IMPROVED BOUNDS FOR THE VARIANCE OF THE BUSY PERIOD OF THE M/G/∞ QUEUE

Y. S. SATHE,* University of Bombay

Abstract

Bounds obtained by Ramalhoto [1] for the variance of the busy period of an M/G/∞ queue are improved.

LOWER AND UPPER BOUNDS

Ramalhoto [1] obtained bounds for the variance of the busy period (BP) of an M/G/∞ queue in terms of the Poisson parameter λ, the mean (α) and the variance (σ²) of the service-time distribution G(·) where α and σ² are finite. These bounds are improved in the present note.

1. Notation and definitions

Let ρ = λα, α²γ² = σ² and U(t) = ∫₀⁺[1 - G(x)] dx. Then U(0) = α and

\[ 2 \int_0^\infty U(t) \, dt = 2 \int_0^\infty t(1 - G(t)) \, dt = \alpha(\gamma^2 + 1). \]

Further, let T be a random variable having p.d.f.

f(t) = \begin{cases} 2t(1 - G(t))/\alpha(\gamma^2 + 1) & t \geq 0 \\ 0 & \text{otherwise.} \end{cases}

The expression for the variance of BP as given in Ramalhoto [1] can be written as

\begin{equation}
\text{Var (BP)} = \lambda^{-2}\{\exp(\rho)[\rho^2(\gamma^2 + 1)e^{\lambda U(T)}] - (\exp(\rho) - 1)^2\}.
\end{equation}

We shall first prove the following lemma.

Lemma. Let \( a_n = E[U(T)^n] \) and \( a_n/b_n = \alpha^n/[(n + 1)(n + 2)] \). Then for any positive integer n,

(i) \[ 2 \int_0^\infty U^n(t) \, dt = na_n(\gamma^2 + 1)a_{n-1} \]

(ii) \[ 2(\gamma^2 + 1)^{-1} \leq b_n \leq 2. \]

Proof. (i) is proved by integration by parts. \( U(t) \geq \max[0, (\alpha - t)] \). Therefore, using...
(i), we get the left-hand side of (ii). Further, \( U(t) \leq \alpha - t(1 - G(t)) \). Multiplying throughout by \( U^n(t) \) and integrating with respect to \( t \) over \( R^+ \), then using (i) we get \((n + 2)a_n \leq n\alpha a_{n-1} \), i.e. \( b_n \leq b_{n-1} \) for all positive integer \( n \). Therefore, \( b_n \leq b_0 = 2 \).

2. Lower and upper bounds for \( \text{Var} (BP) \)

**Proposition.** \( L_1(\lambda^{-2}, \rho, \gamma^2_r) \leq \text{Var} [BP] \leq U_1(\lambda^{-2}, \rho, \gamma^2_r) \) where

\[
L_1(\lambda^{-2}, \rho, \gamma^2_r) = \lambda^{-2} \{ \max \left[ (\exp (2\rho) + \exp (\rho) \rho^2 \gamma^2_r - 2\rho \exp (\rho) - 1), 0 \right] \},
\]

\[
U_1(\lambda^{-2}, \rho, \gamma^2_r) = \lambda^{-2} \{ 2 \exp (\rho)(\gamma^2_r + 1)(\exp (\rho) - 1 - \rho) - (\exp (\rho) - 1)^2 \}.
\]

**Proof.** Since \( \exp [\lambda U(T)] \) is a bounded random variable, therefore,

\[
E[\exp \lambda U(T)] = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n a_n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{\rho^n b_n}{n + 2}.
\]

Hence using (ii)

\[
1 + 2\rho^{-2}(1 + \gamma^2_r)^{-1} \left( \exp (\rho) - 1 - \rho - \frac{\rho^2}{2} \right) \leq E[\exp \lambda U(T)] \leq 2\rho^{-2}(\exp (\rho) - 1 - \rho).
\]

Substituting this in (1), the proposition is proved.

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**Reference**