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# SCATTERING THEORY AND SPECTRAL REPRESENTATIONS FOR GENERAL WAVE EQUATIONS WITH SHORT RANGE PERTURBATIONS

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1. **Introduction.** In this paper we shall develop the scattering theory introduced by Lax and Phillips [5] for the following general wave equation;

$$\text{(1.1)} \qquad \begin{cases} u_{tt} = \alpha(x)\{\partial_i a_{ij}(x)\partial_j u - q(x)u\} & \text{in } \Omega \times R, \\ Bu = 0 & \text{on } \partial\Omega \times R, \end{cases}$$

where $\Omega$ is an exterior domain $R^n (n \geq 3)$ with the smooth boundary $\partial\Omega$ and $B$ is either a Dirichlet boundary condition or of the form $Bu = \nu_i(x)a_{ij}(x)\partial_j u + \sigma(x)u$ with the unit outer normal vector $\nu(x) = (\nu_1, \ldots, \nu_n)$ at $x \in \partial\Omega$. The precise assumptions on $\alpha(x)$, $a_{ij}(x)$, $q(x)$, $\sigma(x)$ are denoted below. If $\Omega$ is an inhomogeneous medium with the density $\rho(x)$, the propagation of waves is described by (1.1) with $\alpha(x) = a(x)^2\rho(x)$, $a_{ij}(x) = \rho^{-1}(x)\delta_{ij}$ and $q(x) = 0$ with the velocity $a(x)$.

The scattering theory of (1.1) in $L^2$-theory is studied by many authors (Ikebe [3] and [4], Mochizuki [8] and [9] and Reed-Simon Chapter XI.10 of [12], etc). On the other hand Lax and Phillips theory of (1.1) is first studied in their book [5] (see also [6] and [13]) with $\alpha = 1$, $a_{ij} = \delta_{ij}$ and $q = 0$. In [14] the developed theory is considered in the case $\alpha = 1$, $a_{ij} = \delta_{ij}$, $q \geq 0$ and $\Omega = R^n$, and shows a completeness of wave operators and an existence of spectral representations. A completeness of wave operators of (1.1) is also established by Lax and Phillips [7] and Phillips [11] either in the case $\alpha = 1$, $a_{ij} = \delta_{ij}$ and $\operatorname{supp} q$ is compact or in the case $\alpha = 1$ and $\Omega = R^n$. In this paper we shall show generalizations of their theorems on a completeness of wave operators and an existence of spectral representations. We also show an invariant principle of wave operators.

We shall state the assumptions on the coefficients of (1.1).

(A.1) The function $\alpha(x)$ is in $C^1(\bar{\Omega})$, real valued, and uniformly positive in $\bar{\Omega}$ and satisfies $\alpha(x) - 1 = 0(|x|^{-1-\delta})$ for some $\delta > 0$.

(A.2) The real symmetric matrix $\big(a_{ij}(x)\big)$ is uniformly positive in $\bar{\Omega}$. The functions $a_{ij}(x)$ are in $C^2(\bar{\Omega})$ and satisfy conditions $a_{ij}(x) - \delta_{ij} = 0(|x|^{-1-\delta}) = \nabla a_{ij}(x)$ for some $\delta > 0$.

(A.3) The real valued function $q(x)$ is in $L^p_{\text{loc}}(\Omega)$ where $p = n/2$ for $n > 4$, $p > 2$ for $n = 4$ and $p = 2$ for $n = 3$, and satisfies $q(x) = 0(|x|^{-2-\delta})$ for some $\delta > 0$, if $|x|$ is sufficiently large.

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(A.4) The function $\sigma(x)$ is a real valued $C^2(\partial\Omega)$ function.

(A.5) The unique continuation property for eigenfunctions associated with nonnegative eigenvalues of $L$ holds, i.e., if $u \in H^2_{\mathrm{loc}}(\bar{\Omega})$ satisfies $(L-\lambda)u = 0$, where $\lambda \geq 0$, and $u(x) = 0$ in some open subset of $\Omega$, then $u = 0$ in $\Omega$.

2. **Perturbed system.** In this section for simplicity we only consider the Neumann boundary condition $Bu = \nu_i a_{ij}\partial_j u + \sigma u = 0$ on $\partial\Omega \times R$. Let $\mathfrak{h}$ be a Hilbert space which is equal to $L^2(\Omega)$ as a set with the inner product $(f,g)_{\mathfrak{h}} = \int_\Omega (f\bar{g}\alpha^{-1})(x)\,dx$, and let $L$ be an operator from $D(L) \subset \mathfrak{h}$ to $\mathfrak{h}$ defined by $Lf = -\alpha(x)\{\partial_i a_{ij}(x)\partial_j - q(x)\}$, where

$$D(L) = \Big\{f \in H^1(\Omega) : Lf \in L^2(\Omega) \text{ as } \mathcal{D}'(\Omega), \\ (Lf, v)_{\mathfrak{h}} = \int_\Omega \{a_{ij}\partial_j f \partial_i \bar{v} + qf\bar{v}\}\,dx + \int_{\partial\Omega} \sigma f \bar{v}\,dS \text{ for all } v \in H^1(\Omega)\Big\}.$$

When $q = 0$, we denote $L$ by $L_1$. Then we have the following

LEMMA 2.1.

*i) The operator L is self adjoint and* $D(L) = \{f \in H^2(\Omega) : Bf = \nu(x)a_{ij}(x)\partial_j f + \sigma(x)f = 0$ *on* $\partial\Omega\}$.

*ii) L has no positive eigenvalues.*

*iii) The number of the non-positive eigenvalues of L is finite and their eigenspaces are finite dimensional spaces.*

The statement i) is proved by the similar argument in the proof of Theorem 3.6 below. The statement ii) is Corollary 1.1 in [10] and the proof of iii) is almost the same as one of Lemma 3.15 in [11].

We introduce data space $H = H_D(\Omega) \times \mathfrak{h}$, where $H_D(\Omega)$ is the completed space of $C_0^\infty(\bar{\Omega})$ by the norm $\|f\|_{H_D(\Omega)}$. The energy form of (1.1) is defined by

$$E(f,g) = \frac{1}{2}\int_\Omega \{a_{ij}\partial_j f_1 \partial_i \bar{g}_1 + qf_1\bar{g}_1 + \alpha^{-1}f_2\bar{g}_2\}\,dx + \frac{1}{2}\int_{\partial\Omega} \sigma f_1 \bar{g}_1\,dS$$

for any $f = (f_1, f_2)$ and $g = (g_1, g_2)$ in $H$. We define an operator $A = \begin{pmatrix} 0 & 1 \\ -L & 0 \end{pmatrix}$ from

$$D(A) = \Big\{f \in H : f_2 \in H^1(\Omega), Lf_1 \in L^2(\Omega) \text{ as } \mathfrak{D}'(\Omega) \text{ and}$$

$$(2.1) \qquad (Lf_1, v)_{\mathfrak{h}} = \int_\Omega \{a_{ij}\partial_j f_1 \partial_i \bar{v} + qf_1\bar{v}\}\,dx + \int_{\partial\Omega} \sigma f_1 \bar{v}\,dS \text{ for } v \in H^1(\Omega)\Big\}$$

to $H$. Similarly $A_1$ is defined by changing $L$ in $A$ with $L_1 = -\alpha(x)\partial_i a_{ij}(x)\partial_j$. The following properties of $A$ and $A_1$ are not difficult.

LEMMA 2.2. *$A$ and $A_1$ are densely defined and closed operators, and $E(Af,g) = -E(f,Ag)$ for any $f,g \in D(A)$. A similar relation holds for the energy form $E_1$ and $A_1$ corresponding to $L_1$.*

Let $\{-\lambda_j^2 : j = 1,\dots,m\}$ denote the negative eigenvalues of $L$, and let $\{p_j \in \mathfrak{h} : j = 1,\dots,m\}$ be the corresponding linearly independent eigenfunctions which span the eigenspace corresponding to all negative eigenvalues. We can choose the data $f_j^{\pm} = (p_j, \pm p_j)$ which are eigenvectors of $A$ with respect to $\pm\lambda_j$, and satisfy the following relations (see pages 48–49 of [7]); for all $j, k$

$$E(f_j^+, f_k^+) = 0 = E(f_j^-, f_k^-), \quad E(f_j^+, f_k^-) = -\lambda_j^2 \delta_{jk}. \tag{2.2}$$

We denote by $\mathcal{P}$ the span of the $\{f_j^{\pm}\}$. Then it is clear from (2.2) that every $f$ in $H$ has a unique decomposition of the form $f = g + p$, where $p \in \mathcal{P}$ and $g$ lies in the following space

$$\mathcal{H}' = \{f \in H : E(f, p) = 0 \text{ for all } p \in \mathcal{P}\}.$$

We denote by $I$ the degenerate space of the energy form $E$;

$$I = \{f \in \mathcal{H}' : E(f, g) = 0 \text{ for all } g \in H\}.$$

By the proof of Lemma 3.15 in [11] we can show the following

LEMMA 2.3. *$I$ is equal to* Ker $A$ *and is a finite dimensional space.*

From this lemma and (A.5) the form

$$K(f, g) = \int_{\Omega \cap \{|x| < R\}} f_1 \bar{g}_1 \, dx,$$

where $R$ is chosen large enough, defines an equivalent norm on $I$ to the one of $H$. We decompose any element $f$ in $\mathcal{H}'$ into the form $f = f' + f''$, where $f' \in I$ and $f''$ belongs to the following space

$$\mathcal{H}'' = \{f \in \mathcal{H}' : K(f, g) = 0 \text{ for all } g \in I\}.$$

Then we have the following proposition (see Lemma 3.19, Corollary 3.29) and 3.21 in [11]).

PROPOSITION 2.4. *i) The form $E'(f) = K(f') + E(f'')$ on $\mathcal{H}'$, where $K(f') = K(f', f')$ and $E(f'') = E(f'', f'')$, defines an equivalent norm on $\mathcal{H}'$ to $\|f\|_H$. ii) The quotient space $\hat{\mathcal{H}} = \mathcal{H}' / I$ is complete in the E-norm. iii) If $f$ belongs to $\mathcal{H}' \cap D(A)$, then for $\|f\|_E^2 = E(f)$*

$$\|Af\|_H \leq C(\|f\|_E + \|Af\|_E).$$

Making use of the above proposition we shall show the following.

PROPOSITION 2.5. *If $|\lambda|$ is sufficiently large, $\lambda$ belongs to the resolvent set of $A_1$.*

PROOF. Let $\mathcal{P}_1$, $\mathcal{H}_1'$, $I_1$ and $\hat{\mathcal{H}}_1$ denote the corresponding spaces to $\mathcal{P}$, $\mathcal{H}'$, $I$ and $\hat{\mathcal{H}}$, respectively, which are similarly defined by changing $L$ in $A$ with $L_1$. Since $\mathcal{P}_1$ and $I_1$ are invariant subspaces of $A_1$ from Lemma 2.3, we can define an operator $\hat{A}_1$ from $D(\hat{A}_1) = \big(D(A_1) \cap \mathcal{H}_1'\big) / I_1 \subset \hat{\mathcal{H}}_1$ to $\hat{\mathcal{H}}_1$. We shall prove that $\hat{A}_1$ is skew self-adjoint in $\hat{\mathcal{H}}_1$. It is easy to show that $D(\hat{A}_1)$ is a dense set of $\hat{\mathcal{H}}_1$. First we shall show that $\hat{A}_1$

is a closed operator. We assume that $\{\hat{f}_n\} \subset D(\hat{A}_1)$ and $\{\hat{A}_1\hat{f}_n\}$ converge to $\hat{f}$ and $\hat{g}$ in the energy norm, respectively. Then from Proposition 2.4 the projections $\{f_n''\}$ and $\{[A_1f_n]''\}$ of $\{f_n\}$ and $\{A_1f_n\}$ to $\mathcal{H}''$, respectively, converge to projections $f''$ and $g''$ of $f$ and $g$, respectively in the $H$-norm. Let $\{g_j = (g_{j1}, 0 : j = 1, \ldots, \ell\}$ denote a base of $I$ such that $K(g_j, g_k) = \delta_{jk}$. Then

$$[A_1f_n]' = [A_1f_n'']' = \sum_{j=1}^{\ell}(f_{n2}'', g_{j1})_{L^2\left(\Omega\cap\{|x|<R\}\right)}g_j, \tag{2.3}$$

where $f_n'' = (f_{n1}', f_{n2}'')$, and $[A_1f_n]'$ and $[A_1f_n'']'$ are projections of $Af_n$ and $Af_n''$ to $I$, respectively. It follows from (2.3) that $\{[A_1f_n]'\}$ is a bounded set of $H$. Since $I_1$ is finite dimensional, we can take a subsequence $\{n_k\}$ of $\{n\}$ such that $[A_1f_{n_k}]'$ converges to $h \in I_1$ in the $H$-norm. From the closedness of $A_1$ we see that $A_1f'' = A_1f = g'' + h$, which means that $\hat{A}_1\hat{f} = \hat{g}$, that is, $\hat{A}_1$ is closed.

Since $\hat{A}_1$ is skew-symmetric from Lemma 2.2, we only show that the range of $\lambda \pm \hat{A}_1$ contains a dense subset of $\hat{\mathcal{H}}_1$, where $\lambda$ is real. Let $g = (g_1, g_2)$ be an element of $\mathcal{H}' \cap \left(H^1(\Omega)\right)^2$. Then $(\lambda \pm A_1)f = g$ is equivalent to $(\lambda^2 + L_1)f_1 = \lambda g_1 \mp g_2$ and $f_2 = \pm(g_1 - \lambda f_1)$. The norm induced by the form $\ell_\lambda(u) = \int_\Omega\{a_{ij}\partial_j u\partial_i\bar{u} + \lambda^2|u|^2\alpha^{-1}\}\,dx + \int_{\partial\Omega}\sigma|u|^2\,dS$ is equivalent to the $H^1(\Omega)$ norm, if $|\lambda|$ is sufficiently large. Thus there exists $f_1 \in H^1(\Omega)$ such that $(\lambda g_1 \mp g_2, v)_{\mathfrak{H}} = \ell_\lambda(f_1, v)$ for all $v \in H^1(\Omega)$. This means that $f = (f_1, f_2) \in D(A_1)$ and $(\lambda \pm A_1)f = g$. Let $f = f' + p$ be the decomposition of $f$ in $\mathcal{H}' \oplus \mathcal{P}$ of $H$. Then we have $(\lambda \pm A_1)f' - g = (\lambda \pm A_1)p$. By the uniqueness of the decomposition it follows that $(\lambda \pm A_1)f' = g$, that is, $(\lambda \pm \hat{A}_1)\hat{f}' = \hat{g}$. The proof of the skew self-adjointness of $\hat{A}_1$ is completed.

In order to prove the statement of Proposition 2.5 we may show that the range of the restricted operator to $\mathcal{H}_1'$ of $\lambda + A_1$ is $\mathcal{H}_1'$, because $\ker(\lambda + A_1) = \{0\}$, if $|\lambda|$ is sufficiently large. For any $g \in \mathcal{H}_1'$ there exists $\hat{f} \in \hat{\mathcal{H}}_1$ such that $(\lambda + \hat{A}_1)\hat{f} = \hat{g}$. This means that there exists $h \in \mathcal{H}_1'$ such that $(\lambda + A_1)f = h$ and $\psi = g - h \in I$. Thus $(\lambda + A_1)(f + \psi/\lambda) = g$. The proof is completed.

Next we shall prove a similar property on $A$.

THEOREM 2.6. *i) The set $D(A)$ is equal to the set $\{f \in H : f_2 \in H^1(\Omega), \partial_x^\alpha f_1 \in L^2(\Omega)$ for $|\alpha| = 2$, $Bf_1 = 0$ on $\partial\Omega\}$. For any $f \in D(A)$*

$$\sum_{|\alpha|=1,2}\|\partial_x^\alpha f_1\|_{L^2(\Omega)} + \sum_{|\alpha|=0,1}\|\partial_x^\alpha f_2\|_{L^2(\Omega)} \le C(\|Af\|_H + \|f\|_H). \tag{2.4}$$

*ii) If $|\lambda|$ is sufficiently large, $\lambda$ belongs to the resolvent set of $A$.*

PROOF. Let $\varphi(x)$ be in $C^\infty(R^n)$ such that $\varphi(x) = 1$ for $|x| > R + 1$ and $\varphi(x) = 0$ for $|x| < R$, where $R$ is sufficiently large. Then for any $f \in D(L_1)$ $L_1(\varphi f_1)$ and $L_1\left((1-\varphi)f_1\right)$ belong to $L^2(\Omega)$. Thus it is well known that $(1-\varphi)f_1 \in H^2(\Omega)$ and $B(1-\varphi)f_1 = 0$ on $\partial\Omega$ (see Section 9, 10 of [1] and Chapter X of [2]). We shall show that $\partial_x^\alpha(\varphi f_1) \in L^2(\Omega)$ for $|\alpha| = 2$. In order to show this we need the following regularity theorem in the weighted space $H_\mu^2(\Omega)$, where $H_\mu^m(\Omega)$ is a Hilbert space with the norm $\|f\|^2_{H_\mu^m(\Omega)} = \sum_{|\alpha|\le m}\|(1+$

$|x|)^\mu \partial_x^\alpha f\|_{L^2(\Omega)}^2$; If $u \in L_\mu^2(\Omega) \cap H_{loc}^2(\bar{\Omega})$, where $L_\mu^2(\Omega) = H_\mu^0(\Omega)$, $L_1 u \in L_\mu^2(\Omega)$ and $Bu = 0$ on $\partial\Omega$, then $u$ belongs to $H_\mu^2(\Omega)$. This is derived by the same way of proving a similar regularity theorem in $H^2(R^n)$. From this theorem and the inequality $\|f\|_{L_\mu^2(\Omega)} \leq C_\mu \|f\|_{H_D(\Omega)}$, where $\mu < -1$ (see the proof of Lemma 1.1 in Chapter IV of [5]), it follows that $\varphi f_1$ belongs to $H_\mu^2(R^n)$. Therefore,

$$\Delta(\varphi f_1) = (1-\alpha)\Delta(\varphi f_1) + \alpha[\partial_i(\delta_{ij} - a_{ij})\partial_j](\varphi f_1) + L_1(\varphi f_1) \tag{2.5}$$

belongs to $L^2(R^n)$ from the assumptions (A.1) and (A.2). By the Fourier transform we see that $\partial_x^\alpha(\varphi f_1)$ ($|\alpha| = 2$) belongs to $L^2(R^n)$. This means that $D(L_1)$ is equal to $\{f \in H : f_2 \in H^1(\Omega), \partial_x^\alpha f_1 \in L^2(\Omega)$ for $|\alpha| = 2, Bf_1 = 0$ on $\partial\Omega\}$.

Next we shall show (2.4) when $q = 0$. From the elliptic estimate for a coercive elliptic boundary value problem (see Chapter X in [2]) it follows that

$$\sum_{|\alpha|=2} \|\partial_x^\alpha(1-\varphi)f_1\|_{L^2(\Omega)} \leq C\{\|L_1 f_1\|_{L^2(\Omega)} + \|f_1\|_{H_D(\Omega)}\}. \tag{2.6}$$

On the other hand from (2.5) it follows that

$$\begin{aligned} \sum_{|\alpha|=2} \|\partial_x^\alpha(\varphi f_1)\|_{L^2(R^n)} &\leq C\|\Delta(\varphi f_1)\|_{L^2(R^n)} \\ &\leq C \sup_{|x|>R}\{|1-\alpha(x)| + |a_{ij} - \delta_{ij}|\} \sum_{|\alpha|=2} \|\partial_x^\alpha(\varphi f_1)\|_{L^2(\Omega)} \\ &\quad + C_R(\|L_1 f_1\|_{L^2(\Omega)} + \|f_1\|_{H_D(\Omega)}), \end{aligned} \tag{2.7}$$

where $C$ does not depend on $\varphi$. From (A.1), (A.2), (2.6) and (2.7) we get (2.4) for $A_1$.

We shall show that if $|\lambda|$ is sufficiently large, $\|(\lambda + A_1)^{-1}\| \leq C|\lambda|^{-1}$. Let us introduce an equivalent norm $N(f)$ on $H$ such that $[N(f)]^2 = \int_\Omega\{a_{ij}\partial_j f_1 \partial_i \bar{f}_1 + \alpha^{-1}|f_2|\}\, dx$. Then from (2.1)

$$[N((\lambda + A_1)g)]^2 \geq C_1\{(|\lambda|^2/2 - C_2|\lambda|)\|\nabla g_1\|_{L^2(\Omega)}^2 + (|\lambda|^2 - C_3|\lambda|)\|g_2\|_{L^2(\Omega)}^2\},$$

which implies that $\|(\lambda + A_1)^{-1}\| \leq C|\lambda|^{-1}$, if $|\lambda|$ is sufficiently large. Thus making use of Proposition 2.5, (2.4) and the argument in the proof of Corollary 3.10 in [11], we can show $D(L) = D(L_1)$. The inequality (2.4) for general $q(x)$ is derived from (2.4) for $q = 0$ and the inequality $\|qf\|_{L^2(\Omega)}^2 \leq \varepsilon \sum_{|\alpha|=2} \|\partial_x^\alpha f\|_{L^2(\Omega)}^2 + C_\varepsilon \|f\|_{H_D(\Omega)}^2$ for ${}^t(f, 0) \in D(A)$, where $\varepsilon$ is an arbitrary positive number (see the proof of Lemma 3.9 in [11]). The proof is completed.

Now we have proved all properties which are used to show the following theorem (see the proof of Theorem 3.22 of [11]).

THEOREM 2.7. *A generates a group of linear operators $U(t)$ on $H$ which is unitary with respect to the energy form $E$.*

Let $j(x)$ be in $C^\infty(R^n)$ such that $j(x) = 1$ for $|x| > \rho + 1$ and $j(x) = 0$ for $|x| < \rho$, where $R^n \setminus \Omega \subset \{x : |x| < \rho\}$. Put $J$ to be $\begin{pmatrix} j(x) & 0 \\ 0 & j(x) \end{pmatrix}$ and denote by $Q'$ the projection from $H$ to $\mathcal{H}'$. Then the wave operators from $H_0$ to $\hat{\mathcal{H}}$ are defined as follows:

$$W_\pm f = s - \lim_{t \to \pm\infty} [Q' U(-t) J U_0(t)]^\wedge, \tag{2.8}$$

where $H_0 = \{f = (f_1, f_2) : f_1 \in H_D(R^n), f_2 \in L^2(R^n)\}$ and $\{U_0(t)\}$ is the unitary group on $H_0$ with the infinitesimal generator $A_0 = \begin{pmatrix} 0 & 1 \\ -\Delta & 0 \end{pmatrix}$. The following theorem is proved by the same ways of proving Theorem 4.1 and Lemma 4.3 of [11].

THEOREM 2.8. *The wave operators $W_\pm$ exist and are isometric from $H_0$ to $\hat{\mathcal{H}}_0$ where $\hat{\mathcal{H}}_0$ is the E-orthogonal subspace of $I_0/I$ in $\hat{\mathcal{H}}$ with $I_0 = \operatorname{Ker} A^2$.*

In Corollary 3.10 we shall show that $W_\pm$ is a unitary operator on $\hat{\mathcal{H}}_0$.

3. **Spectral representations.** First we state several facts which are derived from the principle of limit absorption for $L$ which is a self adjoint operator appeared in Section 2. Let $\mu$ and $\mu'$ be fixed numbers such that $1/2 < \mu' \leq \mu < \delta + 1/2$ and $\mu + \mu' \leq 1 + \delta$, where $\delta$ is appeared in (A.1), and let $\Pi$ be the set $\{\kappa \in \mathbb{C}; \operatorname{Im}\kappa \geq 0, \operatorname{Re}\kappa \neq 0\}$. First we shall state some properties of the generalized resolvent operator of $L$.

THEOREM 3.1. (see Theorem 3.2 in [8] and Proposition 13.7 in [9]). *For any $\kappa \in \Pi$ there exists a bounded operator $R(\kappa)$ from $L^2_\mu(\Omega)$ to $H^2_{-\mu'}(\Omega)$, that is, $R(\kappa) \in \mathcal{B}(L^2_\mu(\Omega), H^2_{-\mu'}(\Omega))$ such that $R(\kappa) \in C(\Pi; \mathcal{B}(L^2_\mu(\Omega), H^2_{\mu'}(\Omega)), R(\kappa) = (L - \kappa^2)^{-1}$ if $\operatorname{Im}\kappa > 0$, and $(L - \kappa^2)R(\kappa)f = f$, $BR(\kappa)f = 0$ on $\partial\Omega$ for all $f \in L^2_\mu(\Omega)$. Moreover $u = R(\kappa)f$ satisifies the following radiation condition; $u \in L^2_{-\mu}(\Omega)$, $\langle A\tilde{x}, \nabla - i\kappa\tilde{x}\rangle u \in L^2_{\mu-1}(\Omega)$, where $A = (a_{ij}(x))$ and $\tilde{x} = x/|x|$.*

In order to define spectral representations for $A$ we need the following operators.

DEFINITION 3.2. i) The operators $V \in \mathcal{B}(H^2_\nu(R^n), L^2_{\nu+1+\delta}(\Omega))$ and its dual operator $V^* \in \mathcal{B}(H^2_\nu(\Omega), L^2_{\nu+1+\delta}(R^n))$ are defined as $Lj - jL_0$ and $j^*L - L_0j^*$, respectively, where $L_0 = -\Delta$ with a domain $H^2(R^n)$ in $L^2(R^n)$, $j \in \mathcal{B}(L^2(R^n), \mathcal{H})$ is a multiplication operator by a function $j(x)$ used in (2.8), and $j^* \in \mathcal{B}(\mathcal{H}, L^2(R^n))$ is the dual operator of $j$.
ii) For any $\sigma \in R\setminus\{0\}$ denoted by $\mathcal{J}(\sigma)$ and $\mathcal{J}^*(\sigma)$ the operator $\mathcal{J}_0(|\sigma|)\{j^* - V^*R(\sigma)\} \in \mathcal{B}(L^2_\mu(\Omega), L^2(S^{n-1}))$ and its dual operator $\{j - R(-\sigma)V\}\,\mathcal{J}_0^*(|\sigma|) \in \mathcal{B}(L^2(S^{n-1}), L^2_{-\mu}(\Omega))$, respectively, where $[\mathcal{J}_0(|\sigma|)f](\omega) = |\sigma|^{(n-1)/2}\tilde{f}(|\sigma|\omega) \in \mathcal{B}\big(L^2_\mu(R^n), L^2(S^{n-1})\big)$ with the inverse Fourier transform $\tilde{f}(\xi)$ of $f(x)$, and $\mathcal{J}_0^*(|\sigma|)$ is the dual operator of $\mathcal{J}_0(|\sigma|)$.

The spectral representations of $L$ are given as follows:

THEOREM 3.3. (see Theorem 2.5 in [4] and Theorem 14.6 in [9]). *For any $(\sigma, \omega) \in R_+ \times S^{n-1}$ we put*

$$[\mathcal{J}_\pm f](\sigma, \omega) = [\mathcal{J}(\pm\sigma)f](\omega) \quad \textit{for } f \in L^2_\mu(\Omega).$$

*Then $\mathcal{J}_\pm$ is able to be uniquely extended from $\mathfrak{h}$ to $L^2\big(R_+; L^2(S^{n-1})\big)$ as a partially isometric operator with $\operatorname{Ker}\mathcal{J}_\pm = E(0)\mathfrak{h}$, where $\{E(\lambda)\}$ is the spectral resolution of $L$. Moreover $\mathcal{J}_\pm$ satisfies the following;*

*i) For any bounded Borel function $\varphi(x)$*

$$[\mathcal{J}_\pm\varphi(L)f](\sigma, \omega) = \varphi(\sigma^2)[\mathcal{J}_\pm f](\sigma, \omega) \textit{ for } f \in \mathfrak{h}; \tag{3.1}$$

*ii) The dual operator* $\mathcal{J}_\pm^* \in \mathcal{B}\Big(L^2\big(R_+; L^2(S^{n-1})\big), \mathfrak{h}\Big)$ *of* $\mathcal{J}_\pm$ *is denoted by the following formula*

$$(3.2)\qquad [\mathcal{J}_\pm^* h](x) = s - \lim_{N\to\infty} \int_{1/N}^{N} [\mathcal{J}^*(\pm\sigma)h(\sigma,\cdot)](x)\, d\sigma \text{ in } \mathfrak{h}.$$

Making use of $\mathcal{J}_\pm$, we define the spectral representations $F_\pm$ of $A$ as follows; for any $f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$

$$(3.3)\quad (F_\pm f)(\sigma,\omega) = \begin{cases} A_n^\pm(\sigma)\{\, i\sigma(\mathcal{J}_\mp f_1)(\sigma,\omega) + (\mathcal{J}_\mp f_2)(\sigma,\omega)\}/2, & \sigma > 0, \\ B_n^\pm(\sigma)\{\, i\sigma(\mathcal{J}_\pm f_1)(-\sigma,-\omega) + (\mathcal{J}_\pm f_2)(-\sigma,-\omega)\}/2, & \sigma < 0, \end{cases}$$

where $A_n^\pm(\sigma) = B_n^\pm(\sigma) = (-i\,\mathrm{sgn}\,\sigma)^{(n-1)/2}$, if $n$ is odd, and $A_n^\pm(\sigma) = -1$ and $B_n^\pm(\sigma) = (-1)^{n/2-2} i$, if $n$ is even. We can show that these definitions of spectral representations are essentially equal to those in [5], [6], [13] and [14].

In order to extend (3.3) on $\mathcal{H}'$ we need three lemmas.

LEMMA 3.4. *Let* $p(x) \in L^2(\Omega)$ *be an eigenvector associated with a negative eigenvalue* $-\lambda^2$. *Then* $p(x)$ *belongs to* $H_\mu^2(\Omega)$ *for all* $\mu \in R$.

PROOF. Let $\varphi(x)$ be in $C^\infty(R^n)$ such that $\varphi(x) = 0$ for $|x| < R$ and $\varphi(x) = 1$ for $|x| > R + 1$, where $\{x : |x| < R\} \subset R^n \setminus \Omega$ and $|q| \le C$ for $|x| > R$. Since $p \in H^2(\Omega)$, $Lp = -\lambda^2 p$, we see that from (A.1) to (A.3)

$$(3.4)\qquad -\Delta(\varphi p) + \lambda^2(\varphi p) = (-\Delta - L)(\varphi p) + (L + \lambda^2)(\varphi p)$$

belongs to $L^2_{1+\delta}(R^n)$. By the Fourier transform of (3.4) it follows that $(|\xi|^2 + \lambda^2)(\varphi p)^\wedge(\xi)$ belongs to $H^{1+\delta}(R^n_\xi)$. This implies that $(\varphi p)^\wedge(\xi) \in H^1(R^n_\xi)$, which is equivalent to $\varphi p(x) \in L^2_1(R^n)$. From a regularity theorem in the weighted space $H^2_\mu(R^n)$ mentioned in the proof of Theorem 2.6 we see that $\varphi p \in H^2_1(R^n)$ and the righthand side of (3.4) belongs to $L^2_{2+\delta}(R^n)$. By taking the Fourier transform of (3.4) we inductively get $(|\xi|^2 + \lambda)(\varphi p)^\wedge(\xi) \in H^{n+\delta}(R^n_\xi)$, and $(\varphi p)^\wedge(\xi) \in H^{n-1}(R^n_\xi)$. Thus we can conclude $(\varphi p)^\wedge(\xi) \in H^n(R^n_\xi)$ for all $n$, which implies the desired property on $p(x)$. The proof is completed.

LEMMA 3.5. *Let* $Y = D(L) \cap L^2_\mu(\Omega)^2 \cap \Big(\big(1 - E(0_-)\big)\mathfrak{h}\Big)^2$, *where* $\mu$ *is an arbitrary positive number. Then the set* $Y \times Y$ *is dense in* $\mathcal{H}'$.

PROOF. From the definition of $D(L)$ for any $g \in \mathcal{H}'$ the exists a sequence $\{g_n\} \subset D(L) \cap \big(C_0^2(\bar\Omega)\big)^2$ such that $g_n$ converges to $g$ in $H$. Put $g_n'' = -\sum_{j=1}^m \{E(g_n, f_j^-) f_j^+ + E(g_n, f_j^+)\} f_j^- / \lambda_j^2$ and $g_n' = g_n - g_n''$. Since $g$ belongs to $\mathcal{H}'$, $E(g_n, f_j^\pm)$ converges to 0 as $n \to \infty$. From these facts and Lemma 3.4 it follows that the sequence $\{g_n'\}$ in $D(L) \cap \big(L^2_\mu(\Omega)\big)^2$ converges to $g$ as $n \to \infty$. Put $g_n = (g_{n1}, g_{n2})$; then we have

$$2E(g_n, f_j^\pm) = -\lambda_j^2 (g_{n1}, p_j)_{\mathfrak{h}} \pm \lambda_j (g_{n2}, p_j)_{\mathfrak{h}}.$$

Therefore $g_n'$ is equal to $(g_{n1} - \sum_{j=1}^m (g_{n1}, p_j)_{\mathfrak{h}} p_j, g_{n2} - \sum_{j=1}^m (g_{n2}, p_j)_{\mathfrak{h}} p_j)$ which belongs to $\big(1 - E(0_-)\mathfrak{h}\big)^2$. The proof is completed.

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LEMMA 3.6. *For any* $f \in Y \times Y$ *put* $h_2 = \big(E(0) - E(0_-)\big)f_2$. *Then*

$$\|F_\pm f\|^2_{L^2(R\times S^{n-1})} = E(f) - (h_2, h_2)_{\mathfrak{h}}. \tag{3.5}$$

PROOF. From (3.3) we have

$$4\|F_\pm f\|^2_{L^2(R\times S^{n-1})} = \int_0^\infty \int \{\,|\sigma \mathcal{J}_\mp f_1|^2 + |\sigma \mathcal{J}_\pm f_1|^2 + |\mathcal{J}_\mp f_2|^2 + |\mathcal{J}_\pm f_2|^2\}\, d\sigma\, d\omega$$
$$+ 2\,\mathrm{Re} \int_0^\infty \int \{\, i\sigma \mathcal{J}_\mp f_1 \overline{\mathcal{J}_\mp f_2} - i\sigma \mathcal{J}_\pm f_1 \overline{\mathcal{J}_\pm f_2})\, d\sigma\, d\omega.$$

Since $f$ belongs to $Y \times Y$, by Theorem 3.3 the last term of the above equality is equal to $2Re\{\, i(L_+^{1/2} f_1, f_2) - i(L_+^{1/2} f_1, f_2)\}$, where $L_+^{1/2} = \int_{0_-}^\infty \lambda^{1/2}\, dE(\lambda)$. It follows that

$$4\|F_\pm f\|^2_{L^2(R^n\times S^{n-1})} = 2(Lg_1, g_1)_{\mathfrak{h}} + 2(g_2, g_2)_{\mathfrak{h}} = 4E(g),$$

where $g_i = \big(1 - E(0)\big)f_i$. If we put $h_i = \big(E(0) - E(0_-)\big)f_i$, then

$$E(g) = E\big(f - (0, h_2)\big) = E(f) - (h_2, h_2)_{\mathfrak{h}},$$

where we use that $(h_1, 0) \in I$ and $E\big(f, (0, h_2)\big) = (h_2, h_2)_{\mathfrak{h}}$. The proof of Lemma 3.6 is completed.

Making use of these lemmas, we can prove the following:

THEOREM 3.7. $F_\pm$ *can be uniquely extended as an isometric operator from* $\hat{\mathcal{H}}_1$ *to* $L^2(R \times S^{n-1})$.

PROOF. From Lemma 3.6 and (3.5) $F_\pm$ can be extended as a bounded operator from $\mathcal{H}'$ to $L^2(R \times S^{n-1})$. Then $F_\pm f = 0$, if $f \in \mathcal{J}$. Thus we can define an operator $F_\pm$ on $\hat{\mathcal{H}}$, which satisfies (3.5) for all $\hat{f} \in \hat{\mathcal{H}}$. We assume that $E(f, g) = 0$ for all $g \in \mathcal{J}_0 = \mathrm{Ker}\, A^2$. Let $\{f_n\}$ be a sequence in $Y \times Y$ such that $f_n$ converges to $f \in H$. Put $f_n' = (f_{n1}, g_{n2})$, where $g_{n2} = \big(1 - E(0)\big)f_{n2}$ and put $h_{n2} = \big(E(0) - E(0_-)\big)f_{n2}$. From $(0, h_{n2}) \in \mathcal{J}_0$ it follows that $\|f - f_n\|_H^2 = \|f - f_n'\|_H^2 + \|h_{n2}\|_{\mathfrak{h}}^2$ and $\|\mathcal{J}_\pm f_n'\|^2_{L^2(R\times S^{n-1})} = E(f_n')$. This means that $f_n'$ converges to $f$ and $\|\mathcal{J}_\pm f\|^2_{L^2(R\times S^{n-1})} = E(f)$. The proof is completed.

Later in Corollary 3.10 we shall show that $F_\pm$ of Theorem 3.7 is unitary from $\hat{\mathcal{H}}_1$ to $L^2(R \times S^{n-1})$.

THEOREM 3.8. *For all* $t \in R$ *and all* $\hat{f} \in \hat{\mathcal{H}}_1$

$$F_\pm \hat{U}(t)\hat{f} = e^{it\sigma} F_\pm f. \tag{3.6}$$

PROOF. Let $L_+$ be $\int_{0_-}^\infty \lambda^{1/2}\, dE(\lambda)$ and put

$$V(t) = \begin{pmatrix} \cos tL_+^{1/2} & L_+^{1/2} \sin tL_+^{1/2} \\ -L_+^{1/2} \sin tL_+^{1/2} & \cos tL_+^{1/2} \end{pmatrix}.$$

Then $d\big(V(t)f\big)/dt = AV(t)f$ for $f \in Y \times Y$. Thus $\hat{f}(t) = [\big(U(t) - V(t)\big)f]^{\wedge}$ satisfies that $d\big(e^{i(ti\hat{A})}\hat{f}(t)\big)/dt = 0$ and $\hat{f}(0) = 0$. It follows that $f(t)$ belongs to Ker $A$, which implies that $df/dt = 0$ and $f(0) = 0$. We see that $U(t)f = V(t)f$ for all $Y \times Y$. Therefore from (3.1) and (3.3) for $\sigma > 0$ and $f \in Y \times Y$

$$\big(F_{\pm}U(t)f\big)(\sigma,\omega) = A_n^{\pm}(\sigma)\{(i\sigma\cos t\sigma - \sigma\sin t\sigma)(\mathcal{J}\pm f_1)(\sigma,\omega) + (i\sin t\sigma + \cos t\sigma)(\mathcal{J}_{\pm}f_2)(\sigma,\omega)\}/2 = e^{it\sigma}(F_{\pm}f)(s,\omega).$$

Similarly for $\sigma < 0$ and $f \in Y \times Y\big(F_{\pm}U(t)f\big)(\sigma,\omega) = e^{it\sigma}(F_{\pm}f)(\sigma,\omega)$. The proof is completed.

The following theorem implies that $W_{\pm}$ and $F_{\pm}$ are unitary.

THEOREM 3.9. *Let $\varphi(\lambda)$ be a real valued function such that there exists a partition, $\cdots < \lambda_{-2}^{\pm} < \lambda_{-1}^{\pm} < \lambda_0^{\pm} < \lambda_1^{\pm} < \lambda_2^{\pm} < \cdots$ of $(0,\pm\infty)$ with the properties; $\lambda_k^{\pm} \longrightarrow 0$ as $\pm k \longrightarrow -\infty$, $\lambda_k^{\pm} \longrightarrow +\infty$ as $\pm k \longrightarrow \infty$, and for $k = 0, \pm 1, \pm 2, \ldots$ $\varphi(\lambda)$ is smooth and $\varphi'(\lambda) > 0$ in $\lambda \in (\lambda_{k-1}, \lambda_k)$. Then we have*

$$s - \lim_{t\to\pm\infty} e^{it\varphi(i\hat{A})}[Q'Je^{-t\varphi(iA_0)}f]^{\wedge} = F_{\pm}^{*}F_{\pm}^{(0)}f \text{ for } f \in H_0. \tag{3.7}$$

*where $\big(F_{\pm}^{(0)}f\big)(\sigma,\omega)$ is similarly defined to (3.3) by changing $(\mathcal{J}_{\pm}f)(\sigma,\omega)$ with $[\mathcal{J}_0(|\sigma|)f](\omega)$.*

COROLLARY 3.10. *The operators $W_{\pm}$ and $F_{\pm}$ are unitary operators from $H_0$ to $\hat{\mathcal{H}}_1$ and from $\hat{\mathcal{H}}_1$ to $L^2(R \times S^{n-1})$, respectively.*

PROOF. We assume that there exists $\hat{g} \in \hat{\mathcal{H}}_1$ such that $(W_{\pm}f, \hat{g})_{\hat{\mathcal{H}}_1} = 0$ for all $f \in H_0$. Then from (3.7) $(F_{\pm}^{(0)}f, F_{\pm}\hat{g})_{L^2(R\times S^{n-1})} = 0$. Since $F_{\pm}^{(0)}$ is unitary (see Theorem 2.1 in p. 100 of [5] and Theorem 5.1 in [6], we see that $F_{\pm}\hat{g} = 0$. From Theorem 3.7 it follows that $\hat{g} = 0$. The proof is completed.

## 4. The proof of Theorem 3.9.

In order to prove Theorem 3.9 we need several lemmas.

LEMMA 4.1. *Let $\ell(\sigma,\omega)$ be an element of $L^2(R \times S^{n-1})$. Then we have the following two assertions;*

*i) If $\sigma\ell(\sigma,\omega) \in L^2(R \times S^{n-1})$, then $F_{\pm}^{*}\ell \in D(A)$ and $\hat{A}F_{\pm}^{*}\ell = F_{\pm}^{*}(i\sigma\ell)$.*

*ii) If supp $\ell \subset \{(\sigma,\omega) : 0 < a < |\sigma| < b\}$, then a Bochner vector-valued integral in $L^2_{-\mu}(\Omega)$ $\int[\mathcal{J}^{*}(\pm\sigma)\ell(\sigma,\cdot)](x)\,d\sigma$ belongs to $\mathfrak{h}$, $\mathcal{J}_{\pm}^{*}\ell \in D(L)$, and*

$$(L\mathcal{J}_{\pm}^{*}\ell)(x) = \int[\mathcal{J}^{*}(\pm\sigma)\sigma^2\ell(\sigma,\cdot)](x)\,d\sigma. \tag{4.1}$$

PROOF. We only show the statement ii). We assume that the support of $\ell \in L^2(R \times S^{n-1})$ satisfies the assumption of the statement ii). Then for any $f \in L^2_{\mu}(\Omega)$ we have

$$\begin{aligned}(f, \mathcal{J}_{\pm}^{*}\ell)_{\mathfrak{h}} &= (\mathcal{J}_{\pm}f, \ell)_{L^2(R\times S^{n-1})} \\ &= \int(f, [\mathcal{J}^{*}(\pm\sigma)\ell(\sigma,\cdot)]_{\mathfrak{h}}\,d\sigma.\end{aligned} \tag{4.2}$$

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Since $\|\mathcal{J}^*(\pm\sigma)\ell(\sigma,\cdot)\|_{L^2_{-\mu}(\Omega)} \leq C\|\ell(\sigma,\cdot)\|_{L^2(S^{n-1})}$, where $C$ only depends on the support of $\ell$, from Theorem 1 in page 133 of [16] $\mathcal{J}^*_\pm(\pm\sigma)\ell(\sigma,\cdot)$ is Bochner integrable with respect to a measure $d\sigma$ on $R$ in $L^2_{-\mu}(\Omega)$. The linear functional $T(h) = (f,h)_{\mathfrak{h}}$ is bounded on $L^2_{-\mu}(\Omega)$ for $f \in L^2_\mu(\Omega)$. Thus by Corollary 2 on page 134 of [16] it follows that

$$\Big(f, \int \mathcal{J}^*(\pm\sigma)\ell(\sigma,\cdot)\,d\sigma\Big)_{\mathfrak{h}} = \int \big(f, \mathcal{J}^*(\pm\sigma)\ell(\sigma,\cdot)\big)_{\mathfrak{h}}\,d\sigma,$$

which implies that from (4.2) $\int \mathcal{J}^*(\pm\sigma)\ell(\sigma,\cdot)\,d\sigma = \mathcal{J}^*_\pm\ell \in \mathfrak{h}$. $\mathcal{J}^*_\pm\ell \in D(L)$ and (4.1) are easily proved. The proof is completed.

The operator $F^*_\pm$ satisifies the following

LEMMA 4.2. *Let $\ell(\sigma,\omega)$ be in $L^2(R\times S^{n-1})$ with* $\operatorname{supp}\ell \subset \{(\sigma,\omega) : 0 < a < |\sigma| < b\}$ *and let $g(x)$ be in $\mathcal{H}'$ such that $\hat{g} = F^*_\pm\ell \in \hat{\mathcal{H}}_1$. Put*

$$h_k = -i^k \int_0^\infty \sigma^{k-2}\big[\bar{A}^\pm_n(\sigma)\mathcal{J}^*(\mp\sigma)\ell(\sigma,\cdot) - (-1)^k\bar{B}^\pm_n(\sigma)\mathcal{J}^*(\pm\sigma)\ell'(\sigma,\cdot)\big]\,d\sigma, \tag{4.3}$$

*where $k = 1,2$, $\ell'(\sigma,\omega) = \ell(-\sigma,-\omega)$ and the integrations of (4.3) are Bochner's integrals. Then $g - (h_1,h_2)$ belongs to $I = \operatorname{Ker} A$.*

PROOF. From (3.1) and (3.3) we see that for $f \in Y \times Y$

$$\begin{aligned} E(f,g) &= \frac{1}{2}\int_0^\infty A^\pm_n(\sigma)(i\sigma\mathcal{J}(\mp\sigma)f_1 + \mathcal{J}(\mp\sigma)f_2, \ell(\sigma,\cdot))_{L^2(S^{n-1})}\,d\sigma \\ &\quad + \frac{1}{2}\int_0^\infty B^\pm_n(-\sigma)(-i\sigma\mathcal{J}(\pm\sigma)f_1, \mathcal{J}(\pm\sigma)f_2, \ell'(\sigma,\cdot))_{L^2(S^{n-1})}\,d\sigma. \end{aligned} \tag{4.4}$$

Put $f_1 = 0$ or $f_2 = 0$ in (4.4); then from Lemma 4.1 it follows that $(f_1, Lg_1)_{\mathfrak{h}} = (f_1, Lh_1)_{\mathfrak{h}}$ and $(f_2,g_2)_{\mathfrak{h}} = (f_2,h_2)_{\mathfrak{h}}$, which implies that $g = (h_1,h_2)$ belongs to $I$ from the properties of $\mathcal{J}_\pm$ stated in Theorem 3.3. The proof is completed.

By the following lemma we can neglect the projection $Q'$ in (2.8).

LEMMA 4.3. *Let $\hat{g}_\pm(t)$ be $F^*_\pm F^{(0)}_\pm e^{-it\varphi(iA_0)}f$. Then $E\big(F^*_\pm F^{(0)}_\pm e^{-it\varphi(iA_0)}f - [Q'Je^{-it\varphi(iA_0)}f]\big) - E\big(g_\pm - Je^{-it\varphi(iA_0)}f\big)$ converges to 0 as $t \longrightarrow \pm\infty$, where $\varphi(\lambda)$ satisfies the assumption of Theorem 3.9.*

PROOF. Let $p(t) = \sum_{k=1}^m \{E(je^{-it\varphi(iA_0)}f, f^-_k)f^+_k + E(Je^{-it\varphi(iA_0)}f, f^+_k)f^-_k]/\lambda^2_k$. Then $Q'Je^{-it\varphi(iA_0)}f = Je^{-it\varphi(iA_0)}f + p(t)$. Since $g_\pm(t)$ belongs to $\mathcal{H}'$, we may show that $\big|E\big(Je^{it\varphi(iA_0)}f, p(t)\big)\big| + \big|E\big(p(t)\big)\big|$ converges to 0 as $t \longrightarrow \pm\infty$. It follows that

$$\big|E\big(Je^{it\varphi(iA_0)}f, p(t)\big)\big| + \big|E\big(p(t)\big)\big| \leq C\{\|f\|_{H_0}\|p(t)\|_H + \|p(t)\|^2_H\}.$$

Thus from the definition of $p(t)$ we may show that $E\big(Je^{it\varphi(iA_0)}f, f^\pm_k\big)$ converges to 0 as $t \longrightarrow \pm\infty$. From the spectral family of $iA_0$ we have

(4.5)

$$2[e^{-it\varphi(iA_0)}f]^\wedge(\xi) = e^{-it\varphi(|\xi|)}X_-(\xi)^t\big(\hat{f}_1(\xi), \hat{f}_2(\xi)\big) + e^{-it\varphi(-|\xi|)}X_+(\xi)^t\big(\hat{f}_1(\xi), \hat{f}_2(\xi)\big)$$

where $X_+(\xi) = \frac{1}{2}\begin{pmatrix} 1 & -i|\xi|^{-1} \\ i|\xi| & 0 \end{pmatrix}$, $X_-(\xi) = \frac{1}{2}\begin{pmatrix} 0 & i|\xi|^{-1} \\ -i|\xi| & 0 \end{pmatrix}$. Since $|E(Je^{-it\varphi(iA_0)}f, f_j^{\pm})| \leq C\|f\|_{H_0}$, where $C$ does not depend on $t$ and $f$, we may show that $E(Je^{-it\varphi(iA_0)}f, f_k^{\pm}) \longrightarrow 0$ as $|t| \longrightarrow \infty$ for $f \in H_0$ with supp$\hat{f}_i \subset \{\xi \in R^n : 0 < a < |\xi| < b\}$. Making use of (4.5) and

$$E(Je^{-it\varphi(iA_0)}f, f_k^{\pm}) = \lambda_k^2\Big([e^{-it\varphi(iA_0)}f]_1^{\wedge}, (jp_k)^{\wedge}\Big)_{L^2(R^n_\xi)} \pm \lambda_k\Big([e^{-it\varphi(iA_0)}f]_2^{\wedge}, (jp_k)^{\wedge}\Big)_{L^2(R^n_\xi)}. \tag{4.6}$$

we can write (4.6) by the sum of the forms $\int_0^\infty e^{-it\varphi(\pm\sigma)}q(\sigma)\,d\sigma$, where $q(\sigma) \in L^1(R_+)$ and supp $q \subset \{\sigma : 0 < a < \sigma < b\}$. Since $\varphi'(\sigma)$ is not zero in each interval, by the Riemann-Lebesgue theorem it follows that (4.6) converges to 0 as $|t| \longrightarrow \infty$. The proof of Lemma 4.3 is completed.

The last lemma to prove Theorem 3.9 is as follows:

LEMMA 4.4. *We assume that the inverse Fourier transforms of components of $f$ are smooth and their supports are contained in $\{\xi : 0 < a < |\xi| < b\}$. Then there exists a positive constant $\gamma > 1$ such that for the operator $V$ defined in i) of Definition 3.2.*

$$\Big\| V\int_a^b e^{-i\left(\mp\sigma^2 s - t\varphi(\sigma)\right)}\mathcal{J}_0^*(\sigma)\sigma^k[F_\pm^{(0)}f](\sigma,\cdot)\,d\sigma\Big\|_{\mathfrak{h}} + \Big\| V\int_a^b e^{-i\left(\pm\sigma^2 s - t\varphi(-\sigma)\right)}\mathcal{J}_0^*(\sigma)\sigma^k[F_\pm^{(0)}f](\sigma,\cdot)\,d\sigma\Big\|_{\mathfrak{h}} \leq C(1+s+|t|)^{-\gamma}. \tag{4.7}$$

where $k$ is an integer, $s > 0$, $t \in R_\pm$ and $C$ depends on $f$, $k$ and $\gamma$.

PROOF. We only consider the term in (4.7) involving the function

$$\tilde{f}_\pm(x,s,t) = \int_a^b e^{-i\left(\mp\sigma^2 s - t\varphi(\sigma)\right)}\mathcal{J}_0^*(\sigma)\sigma^k[F_\pm^{(0)}f](\sigma,\cdot)\,d\sigma.$$

From Hölder's inequality it follows that for $p > 1$

$$\begin{aligned} \|V\tilde{f}_\pm(\cdot,s,t)\|_{\mathfrak{h}} &\leq C\Big(\|V\tilde{f}_\pm(\cdot,s,t)\|_{L^2_{1+\delta}(\Omega)}\Big)^{(p-1)/p} \\ &\quad \times \Big(\|V\tilde{f}_\pm(\cdot,s,t)\|_{L^2_{-(p-1)(1+\delta)}(\Omega)}\Big)^{1/p} \\ &\leq C_1\Big(\|\tilde{f}_\pm(\cdot,s,t,)\|_{H^2(R^n)}\Big)^{(p-1)/p}\Big(\|\tilde{f}_\pm(\cdot,s,t)\|_{H^2_{-p(1+\delta)}(R^n)}\Big)^{1/p}. \end{aligned}$$

First from the definitions of $[\mathcal{J}_0^*(\sigma)h](x)$ and $F_\pm^{(0)}f$, and the assumption of $f$ we see that $\|\tilde{f}_\pm(\cdot,s,t)\|_{H^2(R^n)}$ is bounded by some constant depending on $f$ and not depending on $s$ and $t$. We take a positive integer $j$ such that $p(1+\delta) \geq \mu + j$ and $j > p$, where $\mu > 1/2$. Making use of the equality

$$\{\pm i\big(2\sigma s \pm t\varphi'(\sigma)\big)^{-1}\partial_\sigma\}^m e^{-i\left(\sigma^2 s - t\varphi(\sigma)\right)} = e^{-i\left(\sigma^2 s - t\varphi(\sigma)\right)},$$

and the integration by parts, we can obtain that

$$\|\tilde{f}_\pm(\cdot,s,t)\|_{H^2_{-\mu-j}(R^n)} \le C(s+|t|)^{-m} \sup_{\sigma\in[a,b]} \sum_{\ell=1}^{m} \|\partial_\sigma^\ell \xi(\cdot,\sigma)\|_{H^2_{-\mu-j}(R^n)},$$

where $\xi(x,\sigma) = [\mathcal{J}_0^*(\sigma)\sigma^k F_\pm^{(0)} f(\sigma,\cdot)](x)$ and $C$ does not depend on $t$ and $s$. Thus (4.7) holds for $\gamma = (m-p)/p$. The proof is completed.

PROOF OF THEOREM 3.9. First we remark that from Theorem 2.1 on page 100 of [5] and Theorem 5.1. of [6] $F_\pm^{(0)}$ is a unitary operator from $H_0$ to $L^2(R \times S^{n-1})$ such that $F_\pm^{(0)} U_0(t) = e^{i\sigma t} F_\pm^{(0)}$. Let $f$ be an element in $H_0$ such that the Fourier transform of the components are smooth and $\operatorname{supp}\hat{f}_i \subset \{\xi : 0 < a < |\xi| < b\}$. We note that from (5.7) $F_\pm e^{it\varphi(iA)} = e^{-it\sigma} F_\pm$. From the equality

$$\begin{aligned} &F_\pm^* F_\pm^{(0)} f - e^{it\varphi(i\hat{A})}[Q' J e^{-it\varphi(iA_0)} f]^\wedge \\ &\quad = e^{it\varphi(i\hat{A})}\{F_\pm^* e^{it\varphi(\sigma)} F_\pm^{(0)} f - [Q' J F_\pm^{(0)*} e^{it\varphi(\sigma)} F_\pm^{(0)} f]^\wedge\}, \end{aligned}$$

Lemma 4.2 and Lemma 4.3, we may prove that $E\big(h^\pm(t)\big) \longrightarrow 0$ as $t \longrightarrow \pm\infty$, where $h^\pm(t) = \big(h_1^\pm(t), h_2^\pm(t)\big)$ is defined by

$$\begin{aligned} h_k^\pm(t) = -i^k \int_0^\infty &\{\bar{A}_n^\pm(\sigma)\sigma^{k-2}[\mathcal{J}^*(\mp\sigma) - j(x)\mathcal{J}_0^*(\sigma)]\ell_t(\sigma,\cdot) \\ &+ (-1)^k \bar{B}_n^\pm(-\sigma)\sigma^{k-2}[\mathcal{J}^*(\pm\sigma) - j(x)\mathcal{J}_0^*(\sigma)]\ell_t'(\sigma,\cdot)\}\, d\sigma, \end{aligned} \tag{4.8}$$

with $\ell_t(\sigma,\omega) = e^{it\varphi(\sigma)}(F_\pm^{(0)} f)(\sigma,\omega)$ and $\ell_t'(\sigma,\omega) = \ell_t(-\sigma,-\omega)$. Since from the statement ii) of Lemma 4.1 $h_1^\pm(t)$ belongs to $D(L)$, we see that $2E\big(h^\pm(t)\big) = \big(Lh_1^\pm(t), h_1^\pm(t)\big)_{\mathfrak{h}} + \big(h_2^\pm(t), h_2^\pm(t)\big)_{\mathfrak{h}}$. Then from (3.2) and (4.1) it follows that

$$Lh_1^\pm(t) = -h_3^\pm(t) + (Lj - jL_0)[e^{-it\varphi(iA_0)} f]_1,$$

where $h_3^\pm$ is defined by (4.8) as $k = 3$. We have

$$\begin{aligned} \|(Lj - jL_0)[e^{-it\varphi(iA_0)} f]_1\|_{\mathfrak{h}} &\le C\{\|e^{-it\varphi(iA_0)} f\|_{H_0} + \|A_0 e^{-it\varphi(iA_0)} f\|_{H_0}\} \\ &= C\{\|f\|_{H_0} + \|A_0 f\|_{H_0}\}. \end{aligned}$$

Therefore in order to prove that $E\big(h^\pm(t)\big)$ converges to 0 as $t \longrightarrow \pm\infty$ we have to show

$$\lim_{t\to\pm\infty} \|h_i^\pm(t)\|_{\mathfrak{h}} = 0 \text{ for } i = 1,2,3. \tag{4.9}$$

Since $h_i^\pm(t)$ $(i = 1,2,3)$ have essentially the same form, we only show (4.9) for $i = 1$. By the definition of $\mathcal{J}^*(\pm\sigma)$ we see that

$$\begin{aligned} h_1^\pm(t) = i \int_0^\infty &\{\bar{A}_n^\pm(\sigma)\sigma^{-1} R(\pm\sigma) V \mathcal{J}_0^*(\sigma) e^{it\varphi(\sigma)}(F_\pm^{(0)} f)(\sigma,\cdot) \\ &+ \bar{B}_n^\pm(-\sigma)\sigma^{-1} R(\mp\sigma) V \mathcal{J}_0^*(\sigma) e^{it\varphi(-\sigma)}(F_\pm^{(0)} f)(-\sigma,-\cdot)\}\, d\sigma \end{aligned} \tag{4.10}$$

For $\tau > 0$ we define $h_1^{\mp}(t, \pm\tau)$ by changing $R(\pm\sigma)$ with $R\big((\sigma^2 \pm i\tau)^{1/2}\big)$ in (4.10). Then from $R\big((\sigma^2 \pm i\tau)^{1/2}\big) = \pm i \int_0^\infty e^{\mp i(L-\sigma^2 \mp i\tau)s}\, ds$ it follows that

$$\begin{aligned} h_1^{\pm}(t, \pm\tau) = \mp \int_0^\infty & e^{\mp i(L \mp i\tau)s} [V \int_0^\infty e^{-i(\mp\sigma^2 s - \varphi(\sigma)t)} \\ & \times \bar{A}_n^{\pm}(\sigma)\sigma^{-1} \mathcal{J}_0^*(\sigma)(F_{\pm}^{(0)} f)(\sigma, \cdot)\, d\sigma]\, ds \\ \pm i \int_0^\infty & e^{\pm i(L \pm i\tau)s} [V \int_0^\infty e^{-i\left(\pm\sigma^2 s - \varphi(-\sigma)t\right)} \\ & \times \bar{B}_n^{\pm}(\sigma)\sigma^{-1} \mathcal{J}_0^*(\sigma)(F_{\pm}^{(0)} f)(-\sigma, -\cdot)\, d\sigma]\, ds. \end{aligned}$$

From Lemma 4.4 it follows that

$$\| h_1^{\pm}(t, \pm\tau) \|_{\mathfrak{h}} \le C\langle t\rangle^{-\gamma+1}, \tag{4.11}$$

where $C$ does not depend on $\tau$. On the other hand for any $g \in L^2_\mu(\Omega)$

$$\lim_{\tau \to \infty} \left( h_1^{\pm}(t, \pm\tau), g \right)_{\mathfrak{h}} = \left( h_1^{\pm}(t), g \right)_{\mathfrak{h}}. \tag{4.12}$$

(4.11) and (4.12) imply that $h_1^{\pm}(t, \pm\tau)$ converges weakly to $h_1^{\pm}(t)$. Thus

$$\| h_1^{\pm}(t) \|_{\mathfrak{h}} \le \underline{\lim}_{\tau \to 0} \| h_1^{\pm}(t, \pm\tau) \|_{\mathfrak{h}} \le C|t|^{-\gamma+1}.$$

Since the set consisting of considered $f$ is a dense set of $H_0$, the proof of Theorem 3.9 is completed.

APPENDIX. We shall state a generalization of Theorem 2.1 in [11]. Put $D_{\pm} = \{f \in H_0 : [U_0(t)f](x) = 0 \text{ for } |x| < \pm t\}$ and denote $W_{\pm}D_{\pm}$ by $\mathcal{D}_{\pm}$. Then we can prove the following:

THEOREM A.1. *The spaces $\mathcal{D}_{\pm}$ satisfy the following properties:*
*i) $U(t)\mathcal{D}_+ \subset \mathcal{D}_+$ for $t > 0$, $U(t)\mathcal{D}_- \subset \mathcal{D}_-$ for $t < 0$;*
*ii) $\cap_{t\in R} \hat{U}(t)\hat{\mathcal{D}}_+ = \{0\} = \cap_{t\in R} \hat{U}(t)\hat{\mathcal{D}}_-$;*
*iii) $\cup_{t\in R} \hat{U}(t)\hat{\mathcal{D}}_+$ and $\cup_{t\in R} \hat{U}(t)\hat{\mathcal{D}}_-$ are dense subsets of $\hat{\mathcal{H}}_0$.*

This theorem is proved by a similar argument in Section 4 and 5 of [11].

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