LETTERS TO THE EDITOR

AN ORDERING INEQUALITY FOR EXCHANGEABLE RANDOM VARIABLES

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Abstract
Let $X_1, \ldots, X_n$ be exchangeable random variables with finite variance and two sequences of constants satisfying $a_1 \leq \cdots \leq a_n$, $b_1 \leq \cdots \leq b_n$. Suppose that $a'_1, \ldots, a'_n$ is a rearrangement of $a_1, \ldots, a_n$ and that $g(x)$ is a non-decreasing function. Then

$$E \sum a'_i X_i g(\sum b_i X_i) \leq E \sum a_i X_i g(\sum b_i X_i).$$

1. Introduction

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two sequences with the same ordering, for example, both non-decreasing. Then a classic inequality (Hardy et al. (1934), Chapter 10) tells us that, if $a'_1, \ldots, a'_n$ is a rearrangement of $a_1, \ldots, a_n$, then $\sum a'_i b_i \leq \sum a b_i$. Indeed this classic inequality shows that to establish

(1) $$E \sum a'_i X_i g(\sum b_i X_i) \leq E \sum a_i X_i g(\sum b_i X_i)$$

we have only to prove, under the conditions of the theorem, that

(2) $$EX_1 g(\sum b_i X_i) \leq EX_2 g(\sum b_i X_i) \leq \cdots \leq EX_n g(\sum b_i X_i).$$

This will be done in Section 2.

In the course of proving (see Watson (1985)) a multivariate result concerned with rank-$s$ orthogonal projectors $Q$ uniformly distributed on the appropriate Grassmann manifold, it was observed that since the diagonal elements $Q_{ii}$ of $Q$ have an exchangeable distribution, the proof would go through if the inequality (1), with $g$ replaced by exp, were true. This application is sketched in Section 3. The result may be useful outwith multivariate analysis.

2. Proof of (2)

We begin with a special case which is entirely analogous to the deterministic result.

Lemma 1. For exchangeable real random variables $X_1, \ldots, X_n$ constants $a_1 \leq \cdots \leq a_n$, $b_1 \leq \cdots \leq b_n$,

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Proof. The left-hand side of (3) may be written

\[ E(\sum a_i X_i) (\sum b_j X_j) \leq E(\sum a_i X_i) (\sum b_j X_j) \]

where, by exchangeability, \( EX_i^2 = EX_j^2 = c, \) \( EX_iX_j = d, \) \( E(X_i - X_j)^2 = 2(c - d) \geq 0. \) But since \( a_i \equiv a, \) and \( a_j \equiv a \), \( (3) \) is proved.

To prove (2), there is no loss of generality in assuming that \( P(X_1 > X_2) = P(X_1 < X_2) = \frac{1}{2}, \) for the only alternative to this is \( X_1 = \cdots = X_n \) when there is nothing to prove. Further, it suffices to prove that

\[ E(X_2 - X_1) g(\sum b_j X_j) \geq 0, \]

the first inequality in (2), since all others follow from the same argument. But the left-hand side of (4) is \( P(X_2 > X_1) \) times the sum of two conditional expectations,

\[ E \{ (X_2 - X_1) g(\sum b_j X_j) \mid X_2 > X_1 \} + E \{ (X_2 - X_1) g(\sum b_j X_j) \mid X_2 < X_1 \}. \]

By the exchangeability of \( X_1 \) and \( X_2, \) \( X_1 \) and \( X_2 \) may be interchanged in the second term of (5) which can then be rewritten as

\[ E \{ (X_2 - X_1) \{ g(b_1 X_1 + b_2 X_2 + \sum b_i X_i) - g(b_1 X_1 + b_2 X_2 + \sum b_i X_i) \} \mid X_2 > X_1 \}, \]

where \( \sum b_i X_i \) is \( \sum b_i X_i \) excluding the first two terms. Clearly the first factor \( X_2 - X_1 \) is positive. The second factor is non-negative because \( b_1 \equiv b_2, \) \( X_1 < X_2 \) implies that \( b_1 X_1 + b_2 X_2 \equiv b_1 X_2 + b_2 X_1 \) and \( g \) is a non-decreasing function. Hence the theorem is proved.

3. Remarks

The seemingly trivial inequality (1), or its equivalent form (2), enables us to prove easily results which are otherwise rather baffling. For example, let \( M \) be a symmetric \( q \times q \) matrix with eigenvalues \( \lambda_1(M) \equiv \cdots \equiv \lambda_q(M), \) \( Q \) a symmetric \( q \times q \) idempotent matrix of rank \( s < q \) uniformly distributed on its Grassmann manifold, \( G, \) and define

\[ N = \int_G Q \exp \text{trace} (MQ) \Delta(dQ), \]

where \( \Delta(dQ) \) is the invariant measure on \( G \) integrating to unity. It may be shown without too much trouble that \( N \) and \( M \) commute and so have the same eigensubspaces. It then follows that

\[ \lambda_s(N) = \int_G Q \exp \sum \lambda_s(M) Q \Delta(dQ). \]

We may use the inequality of this paper, and the ‘classic’ inequality, to show that
\( \lambda_1(N) \geq \cdots \geq \lambda_q(N) \). For let \( w_1, \ldots, w_q \) be a reordering of the \( \lambda_i(M) \) and consider

\[
\sum w_i \lambda_i(N) = \int_G \sum w_i Q_u \exp \sum \lambda_i(M) Q_u \Delta(dQ).
\]

By symmetry, the \( Q_u \) are exchangeable. The inequality (1) then tells us that the right-hand side of (6) is a maximum if \( w_1 \geq \cdots \geq w_q \). Since this is true of the left-hand side of (7), the ‘classic’ inequality requires that \( \lambda_1(N) \geq \cdots \geq \lambda_q(N) \).

References
