ASYMPTOTICS OF TWO INTEGRALS FROM OPTIMIZATION THEORY AND GEOMETRIC PROBABILITY

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Abstract

Asymptotic series are derived for two integrals using a Gaussian identity and Laplace's method, demonstrating an improvement over earlier methods.

LAPLACE'S METHOD; OPTIMIZATION

Anderssen et al. (1976) obtain various bounds and approximations for the expected distance

\[ m_k = \int_0^1 \cdots \int_0^1 (x_1^2 + \cdots + x_k^2) dx_1 \cdots dx_k \]

from the origin of a point uniformly distributed in the cube \([0, 1]^k\). They evaluate \(m_1\), \(m_2\) and \(m_3\) exactly. Otherwise their computationally most efficient formula, by far, is the asymptotic series

\[ m_k = (k/3)^{k/2} (1 - 1/10k - 13/280k^2 - 101/2800k^3 - 37533/1232000k^4) + O(k^{-3}) \]

as \(k \to \infty\). Terms up to \(k^{-3}\) give, for example, \(m_4\) accurate to five figures, \(m_{10}\) accurate to six figures and \(m_{20}\) accurate to seven figures. Their derivation of (2) is, however, cumbersome. We give a simple derivation based on Laplace's method.

The authors also study the expected interpoint distances

\[ M_k = \int_0^1 \cdots \int_0^1 (x_1 - y_1)^2 + \cdots + (x_k - y_k)^2 dx_1 dy_1 \cdots dx_k dy_k, \]

but do not give an asymptotic series like (2), presumably because of the work required using their method. We give a simple derivation of such a series, again using Laplace's method.

Since

\[ \lambda^{1/2} = (2/\pi)^{1/2} \int_0^\infty ds \exp \left( -\frac{1}{2} \lambda s^2 \right), \]

we can write

\[ m_k = (2/\pi)^{1/2} k \int_0^\infty f'(\frac{1}{2}s^2)f(\frac{1}{2}s^2)^{-k-1} ds, \]

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where

\[ f(t) = \int_0^1 \exp \left( t(x^2) \right) dx. \]  

Since \( f(t) \) has a maximum at \( t = 0 \), and \( f'(0) = \frac{1}{3} \), we write

\[ f(t) = \exp \left( \frac{t}{3} \right) \int_0^1 \{1 + t(x^2 - \frac{1}{3}) + \frac{1}{6}t^2(x^2 - \frac{1}{3})^2 + \cdots \} dx \]

= \exp \left( \frac{t}{3} \right) (1 + 2t^2/45 + \cdots).  

Similarly

\[ f'(t) = \exp \left( \frac{t}{3} \right) (\frac{1}{3} + 4t/45 + \cdots). \]  

Finally

\[ m_k = (2/\pi)^{1/2} \int_0^\infty \exp \left( -ks^2/6 \right) (1 - \frac{1}{4}s^2 + \cdots) ds \]

\[ = \left( \frac{k}{3} \right)^{1/2} (1 - 1/10k + \cdots). \]  

Turning now to (3) we have, similarly,

\[ M_k = (2/\pi)^{1/2} \int_0^\infty g'(\frac{1}{2}s^2) g(-\frac{1}{2}s^2) k^{-1} ds, \]

where

\[ g(t) = \int_0^1 \int_0^1 \exp \left( t(x - y)^2 \right) dx dy, \]

which has a maximum at \( t = 0 \), where \( g' = \frac{1}{6} \). Thus we write

\[ g(t) = \exp \left( \frac{1}{6} \right) \sum_{n=0}^\infty t^n I_n. \]

where

\[ I_n = \frac{1}{n!} \int_0^1 \int_0^1 \{(x - y)^2 - \frac{1}{3}\}^n dx dy, \]

and

\[ g'(t) = \exp \left( \frac{1}{6} \right) \sum_{n=0}^\infty t^n J_n, \]

where

\[ J_n = I_n/6 + (n + 1)I_{n+1}. \]

Then \( I_0 = 1, I_1 = 0, I_2 = 7/360, I_3 = 11/5670, J_0 = 1/6, J_1 = 7/180 \) and \( J_2 = 137/15120 \).

Now putting (12) and (13) in (10) and using standard formulae for moments of a normal density gives

\[ M_k = (k/6)^{1/2} (1 - 7/40k - 65/896k^2 + \cdots). \]

Anderssen et al. (1976) compute \( M_1, M_2 \) exactly and \( M_3, \cdots, M_{10} \) by a slowly
convergent series method. They also obtain an upper bound

\[ M_k \approx \frac{k}{6}\{1 + 2(1 - 3/5k)^{3/2}\}. \]

The table lists the \( M_1, \cdots, M_{10} \) from Anderssen et al. and their deviations from (14) (as shown) and (15) denoted \( (14) - M_k \) and \( (15) - M_k \) respectively. This illustrates the accuracy of (14), for \( k \) not too small, while its efficiency is obvious.

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References

ANDERSSEN, R. S., BRENT, R. P., DALEY, D. J. AND MORAN, P. A. P. (1976) Concerning \( \int_0^1 \cdots \int_0^1 (x_1^2 + \cdots + x_k^2)^{1/2} dx_1 \cdots dx_k \) and a Taylor series method. SIAM J. Appl. Math. 30, 22–30.