0}$ rely on an $\mathbb{F}$-adapted stochastic economic factor $(y_t)_{t \geq 0}$ (see, e.g. [23]), i.e. there exist a nonnegative measurable function $\lambda(\cdot)$ on $\mathbb{R}^+$ and a $(0, 1]$-valued measurable function $\eta(\cdot)$ such that

$$\lambda_t = \lambda(y_t), \qquad \eta_t = \eta(y_t), \qquad t \geq 0,$$

where the $(y_t)_{t \geq 0}$ is described by

$$dy_t = \mu(y_t) \, dt + d\breve{W}_t, \qquad t > 0, \ y_0 = y. \tag{2.4}$$

Remark 2.1. We suppose that the economic factor is independent of the default-free risky asset in (2.6), below. If the economic factor is correlated with the default-free risky asset, there will be an additional mixed partial derivative term in the HJB equation (3.3), below, and this term dose not have the essential effect on the problem (see, e.g. [11]).

Let $\rho \in (0, 1)$ denote the constant loss rate when a default occurs. Here we adopt the 'recovery of market value' scheme, which means that at the default time the bond loses a fraction ρ of its value (see, e.g. Equation (9) of [9]). Then we can suggest price dynamics $(p_t)_{t \geq 0}$ for a defaultable perpetual bond that pays a constant coupon $\widetilde{C}$ per unit time as follows:

$$dp_t = r p_t \, dt + \rho \lambda_t p_t (1 - z_t)\left(\frac{1}{\eta_t} - 1\right) dt - (1 - z_t)\widetilde{C} \, dt - \rho p_{t-} \, dm_t, \tag{2.5}$$

where $(m_t)_{t \geq 0}$ is the $(P, \mathbb{G})$-martingale defined in (2.2) and $r \in \mathbb{R}^+$ is the constant interest rate. (The derivation of the dynamics when the market parameters are constant is given in Appendix A. Note that Example 2.2.4 of [5] indicates that $\rho < 1$ is possible. Here we directly randomize the market parameters in the dynamics similarly to the method used for the stochastic volatility model (see, e.g. Heston [12]). One must be careful when the market parameters are random, since then the derivation in Appendix A will be not valid.) Moreover, the money market account $(\beta_t)_{t \geq 0}$ and the default-free risky asset $(\gamma_t)_{t \geq 0}$ are described by the price dynamics

$$\begin{aligned} d\beta_t &= r\beta_t \, dt, & \beta_0 &= 1, \\ d\gamma_t &= b\gamma_t \, dt + a\gamma_t \, dW_t, & \gamma_0 &= \gamma > 0, \end{aligned} \tag{2.6}$$

where the coefficients a and b are constants. (The reasons for assuming a constant interest rate are two-fold. First, we are mainly concerned with the influence of the default risk in the portfolio argument, so we do not pay much attention to the interest rate risk. Second, the default-free portfolio arguments with the stochastic interest rate have been investigated in [11] and [22]. The context of this paper may be extended to the stochastic interest rate case, but some additional technique should be adopted and the respective argument will probably be more complicated.)

We make the following technical assumptions.

(H1) $\mu(y) \in C^1(\mathbb{R})$ and there exist constants $\underline{C}, \overline{C}$ such that $\underline{C} \leq \mu_y(y) := d\mu(y)/dy \leq \overline{C}$ for all $y \in \mathbb{R}$.

(H2) There exist constants $C > 0$ and $\delta \geq 1$ such that

$$\lambda(y) \leq C + C|y|^{\delta}, \qquad y \in \mathbb{R}.$$

(H3) The quantity $\eta_m := \inf_{y \in \mathbb{R}} \eta(y)$ is strictly positive.

We conclude this section with the following lemma.

Lemma 2.1. *Under assumption (H1), (2.4) admits a unique strong solution* $(y_t)_{t \geq 0}$ *such that, for some* $\alpha_m > 2m\overline{C}$ $(m \geq 1)$,

$$\lim_{T \to \infty} e^{-\alpha_m T} \mathrm{E}[y_T^{2m}] = 0, \qquad \lim_{T \to \infty} e^{-\alpha_m T} \mathrm{E}\left[\int_0^T y_s^{2m} \, \mathrm{d}s \right] = 0.$$

Proof. Note that, by (H1), the drift $\mu(\cdot)$ is globally Lipschitzian on $\mathbb{R}$. Then the conclusion follows from Lemma 3.2 of [22]. □

3. The optimal portfolio problem with a defaultable perpetual bond

In this section we explore the stochastic portfolio problem with a defaultable perpetual bond. The goal is to maximize the infinite-horizon expected discounted log utility of consumption.

For each time $t \geq 0$, let x_t be the wealth at time t, and let κ_t and ℓ_t be the respective proportions of wealth x_t in the defaultable bond $(p_t)_{t \geq 0}$ and the default-free risky asset $(\gamma_t)_{t \geq 0}$. Then $1 - \kappa_t - \ell_t$ is the t-time proportion of wealth x_t in the money market account $(\beta_t)_{t \geq 0}$. In addition, the investor can choose a consumption rate $c_t > 0$ at time $t \geq 0$. Suppose that the initial wealth $x_0 > 0$. Then, by virtue of the self-financing investment policy (see, e.g. [14]), the dynamics of the wealth process $(x_t)_{t \geq 0}$ follow

$$\mathrm{d}x_t = \frac{(1 - \kappa_t - \ell_t)x_t}{\beta_t} \, \mathrm{d}\beta_t + \frac{\kappa_t x_{t-}}{p_{t-}} \, \mathrm{d}p_t + \frac{\ell_t x_t}{\gamma_t} \, \mathrm{d}\gamma_t + \frac{\kappa_t x_t}{p_t}(1 - z_t)\widetilde{C} \, \mathrm{d}t - c_t x_t \, \mathrm{d}t, \qquad t > 0. \tag{3.1}$$

Now summarizing (2.5), (2.6), and (3.1), we have the wealth dynamics

$$\frac{\mathrm{d}x_t}{x_{t-}} = r(1 - \ell_t) \, \mathrm{d}t + b\ell_t \, \mathrm{d}t + \rho\kappa_t(1 - z_t)\lambda(y_t)\left(\frac{1}{\eta(y_t)} - 1 \right) \mathrm{d}t - c_t \, \mathrm{d}t + a\ell_t \, \mathrm{d}W_t$$
$$\quad - \rho\kappa_{t-}(1 - z_{t-}) \, \mathrm{d}m_t, \qquad t > 0. \tag{3.2}$$

Remark 3.1. The difference between our analysis and Equation (4.5.14) of [14, Chapter 4] (or Lemma 3 of [4]) is that the default process $(\lambda_t)_{t \geq 0}$ and the default risk premium $(1/\eta_t)_{t \geq 0}$ rely on a stochastic factor process (which has been assumed for some default-free stochastic portfolio optimization arguments; see, e.g. [10] and [11]), rather than on two constants.

Our aim is to seek an optimal allocation $(\kappa_t, \ell_t)_{t \geq 0}$ and an optimal consumption rate $(c_t)_{t \geq 0}$ for the wealth $(x_t)_{t \geq 0}$ to maximize the infinite-horizon expected discounted utility of consumption. To this aim, we restrict the allocation $(\kappa_t, \ell_t)_{t \geq 0}$ and the consumption rate $(c_t)_{t \geq 0}$ to some admissible control set $\mathcal{A}(\mathbb{G})$.

Definition 3.1. A càdlàg $\mathbb{G}$-adapted Markov control $(\kappa_t, \ell_t, c_t)_{t \geq 0}$ is in the admissible control space $\mathcal{A}(\mathbb{G})$ if the following conditions hold:

$$\kappa_t \in \left[0, \frac{1}{\rho} \right), \qquad \ell_t \in \mathbb{R}, \qquad c_t > 0, \quad \text{for all } t \geq 0.$$

Remark 3.2. In Definition 3.1, the restriction $\kappa. \in [0, 1/\rho)$ (which ensures that $x_t > 0$ for all $t \geq 0$) is the so-called avoiding bankruptcy condition (see Remark 5.2.4 of [14]), and each element of the admissible control space $\mathcal{A}(\mathbb{G})$ is a bankruptcy avoiding portfolio.

We consider the log utility function on $(0, \infty)$, i.e. $U(x) = \log x$, $x > 0$.

Remark 3.3. The log utility function $U(x)$ corresponds to the HARA utility function in the limiting case of relative risk aversion equal to 1.

Let $\alpha > 0$ be a discount factor. For an admissible control $(\kappa_t, \ell_t, c_t)_{t \geq 0}$ and an initial value $(x, y, z) \in (0, \infty) \times \mathbb{R} \times \{0, 1\}$, define the objective functional on an infinite time horizon by

$$
\begin{aligned}
J(x, y, z, \kappa., \ell., c.) &= \mathrm{E}_{x,y,z}\left[\int_0^\infty \mathrm{e}^{-\alpha t} U(c_t x_t)\,\mathrm{d}t\right]\\
&:= \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-\alpha t} U(c_t x_t)\,\mathrm{d}t \;\middle|\; x_0 = x,\ y_0 = y,\ z_0 = z\right].
\end{aligned}
$$

Our goal is to maximize the objective functional $J(x, y, z, \kappa., \ell., c.)$ for all admissible $(\kappa_t, \ell_t, c_t)_{t \geq 0}$. The value function is given by

$$
v(x, y, z) := \sup_{(\kappa., \ell., c.) \in \mathcal{A}(\mathbb{G})} J(x, y, z, \kappa., \ell., c.)
$$

for each $(x, y, z) \in (0, \infty) \times \mathbb{R} \times \{0, 1\}$. Applying Lemma 1.5 (the Bellman principle) of [13] to the value function defined above, we have

$$
v(x, y, z) = \sup_{(\kappa., \ell., c.) \in \mathcal{A}(\mathbb{G})} \mathrm{E}_{x,y,z}\left[\int_0^{\zeta_t} \mathrm{e}^{-\alpha s} U(c_s x_s)\,\mathrm{d}s + \mathrm{e}^{-\alpha \zeta_t} v(x_{\zeta_t}, y_{\zeta_t}, z_{\zeta_t})\right],
$$

for all $\mathbb{G}$-stopping times $\zeta_t := \zeta \wedge t$, $t \geq 0$. Hence, the HJB equation associated with $v(x, y, z)$ is given by

$$
\begin{aligned}
\alpha v(x, y, z) = {}& rx v_x(x, y, z)\\
&+ \sup_{(\kappa, \ell, c) \in A_1 \times A_2 \times A_3}\Big[U(cx) + (b - r)\ell x v_x(x, y, z) - cx v_x(x, y, z)\\
&\qquad\qquad + \tfrac{1}{2} a^2 \ell^2 x^2 v_{xx}(x, y, z)\\
&\qquad\qquad + (1 - z)\lambda(y)\left(\frac{\rho}{\eta(y)}\right)\kappa x v_x(x, y, z)\\
&\qquad\qquad + (v(x - x\kappa\rho, y, 1) - v(x, y, 0))(1 - z)\lambda(y)\Big]\\
&+ \mu(y) v_y(x, y, z) + \tfrac{1}{2} v_{yy}(x, y, z) \qquad\qquad\qquad (3.3)
\end{aligned}
$$

for $(x, y, z) \in (0, \infty) \times \mathbb{R} \times \{0, 1\}$, where $v_x := \partial v / \partial x$, $v_y := \partial v / \partial y$, $v_{xx} := \partial v^2 / \partial x^2$, and $v_{yy} := \partial v^2 / \partial y^2$.

Since $z = 0$ or 1, we consider two cases:

$$
\widetilde{v}(x, y) := v(x, y, 0) \quad \text{(the pre-default case)}
$$

and

$$
\overline{v}(x, y) := v(x, y, 1) \quad \text{(the post-default case)}.
$$

Since the post-default case has been well studied, we only provide results for which the post-default value function is given by

$$\overline{v}(x, y) = \frac{1}{\alpha} \log x + R_\alpha, \tag{3.4}$$

where

$$R_\alpha := \frac{1}{\alpha^2} \left(\frac{(b-r)^2}{2a^2} + r \right) + \frac{\log \alpha - 1}{\alpha}, \qquad y \in \mathbb{R}, \tag{3.5}$$

and the optimal control strategies are given by

$$\overline{\ell}^* = \frac{b-r}{a^2}, \qquad \overline{\kappa}^* = 0, \qquad \overline{c}^* = \alpha. \tag{3.6}$$

In the following, we will concentrate on the pre-default case. According to (3.3), $\widetilde{v}$ obeys the following dynamics:

$$\begin{aligned}
\alpha \widetilde{v} = {}& rx\widetilde{v}_x + \mu(y)\widetilde{v}_y + \tfrac{1}{2}\widetilde{v}_{yy} \\
&+ \sup_{(\ell,c)\in A_2 \times A_3} \left[U(cx) + (b-r)\ell x \widetilde{v}_x - cx\widetilde{v}_x + \tfrac{1}{2}a^2\ell^2 x^2 \widetilde{v}_{xx} \right] \\
&+ \sup_{\kappa \in A_1} \left[\lambda(y)\left(\frac{\rho}{\eta(y)} \right)\kappa x \widetilde{v}_x + (\overline{v}(x - x\kappa\rho, y) - \widetilde{v}(x, y))\lambda(y) \right].
\end{aligned} \tag{3.7}$$

It is not hard to verify that $\widetilde{v}(x, y)$ admits the form

$$\widetilde{v}(x, y) = \frac{1}{\alpha} \log x + \widetilde{\omega}(y), \qquad (x, y) \in (0, \infty) \times \mathbb{R}.$$

Substituting this into (3.7), it follows that $\widetilde{\omega}(y)$ is governed by

$$\begin{aligned}
&\tfrac{1}{2}\widetilde{\omega}_{yy} + \mu(y)\widetilde{\omega}_y - (\alpha + \lambda(y))\widetilde{\omega} + \log \alpha - 1 + \lambda(y)R_\alpha \\
&+ \frac{1}{\alpha}\left(r + \frac{(b-r)^2}{2a^2} + \frac{\lambda(y)(1 - \eta(y))}{\eta(y)} + \lambda(y)\log(\eta(y)) \right) = 0,
\end{aligned} \tag{3.8}$$

and the maximum points in (3.7) are given by

$$\widetilde{\ell}^* = \frac{b-r}{a^2}, \qquad \widetilde{\kappa}^* = \frac{1 - \eta(y)}{\rho}, \qquad \widetilde{c}^* = \alpha.$$

Actually, in the next section we will prove that $(\widetilde{\ell}^*, \widetilde{\kappa}^*, \widetilde{c}^*)$ is the optimal control strategy.

Next we adopt the so called sub/super-solution method to obtain the existence and uniqueness of the solution to Equation (3.8). (An advantage of the sub/super-solution method is that we can obtain some proper bounds for the unique classical solution to (3.8). Since the linear equation (3.8) has *variable coefficients*, the 'fundamental set of solutions' approach seems to be unavailable; see also [22].) Rewrite (3.8) as

$$\frac{1}{2}\widetilde{\omega}_{yy} + \mu(y)\widetilde{\omega}_y - (\alpha + \lambda(y))\widetilde{\omega} + \alpha R_\alpha + \frac{\lambda(y)}{\alpha}h(\eta(y)) = 0, \tag{3.9}$$

where

$$h(x) := \frac{1-x}{x} + \log x + \alpha R_\alpha, \qquad x \in (0, 1]. \tag{3.10}$$

Define

$$Lw = -\tfrac{1}{2}w_{yy} - \mu(y)w_y,$$

$$f(y, w) = -(\alpha + \lambda(y))w + \alpha R_\alpha + \frac{\lambda(y)}{\alpha}h(\eta(y)).$$

Recall that $\underline{w}(y)$ is said to be a sub-solution of (3.9) on the real line if $L\underline{w} \le f(y, \underline{w})$, and that $\overline{w}(y)$ is said to be a super-solution of (3.9) on the real line if $L\overline{w} \ge f(y, \overline{w})$). Moreover, if $\underline{w}(y) \le \overline{w}(y)$ for all $y \in \mathbb{R}$ then $(\underline{w}, \overline{w})$ is called a sub/super-solution ordered pair of (3.9) (see Definition 3.1 of [11]). Now we have the following result.

Proposition 3.1. *Let the default intensity function $\lambda(y) \ge 2\overline{C} - \alpha$ for all $y \in \mathbb{R}$. Then, under assumptions (H1)–(H3), (3.9) admits a classical solution $\widetilde{\omega}(y)$ such that*

$$R_\alpha \le \widetilde{\omega}(y) \le \overline{\widetilde{\omega}}(y) := C_1 y^2 + C_2 \quad \text{for all } y \in \mathbb{R},$$

for some $C_1 > 0$ and $C_2 > R_\alpha$, where R_α is defined in (3.5).

   Proof. We use the so-called sub/super-solution of partial differential equations (see, e.g. [11]). Note that, for all $x \in (0, 1]$, it holds that

$$0 \le x^{-1}(1 - x) + \log x \le x^{-1}.$$

As a consequence, the mapping $x \to h(x)$ defined by (3.10) satisfies

$$\alpha R_\alpha \le h(x) < \alpha R_\alpha + x^{-1} \quad \text{for all } x \in (0, 1], \tag{3.11}$$

so we have

$$\frac{\alpha R_\alpha + (\lambda(y)/\alpha)h(\eta(y))}{\alpha + \lambda(y)} \ge \frac{\alpha R_\alpha + (\lambda(y)/\alpha)\alpha R_\alpha}{\alpha + \lambda(y)} = R_\alpha.$$

Since R_α is a constant, $LR_\alpha = 0$. Then, for each constant $C \le R_\alpha$, it is a sub-solution of (3.9).
   On the other hand, for $y \in \mathbb{R}$ fixed, there exists some $\theta \in [0, y]$ such that

$$\begin{aligned}
L\overline{\widetilde{\omega}}(y) &= -C_1 - 2C_1 y\mu(y) \\
&= -C_1 - 2C_1 y[\mu(y) - \mu(0) + \mu(0)] \\
&= -C_1 - 2C_1 y[y\mu_y(\theta) + \mu(0)] \\
&\ge -2C_1\overline{C}y^2 - 2C_1\mu(0)y - C_1.
\end{aligned}$$

It follows from (3.11) that

$$\begin{aligned}
f(y, \overline{\widetilde{\omega}}(y)) &= -(\alpha + \lambda(y))[C_1 y^2 + C_2] + \alpha R_\alpha + \frac{\lambda(y)}{\alpha}h(\eta(y)) \\
&\le -(\alpha + \lambda(y))C_1 y^2 + \frac{\lambda(y)}{\alpha\eta_m} + (\alpha + \lambda(y))(R_\alpha - C_2).
\end{aligned}$$

Taking $C_2 > R_\alpha$ large enough (since $\lambda(\cdot)$ is nonnegative), it holds that

$$L\overline{\widetilde{\omega}}(y) \ge f(y, \overline{\widetilde{\omega}}(y)) \quad \text{for all } y \in \mathbb{R}.$$

This shows that $\overline{\widetilde{\omega}}(y)$ is a super-solution of (3.9). Hence, $(R_\alpha, \overline{\widetilde{\omega}}(y))$ is a sub/super-solution ordered pair of (3.9). Now define

$$\overline{H}(y, p, q) := -\mu(y)q + (\alpha + \lambda(y))p - \alpha R_\alpha - \frac{\lambda(y)}{\alpha}h(\eta(y)).$$

It is not hard to conclude that $p \to \overline{H}(y, p, q)$ is strictly increasing. On the other hand, let $\triangle = [\underline{y}, \overline{y}]$ be an arbitrary finite interval on the real line, and let $\Phi = \max\{\sup_{y \in \triangle} |\widetilde{\omega}(y)|, |R_\alpha|\}$. Then it follows from assumptions (H1) and (H2) that there exists $\theta \geq 0$ such that, for all $y \in \triangle$ and $|p| \leq 3\Phi$,

$$
\begin{aligned}
|\overline{H}(y, p, q)| &\leq (|\mu_y(\theta)||y| + |\mu(0)|)|q| + (\alpha + C + C|y|^\delta)(|p| + R_\alpha) \\
&\quad + \frac{1}{\alpha}(C + C|y|^\delta)\eta_m^{-1} \\
&\leq (\overline{C}|y| + |\mu(0)|)|q| + (\alpha + C + C|y|^\delta)(|p| + \alpha R_\alpha) \\
&\quad + \frac{1}{\alpha}(C + C|y|^\delta)\eta_m^{-1} \\
&\leq \tfrac{1}{2}q^2 + \tfrac{1}{2}(\overline{C}|y| + |\mu(0)|)^2 + (\alpha + C + C|y|^\delta)(|p| + \alpha R_\alpha) \\
&\quad + \frac{1}{\alpha}(C + C|y|^\delta)\eta_m^{-1} \\
&\leq \tfrac{1}{2}q^2 + \Psi,
\end{aligned}
$$

where

$$
\Psi = \frac{1}{2}(\overline{C}y_m + |\mu(0)|)^2 + (\alpha + C + Cy_m^\delta)(3\Phi + \alpha R_\alpha) + \frac{1}{\alpha}(C + Cy_m^\delta)\eta_m^{-1}
$$

with $y_m := \max\{|\underline{y}|, |\overline{y}|\}$. Thus, the proposition follows from Lemma 3.9 and Theorem 3.8 of [11].

4. Verification theorem

In this section we prove a verification theorem, in which we verify that the pre-default value function is $\widetilde{v}(x, y) = (1/\alpha) \log x + \widetilde{\omega}(y)$, where $\widetilde{\omega}(y)$ is given in Proposition 3.1, and that the optimal control strategy is given by

$$
\kappa_t^* = \kappa^*(y_t) = \frac{1 - \eta(y_t)}{\rho}, \qquad \ell_t^* = \frac{b - r}{a^2}, \qquad c_t^* = \alpha. \tag{4.1}
$$

Firstly, we have the following lemma.

Lemma 4.1. *Suppose that assumptions (H1)–(H3) hold and that $\alpha > \delta\overline{C}$. Then the triplet $(\kappa_t^*, \ell_t^*, c_t^*)_{t \geq 0}$ given in (4.1) is an admissible control, i.e. $(\kappa_t^*, \ell_t^*, c_t^*)_{t \geq 0} \in \mathcal{A}(\mathbb{G})$. Moreover, the following properties hold:*

(a) $\lim_{T \to \infty} e^{-\alpha T} E[\int_0^T \ell_s^2 \, ds] = 0$,

(b) $\lim_{T \to \infty} e^{-\alpha T} E[\int_0^{T \wedge \tau} \lambda(y_s) \log^2(1 - \rho\kappa_s) \, ds] = 0$,

(c) $c_t \leq N$ *for some $N > 0$, P-almost surely (P-a.s.).*

Proof. Obviously, the $(\ell_t^*)_{t \geq 0}$ satisfy (a). Note that, for $y \in \mathbb{R}$, $\eta(y) \in (0, 1]$, and so $\kappa_t^* \in [0, 1/\rho)$ for each $t \geq 0$. Also, c_t^* obviously satisfies (c). On the other hand, it follows from assumption (H3) that $1 \leq \eta^{-1}(y) \leq \eta_m^{-1}$ for all $y \in \mathbb{R}$. Thus, in light of assumption (H2)

and Lemma 2.1,

$$
\begin{aligned}
0 &\le \mathrm{e}^{-\alpha T}\mathrm{E}\left[\int_0^{T\wedge\tau} \lambda(y_t)\log^2(1-\rho\kappa_t^*)\,\mathrm{d}t\right] \\
&= \mathrm{e}^{-\alpha T}\mathrm{E}\left[\int_0^{T\wedge\tau} \lambda(y_t)\log^2\left(\frac{1}{\eta(y_t)}\right)\mathrm{d}t\right] \\
&\le C\log^2(\eta_m)\mathrm{e}^{-\alpha T}\mathrm{E}\left[\int_0^{T}(1+|y_t|^\delta)\,\mathrm{d}t\right] \\
&\to 0
\end{aligned}
$$

as $T\to\infty$. This proves that $(\kappa_t^*)_{t\ge0}$ satisfies (b).

Now we are in a position to state the main result of the paper.

Theorem 4.1. (Verification theorem.) *Suppose that assumptions (H1)–(H3) hold. Assume that $\alpha > 2\max\{\delta, 2\}\overline{C}$ (where $\overline{C}$ is given in (H1) and $\delta \ge 1$ is given in (H2)). Define a function on $(0,\infty)\times\mathbb{R}\times\{0,1\}$ by*

$$
\hat{v}(x, y, z) = \frac{1}{\alpha}\log x + zR_\alpha + (1-z)\widetilde{\omega}(y), \tag{4.2}
$$

where R_α is presented in (3.5) and $\widetilde{\omega}(y)$ is a classical solution to (3.9).

(a) *For all admissible control policies $(\kappa_t, \ell_t, c_t)_{t\ge0}\in\mathcal{A}(\mathbb{G})$, it holds that*

$$
\hat{v}(x, y, z) \ge \mathrm{E}_{x,y,z}\left[\int_0^\infty \mathrm{e}^{-\alpha t}U(c_t x_t)\,\mathrm{d}t\right],
$$

with $(x, y, z)\in(0,\infty)\times\mathbb{R}\times\{0,1\}$.

(b) *Define*

$$
\ell_t^* = \frac{b-r}{a^2}, \qquad t\ge0, \tag{4.3}
$$

$$
c_t^* = \alpha, \qquad t\ge0, \tag{4.4}
$$

and

$$
\kappa_t^* = \kappa^*(y_t) =
\begin{cases}
\dfrac{1-\eta(y_t)}{\rho}, & 0\le t < \tau, \\
0, & t\ge\tau.
\end{cases}
\tag{4.5}
$$

Then $(\kappa_t^*, \ell_t^*, c_t^*)_{t\ge0}\in\mathcal{A}(\mathbb{G})$ and its value function $v = \hat{v}$, i.e. for $(x, y, z)\in(0,\infty)\times\mathbb{R}\times\{0,1\}$,

$$
v(x, y, z) := \mathrm{E}_{x,y,z}\left[\int_0^\infty \mathrm{e}^{-\alpha t}U(c_t^* x_t^*)\,\mathrm{d}t\right] = \hat{v}(x, y, z), \tag{4.6}
$$

where $(x_t^*)_{t\ge0}$ is the wealth process satisfying (3.1) with $(\kappa_t, \ell_t, c_t)_{t\ge0}$ replaced by $(\kappa_t^*, \ell_t^*, c_t^*)_{t\ge0}$.

Proof. Recall x_t, y_t, and z_t in Sections 2 and 3. Let $\hat{v}$ be as defined in (4.2). Then the Itô formula yields

$$
\begin{aligned}
\mathrm{d}\hat{v}(x_t, y_t, z_t) &= x_t \hat{v}_x \left[r - c_t + \ell_t(b - r) + \kappa_t(1 - z_t)\lambda(y_t)\left(\frac{\rho}{\eta(y_t)}\right) \right] \mathrm{d}t \\
&\quad + \left[\tfrac{1}{2} x_t^2 \hat{v}_{xx} \ell_t^2 a^2 + \tfrac{1}{2} \hat{v}_{yy} + \mu(y_t)\hat{v}_y \right] \mathrm{d}t \\
&\quad + \left[\hat{v}(x_t - x_t \kappa_t(1 - z_t)\rho, y_t, z_t + 1) - \hat{v}(x_t, y_t, z_t) \right] \\
&\qquad \times (1 - z_t)\lambda(y_t)\,\mathrm{d}t + \mathrm{d}M_t, \qquad t \geq 0,
\end{aligned}
\tag{4.7}
$$

where

$$
\begin{aligned}
M_t &= \int_0^t x_s \hat{v}_x(x_s, y_s, z_s)\ell_s a\,\mathrm{d}W_s + \int_0^t a\hat{v}_y(x_s, y_s, z_s)\,\mathrm{d}\breve{W}_s \\
&\quad + \int_0^{t+} [\hat{v}(x_{s-} - x_{s-}\kappa_{s-}(1 - z_{s-})\rho, y_s, z_{s-} + 1) - \hat{v}(x_{s-}, y_s, z_{s-})]\,\mathrm{d}m_s.
\end{aligned}
$$

By the assumptions in Section 2, $(M_t)_{t \geq 0}$ is a (P, $\mathbb{G}$)-adapted càdlàg martingale. Note that

$$
x\hat{v}_x = \alpha^{-1}, \qquad x^2 \hat{v}_{xx} = -\alpha^{-1}.
$$

Consequently, by some standard calculus and (3.8), we have

$$
\begin{aligned}
\mathrm{d}\hat{v}(x_t, y_t, z_t) &\leq \alpha[z_t R_\alpha + (1 - z_t)\widetilde{\omega}(y_t)]\,\mathrm{d}t + \mathrm{d}M_t \\
&= [\alpha\hat{v}(x_t, y_t, z_t) - U(c_t x_t)]\,\mathrm{d}t + \mathrm{d}M_t,
\end{aligned}
\tag{4.8}
$$

where $(\kappa_t^*, \ell_t^*, c_t^*)_{t \geq 0}$ is given in (4.1). So

$$
\begin{aligned}
\mathrm{d}[\mathrm{e}^{-\alpha t}\hat{v}(x_t, y_t, z_t)] &= \mathrm{e}^{-\alpha t}\,\mathrm{d}\hat{v}(x_t, y_t, z_t) - \alpha\mathrm{e}^{-\alpha t}\hat{v}(x_t, y_t, z_t)\,\mathrm{d}t \\
&\leq \mathrm{e}^{-\alpha t}[\alpha\hat{v}(x_t, y_t, z_t) - U(c_t x_t)]\,\mathrm{d}t + \mathrm{e}^{-\alpha t}\,\mathrm{d}M_t - \alpha\mathrm{e}^{-\alpha t}\hat{v}(x_t, y_t, z_t)\,\mathrm{d}t.
\end{aligned}
$$

This implies that, for $T > 0$,

$$
\hat{v}(x, y, z) \geq \mathrm{E}_{x,y,z}\left[\int_0^T \mathrm{e}^{-\alpha s} U(c_s x_s)\,\mathrm{d}s \right] + \mathrm{E}_{x,y,z}[\mathrm{e}^{-\alpha T}\hat{v}(x_T, y_T, z_T)].
\tag{4.9}
$$

On the other hand, from Itô's formula (see, e.g. [24, p. 78]), it follows that

$$
\begin{aligned}
\log x_T &= \log x_0 + \int_0^T \ell_s a\,\mathrm{d}W_s + \int_0^T \log(1 - \kappa_{s-}\rho(1 - z_{s-}))\,\mathrm{d}m_s \\
&\quad + \int_0^T \left[r - c_s + \ell_s(b - r) + \kappa_s(1 - z_s)\lambda(y_s)\left(\frac{\rho}{\eta(y_s)}\right) - \frac{1}{2}\ell_s^2 a^2 \right]\mathrm{d}s \\
&\quad + \int_0^{T \wedge \tau} \lambda(y_s)\log(1 - \kappa_s\rho(1 - z_s))\,\mathrm{d}s.
\end{aligned}
\tag{4.10}
$$

Once again, since $(\kappa_t, \ell_t, c_t)_{t \geq 0} \in \mathcal{A}(\mathbb{G})$, we have, for $x > 0$,

$$
\limsup_{T \to \infty} \mathrm{e}^{-\alpha T} \mathrm{E}_x\left[\int_0^T -\frac{1}{2} a^2 \ell_s^2\,\mathrm{d}s \right] \geq -\frac{1}{2} a^2 \lim_{T \to \infty} \mathrm{e}^{-\alpha T} \mathrm{E}_x\left[\int_0^T \ell_s^2\,\mathrm{d}s \right] = 0.
$$

It follows that

$$\limsup_{T\to\infty} e^{-\alpha T} E_x \left[\int_0^T (r + \ell_s(b-r))\, ds \right]$$

$$\geq \lim_{T\to\infty} \left[rT - \frac{1}{2}(b-r)^2 T \right] e^{-\alpha T} - \frac{1}{2} \lim_{T\to\infty} e^{-\alpha T} E_x \left[\int_0^T \ell_s^2\, ds \right]$$

$$= 0.$$

Moreover, it holds that

$$\limsup_{T\to\infty} e^{-\alpha T} E_x \left[\int_0^T -c_s\, ds \right] \geq -N \lim_{T\to\infty} e^{-\alpha T} T = 0.$$

On the other hand, by virtue of Lemma 2.1,

$$\limsup_{T\to\infty} e^{-\alpha T} E_x \left[\int_0^T \kappa_s(1-z_s)\lambda(y_s)\frac{\rho}{\eta(y_s)}\, ds \right]$$

$$\geq \liminf_{T\to\infty} e^{-\alpha T} E_x \left[-\int_0^{T\wedge\tau} \kappa_s \frac{\rho}{\eta(y_s)}\lambda(y_s)\, ds \right]$$

$$\geq \frac{1}{2}\liminf_{T\to\infty} e^{-\alpha T} E_x \left[-\int_0^{T\wedge\tau} \kappa_s^2 \lambda(y_s)\, ds \right]$$

$$+ \frac{1}{2}\liminf_{T\to\infty} e^{-\alpha T} E_x \left[-\int_0^{T\wedge\tau} \left(\frac{\rho}{\eta(y_s)} \right)^2 \lambda(y_s)\, ds \right]$$

$$\geq \frac{1}{2}\left[\frac{1}{\rho^2} + \frac{\rho^2}{\eta_m^2} \right] \lim_{T\to\infty} e^{-\alpha T} E_x \left[-CT - C\int_0^T |y_s|^\delta\, ds \right]$$

$$= 0$$

and

$$\limsup_{T\to\infty} e^{-\alpha T} E_x \left[\int_0^T \lambda(y_s)(1-z_s)\log(1-\kappa_s\rho(1-z_s))\, ds \right]$$

$$\geq \frac{1}{2}\lim_{T\to\infty} e^{-\alpha T} E_x \left[-\int_0^{T\wedge\tau} \lambda(y_s)\log^2(1-\kappa_s\rho)\, ds \right]$$

$$+ \frac{1}{2}\lim_{T\to\infty} e^{-\alpha T} E_x \left[-\int_0^T (C + C|y_s|^\delta)\, ds \right]$$

$$= 0.$$

Based on the above derivations, part (a) follows from Proposition 3.1 and (4.9). Recall (3.8). Then a similar argument as that used for (4.9) shows that

$$\hat{v}(x, y, z) = E_{x,y,z}\left[\int_0^T e^{-\alpha s}\log(c_s^* x_s^*)\, ds \right] + e^{-\alpha T} E_{x,y,z}[\hat{v}(x_T^*, y_T, z_T)]. \qquad (4.11)$$

Combining the survival probability (2.3) and Proposition 3.1, we have

$$
\begin{aligned}
\mathrm{E}_{x,y,z}[\hat{v}(x_T^*, y_T, z_T)] \\
&= \frac{1}{\alpha}\mathrm{E}[\log x_T^*] + R_\alpha \mathrm{P}(\tau \le T) + \mathrm{E}_{x,y,z}[(1 - z_T)\widetilde{\omega}(y_T)] \\
&\le \frac{1}{\alpha}\mathrm{E}[\log x_T^*] + R_\alpha + \mathrm{E}_{x,y,z}[C_1 y_T^2 + C_2],
\end{aligned}
$$

where the constants C_1 and C_2 are given in Proposition 3.1. Similarly as in the proof of part (a), using Lemma 2.1,

$$
\liminf_{T \to \infty} \mathrm{e}^{-\alpha T}\mathrm{E}_{x,y,z}[\hat{v}(x_T^*, y_T, z_T)] \le 0. \tag{4.12}
$$

Thus, apply (a), the Fatou lemma, and (4.10)–(4.11) to conclude that (b) holds. This completes the proof.

Appendix A. Price dynamics for a defaultable bond

In this appendix we derive the price dynamics for a perpetual defaultable bond that pays a constant coupon $\widetilde{C}$ per unit time. Here the spot interest rate r and the loss rate, given default $\rho \in (0, 1)$, are two positive constants.

Let Q be the risk-neutral probability measure. Suppose that the P-default intensity λ and the default risk premium $1/\eta$ are two positive constants. Then the Q-default intensity is $\lambda^Q = \lambda/\eta$.

Under the recovery of market value scheme, the pre-default value of the bond at time t is given by

$$
V_t = \mathrm{E}^Q\!\left[\int_t^\infty \widetilde{C}\mathrm{e}^{-r(s-t)}\mathbf{1}_{\{\tau > s\}}\,\mathrm{d}s \,\Big|\, \mathscr{G}_t\right] + \mathrm{E}^Q[\mathrm{e}^{-r(\tau - t)}(1 - \rho)V_{\tau-} \mid \mathscr{G}_t]\mathbf{1}_{\{\tau > t\}}. \tag{A.1}
$$

From Lemma 5.1.2 of [5] we have

$$
V_t = \frac{\widetilde{C}}{r + \lambda^Q} + (1 - \rho)\lambda^Q \mathrm{E}^Q\!\left[\int_t^\infty \mathrm{e}^{-(r+\lambda^Q)(s-t)}V_s\,\mathrm{d}s \,\Big|\, \mathscr{F}_t\right] \quad \text{on } \{\tau > t\}. \tag{A.2}
$$

Differentiating (A.2) with respect to t yields

$$
\begin{aligned}
\mathrm{d}V_t &= -(1 - \rho)\lambda^Q V_t\,\mathrm{d}t + (r + \lambda^Q)\!\left(V_t - \frac{\widetilde{C}}{r + \lambda^Q}\right)\mathrm{d}t \\
&= \widetilde{r}V_t\,\mathrm{d}t - \widetilde{C}\,\mathrm{d}t \quad \text{on } \{\tau > t\}, \tag{A.3}
\end{aligned}
$$

with the adjusted interest rate $\widetilde{r} = r + \rho\lambda^Q$. By some standard calculus,

$$
V_t = \frac{\widetilde{C}}{\widetilde{r}} + \left(V_0 - \frac{\widetilde{C}}{\widetilde{r}}\right)\mathrm{e}^{\widetilde{r}t}. \tag{A.4}
$$

We now define the price process for a perpetual defaultable bond under Q as follows (see the price process (1) in [4]):

$$
p_t = \mathbf{1}_{\{\tau > t\}}V_t + \mathbf{1}_{\{\tau \le t\}}(1 - \rho)V_\tau \mathrm{e}^{r(t-\tau)}. \tag{A.5}
$$

Recall that $z_t = \mathbf{1}_{\{\tau \le t\}}$, and note that $\mathrm{d}z_t = (1 - z_t)\,\mathrm{d}z_t$, $V_t\,\mathrm{d}z_t = V_\tau\,\mathrm{d}z_t$, and $\mathrm{e}^{r(t-\tau)}\,\mathrm{d}z_t = \mathrm{d}z_t$. Applying Itô's formula to (A.5), we obtain

$$
\mathrm{d}p_t = rp_t\,\mathrm{d}t + \rho\lambda^Q(1 - z_t)p_t\,\mathrm{d}t - (1 - z_t)\widetilde{C}\,\mathrm{d}t - \rho p_t\,\mathrm{d}z_t. \tag{A.6}
$$

Moreover, the price dynamics of $(p_t)_{t \geq 0}$ under the physical probability measure P is

$$\mathrm{d}p_t = r p_t \, \mathrm{d}t + \rho \lambda \left(\frac{1}{\eta} - 1 \right)(1 - z_t) p_t \, \mathrm{d}t - (1 - z_t)\widetilde{C} \, \mathrm{d}t - \rho p_t \, \mathrm{d}m_t, \qquad \text{(A.7)}$$

where $m_t = z_t - \int_0^t (1 - z_s)\lambda \, \mathrm{d}s$ is a P-martingale.

Appendix B. Numerical results

In this appendix we present a parametric sensitivity analysis for the optimal control $(\kappa_t^*)_{t \geq 0}$ of the defaultable bond and the value function v obtained in (4.2) using some numerical simulations.

For ease of exposition, we adopt the following abbreviations.

OPDB	Optimal investment proportion of the defaultable bond.
LR	Loss rate $\rho \in (0, 1]$ when a default occurs.
DRP (DRPF)	Default risk premium (function) $1/\eta = 1/\eta(y) \geq 1$.
SF	Stochastic factor $y \in \mathbb{R}$.
PDV	Post-default value $\overline{v}(x, y) = \hat{v}(x, y, 1)$.
DF	Discount factor $\alpha > 0$.
WEA	The wealth $x > 0$.
VF	The value function $\hat{v}(x, y, z)$.

Suppose that $\lambda(y) = \varepsilon + |y| (\varepsilon > 0)$, $\mu(y) = \overline{C}(1 - y)$, and that the DRPF is a constant, i.e. $\eta(y) \equiv \eta \in (0, 1]$. We quote some parameters from [4]:

ρ	α	r	a	$\overline{C}$	η	b	ε
0.52, 0.62, 0.9	$2\overline{C} + \varepsilon$	0	0.15	0.5	0.395 257	0.067 65	0.01

We begin by discussing the parametric sensitivity in the optimal control κ^* for the defaultable bond. Recall (4.5). The optimal control κ^* is completely characterized by the LR and DRP, and it is independent of the risk-free interest rate r and of the default intensity process λ_t (different from that in [14] for the exponential utility). From Figures 1 and 2, we find that, for a fixed DRP, the OPDB increases while the LR decreases; for a fixed LR, the investor increases the amount of OPDB when the DRP increases (a similar phenomena appears in [14]). However, if the DRPF relies on some SF (not constant) then the OPDB is a joint function of the LR and SF variables. Figure 3 displays the relationship between the OPDB, LR, and SF for a DRPF. (The choice of the DRPF in Figure 3 is not experiential, we use it only to illuminate the analysis.)

We now investigate the parametric sensitivity of the value function $\hat{v}(x, y, z)$ given by (4.2). Owing to the form of the equation,

$$\frac{1}{2}\widetilde{\omega}_{yy} + \overline{C}(1 - y)\widetilde{\omega}_y - (\alpha + \varepsilon + |y|)\widetilde{\omega} + \alpha R_\alpha + \frac{\varepsilon + |y|}{\alpha}h(\eta) = 0, \qquad \text{(B.1)}$$

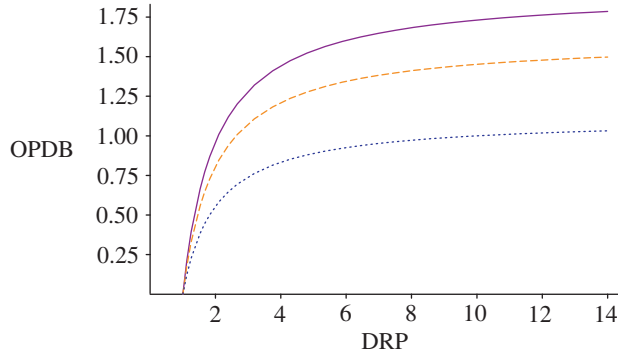with $y \in \mathbb{R}$, we do not expect (B.1) to admit a closed-form solution.

FIGURE 1: The OPDB versus the DRP for LR $= 0.52, 0.62, 0.9$. The solid, dashed, and dotted lines correspond to $R = 0.52, 0.6$, and 0.9, respectively.
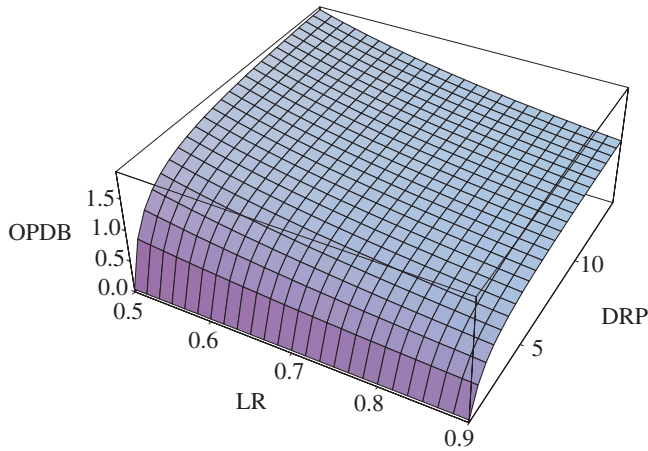

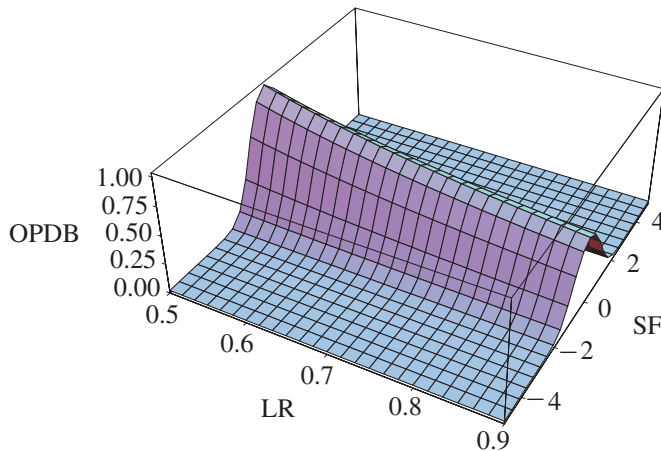FIGURE 2: The OPDB versus the LR and DRP.

FIGURE 3: The OPDB versus the LR and SF with $\eta_m = \frac{1}{2}$ and DRPF $\eta^{-1}(y) = 1 + e^{-y^2}$.

Thanks to Proposition 3.1, we can obtain the lower and upper bounds:

$$
\begin{aligned}
\hat{v}(x, y, z) &= \frac{1}{\alpha} \log x + z R_\alpha + (1 - z)\widetilde{\omega}(y) \\
&\geq \frac{1}{\alpha} \log x + z R_\alpha + (1 - z) R_\alpha \\
&= \frac{1}{\alpha} \log x + R_\alpha \\
&= \overline{v}(x, y),
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\hat{v}(x, y, z) &\leq \frac{1}{\alpha} \log x + R_\alpha + C_2 + C_1 y^2 \\
&= \overline{v}(x, y) + C_1 y^2 + C_2,
\end{aligned}
\tag{B.3}
$$

for all $(x, y, z) \in (0, \infty) \times \mathbb{R} \times \{0, 1\}$, where $\overline{v}(x, y)$ is the post-default value function defined in (3.4), and the two constants $C_1 > 0$ and $C_2 > R_\alpha$ are defined in Proposition 3.1. Here we switch to study sensitivity of these bounds with respect to the parameters WEA, SF, and DF. Figures 4 and 5 exhibit the relationships between the PDV and WEA, and the PDV, DF, and WEA, respectively. Figure 6 depicts the lower bound (B.2) and the upper bound (B.3) of the value function $\hat{v}$ with respect to the parameters SF and WEA.
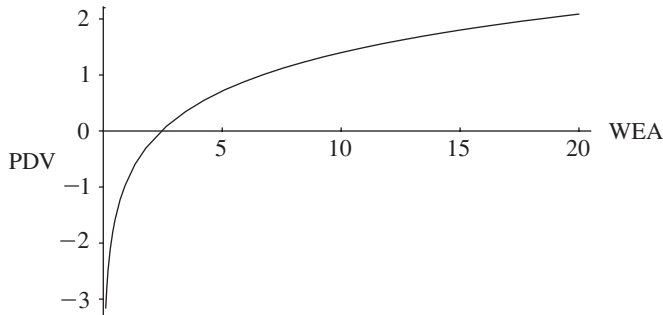

FIGURE 4: The PDV versus the WEA with DF $\alpha = \varepsilon + 2\overline{C}$.
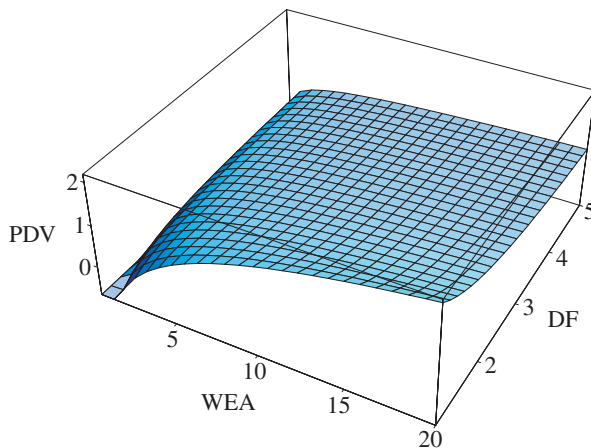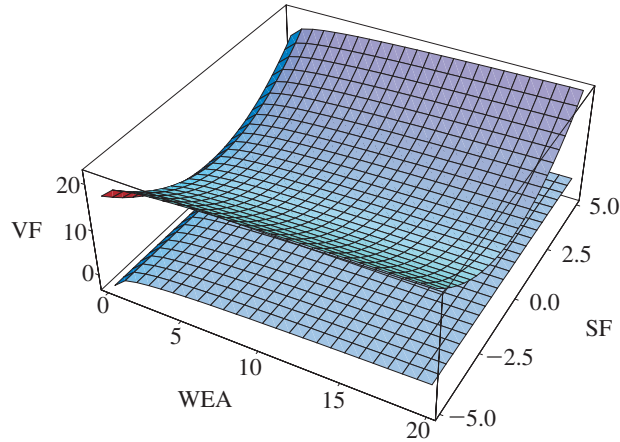

FIGURE 5: The PDV versus the DF and WEA.

FIGURE 6: Upper and lower bounds for the of VF versus SF and WEA, with $C_2 > R_\alpha$ and DF $\alpha = \varepsilon + 2\overline{C}$.

Acknowledgements

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