ON SOME BANACH SPACE SEQUENCES

Roshdi Khalil

We introduce the Banach space of vector valued sequences
\( \ell^p_q(E) \), \( 1 \leq p, q \leq \infty \), where \( E \) is a Banach space. Then we
study the relation between \( \ell^p_q(E) \) and the Schur multipliers of
\( \ell^p \hat{\otimes} E \), where \( E \) is taken to be some \( \ell^r \).

0. Introduction

Let \( E \) be a Banach space. Cohen [3], used the spaces \( \ell^p(E), \ell^p(E) \)
together with the space he introduced \( \ell^p(E) \), to study \( p \)-summing
operators, and their dual ideal (see [11]). Apiola [1], studied the
duality relationships between the spaces \( \ell^p(E), \ell^p(E) \) and \( \ell^p(E) \).

In this paper we introduce the space \( \ell^p_q(E) \), and find its dual.
Further, we investigate the relationship between such spaces and the Schur
multipliers [2], on discrete spaces.

Throughout the paper, if \( E \) and \( F \) are Banach spaces, then \( E \hat{\otimes} F \)
and \( E \tilde{\otimes} F \) will denote the completion of the projective tensor product of
\( E \) with \( F \), and the injective tensor product, respectively [4]. Let
\( \phi \in E \hat{\otimes} F \); then \( \| \phi \|_\pi \) designates the projective norm and \( \| \phi \|_\epsilon \) that of
the injective norm. The dual of \( E \) will be denoted by \( E^* \) for any Banach
space \( E \). The set of natural numbers is denoted by \( \mathbb{N} \), and the complex
numbers by \( \mathbb{C} \). Let \( \ell^p \) be the space of \( p \)-summable sequences,
\( 1 \leq p \leq \infty \).

Received 7 October 1981.

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1. The space $l^{p'_q}(E)$ and its dual

Let $E$ be a Banach space. Then $l^p(E)$ will denote the space of all functions $f : N \to E$, such that $\sum_{n=1}^{\infty} |\langle f(n), x^* \rangle|^p < \infty$, $x^* \in E^*$. The space $l^p(E)$ becomes a Banach space when one introduces the norm

$$\|f\|_{l^p(E)} = \sup_{x^*} \left\{ \left( \sum_{i=1}^{\infty} |\langle f(i), x^* \rangle|^p \right)^{1/p}, \|x^*\| \leq 1, x^* \in E^* \right\},$$

for all $f \in l^p(E)$, [3]. Grothendieck, [5], showed that $l^p(E)$ is isometrically isomorphic to $(l^{p'}(F)^*)^*$, where $F^* = E$, and $1/p + 1/p' = 1$.

Cohen, [3], introduced the space $l^{p'}(E)$ to be the space of all functions $f : N \to E$ such that $\sum_{i=1}^{\infty} |\langle f(i), g(i) \rangle|^p < \infty$, for all $g \in l^{p'}(E^*)$. The norm of $f$ is given by

$$\|f\|_{l^{p'}(E^*)} = \sup_{g} \left\{ \sum_{i=1}^{\infty} |\langle f(i), g(i) \rangle|, g \in l^{p'}(E^*) \text{ and } \|g\|_{l^{p'}(E^*)} \leq 1 \right\}.$$

The space $l^p(E)$ was shown to induce the injective norm on $l^p(\otimes E)$, [3], and Cohen showed that $l^p(E)$ induces the projective norm on $l^p(\otimes E)$.

Further, Apioia, [J], showed that $(l^p(E))^* \equiv l^{p'}(E^*)$ and $(l^{p'}(E))^* \equiv l^{p}(E^*)$.

Now we introduce the space $l^{p', q}(E)$ to be the space of all functions $f : N \to E$ such that $\sum_{i=1}^{\infty} |\langle f(i), g(i) \rangle|^p < \infty$ for all $g \in l^{q'}(E^*)$. If $f \in l^{p', q}(E)$, then we define

$$\|f\|_{l^{p', q}(E)} = \sup_{g} \left\{ \sum_{i=1}^{\infty} |\langle f(i), g(i) \rangle|^p \right\}^{1/p},$$

where $g \in l^{q'}(E^*)$ and $\|g\|_{l^{q'}(E^*)} \leq 1$.
**Lemma 1.1.** The function \( \| \cdot \|_{\sigma(p,q)} \) is a norm on \( l^{p,q}(E) \).

Proof. It is enough to show that \( \| f \|_{\sigma(p,q)} < \infty \) for all \( f \in l^{p,q}(E) \). The rest of the properties of the norm are easy to verify.

Let \( f \in l^{p,q}(E) \). Define the bilinear form
\[
\hat{f} : l^{p'} \times l^{q'}(E^*) \to \mathbb{C},
\]
\[
f(a, g) = \sum_{i=1}^{\infty} a(i) \langle f(i), g(i) \rangle.
\]
It is not hard to check that \( \hat{f} \) is separately continuous on \( l^{p'} \times l^{q'}(E^*) \). Hence, [2, p. 172], \( \hat{f} \) is jointly continuous, and consequently \( \| \hat{f} \|_{\sigma(p,q)} < \infty \) for all \( f \in l^{p,q}(E) \).

**Theorem 1.2.** The space \( l^{p,q}(E) \) with the \( \sigma(p,q) \) norm is a Banach space.

Proof. Let \( f_n \in l^{p,q}(E) \) such that \( \sum_{n=1}^{\infty} \| f_n \|_{\sigma(p,q)} < \infty \). It is enough to show that \( \| \sum_{n=1}^{\infty} f_n \| < \infty \), [12]. We first prove this for the case \( p = 1 \). Since \( E \) is a Banach space, then every absolutely summable sequence in \( E \) is summable. It follows that for each natural number \( i \), the series \( \sum_{n=1}^{\infty} f_n(i) \) is convergent in \( E \). Define \( F : N \to E \) by
\[
F(i) = \sum_{n=1}^{\infty} f_n(i).
\]
Let \( g \in l^{q'}(E^*) \) and \( \| g \|_{\varepsilon(q')} \leq 1 \). We have to prove that
\[
\sum_{i=1}^{\infty} \langle F(i), g(i) \rangle < \infty .
\]
\[
\sum_{i=1}^{\infty} |\langle F(i), g(i) \rangle| \\
= \sum_{i=1}^{\infty} \left| \left( \sum_{n=1}^{\infty} f_n(i), g(i) \right) \right| \\
= \sum_{i=1}^{\infty} \left| \sum_{n=1}^{\infty} \langle f_n(i), g(i) \rangle \right| \quad \text{(since } \sum_{n=1}^{\infty} \|f_n(i)\| < \infty \text{)} \\
\leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\langle f_n(i), g(i) \rangle| \quad \text{(since } \sum_{n=1}^{\infty} \langle f_n(i), g(i) \rangle < \infty \text{)}.
\]

If \( \nu \) is the counting measure on the set of natural numbers \( N \), then

\[
\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\langle f_n(i), g(i) \rangle|
\]

can be considered as

\[
\int_{N} \sum_{n=1}^{\infty} |\langle f_n(i), g(i) \rangle| d\nu(i).
\]

As a consequence of the monotone convergence theorem we get

\[
\int_{N} \sum_{n=1}^{\infty} |\langle f_n(i), g(i) \rangle| d\nu(i) = \sum_{n=1}^{\infty} \int_{N} |\langle f_n(i), g(i) \rangle| d\nu(i).
\]

It follows that

\[
\sum_{i=1}^{\infty} |\langle F(i), g(i) \rangle| \\
\leq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\langle f_n(i), g(i) \rangle| < \infty \quad \text{(since } \sum_{n=1}^{\infty} \|f_n\|_q < \infty \text{)}.
\]

Hence \( \sum_{i=1}^{\infty} |\langle F(i), g(i) \rangle| < \infty \) for all \( g \in l^q(E^*) \) with \( \|g\|_{E(q')} < \infty \).

Consequently \( F \in l^1,q(E) \), and so \( \sum_{n=1}^{\infty} f_n \in l^1,q(E) \).

For general \( p \), the result follows from the fact that

\[
\|f\|_{g(p,q)} = \sup_{\theta,g} \left| \sum_{i=1}^{\infty} \theta(i) \langle f(i), g(i) \rangle \right|,
\]
where \( \theta \in \ell^p \), \( g \in \ell^q(E^*) \) and \( \|\theta\|_p, \|g\|_{(q')} \leq 1 \). Hence the proof of the theorem is complete.

Let \( \ell^p \otimes \ell^q(E^*) \) be the set of all elements of the form \( a \cdot f \) such that \( a \in \ell^p \), \( f \in \ell^q(E^*) \) and \( (a \cdot f)(i) = a(i) \cdot f(i) \).

**Theorem 1.3.** A linear functional \( F \) on \( \ell^p \otimes \ell^q(E) \) is bounded if and only if \( F \) is of the form \( a \cdot f \), for some \( a \in \ell^p \) and \( f \in \ell^q(E^*) \).

**Remark.** The space \( \ell^1 \otimes \ell^q(E) \) is just \( \ell^q(E) \) in Cohen [3]. Apiola, [1], proved that \( (\ell^1, \ell^q(E))^* \) is isometrically isomorphic to \( \ell^q(E^*) \) which is in turn isomorphic to \( \ell^\infty \otimes \ell^q(E^*) \).

**Proof of Theorem 1.3.** Let \( a \in \ell^p \) and \( f \in \ell^q(E^*) \). Consider the linear functional \( F : \ell^p \otimes \ell^q(E) \to \mathbb{C} \) defined by

\[
F(g) = \sum_{i=1}^{\infty} a(i) \langle f(i), g(i) \rangle .
\]

Then

\[
|F(g)| \leq \|a\|_p \cdot \|f\|_{\ell^q(E^*)} \cdot \|g\|_{\sigma(p,q)} .
\]

Hence \( F \) is bounded and \( \|F\| \leq \|a\|_p \cdot \|f\|_{\ell^q(E^*)} \).

Conversely, let \( F \in (\ell^p \otimes \ell^q(E))^* \). Hence \( |F(f)| \leq \lambda \cdot \|f\|_{\sigma(p,q)} \) for some constant \( \lambda \). Let \( e_i \) be the natural embedding of \( E \) in \( \ell^p \otimes \ell^q(E) \), so

\[
e_i(x)(j) = \begin{cases} x, & i = j, \\ 0, & i \neq j. \end{cases}
\]

Put \( x_i^* = F \circ e_i \). Clearly \( x_i^* \in E^* \), and if \( f \in \ell^p \otimes \ell^q(E) \), then

\[
F(f) = \sum_{i=1}^{\infty} \langle f(i), x_i^* \rangle .
\]

Assume \( F \) to be of norm one; then there is an \( a \in \ell^p \) such that
Now let $D$ be the unit disc and $\pi D$ be the countable product of $D$ with itself. Since $D$ is compact, then $\pi D$ is compact. Let $B_1$ be the unit ball of $l^{q'}(E^*)$. As a dual of $l^{1,q}(E)$, $[1]$, $B_1$ is compact with respect to the $\omega^*$-topology, and so is the product space $\pi D \times B_1$. Let $C(\pi D \times B_1)$ be the space of continuous functions on $\pi D \times B_1$. Consider the map

$$\psi : l^{p,q}(E) \rightarrow C(\pi D \times B_1) ,$$

$$\psi(f) = G ,$$

where

$$G(\theta, u) = \sum_{i=1}^{\infty} a(i) \theta(i) f(i), u(i) ,$$

for all $f \in l^{p,q}(E)$ and $\theta \in \pi D$, and $u \in B_1$. It follows that

$$\|G\|_\infty = \sup_{\theta,u} |G(\theta, u)| = \sup_{\theta,u} \left| \sum_{i=1}^{\infty} a(i) \theta(i) f(i), u(i) \right| = \sup_{u} \sum_{i=1}^{\infty} |a(i) f(i), u(i)| .$$

Hence $|F(f)| \leq \|\psi(f)\|$ . This implies that $\ker \psi \subseteq \ker F$.

This implies that there exists an $\tilde{F} : C(\pi D \times B_1) \rightarrow \mathbb{C}$ such that

$$\tilde{F} \circ \psi = F .$$

The Riesz representation theorem implies that there exists a regular Borel measure $\mu$ on $\pi D \times B_1$ such that
Let $f_n$ denote the function $f_n : N \to E$

$$f_n(i) = \begin{cases} f(i), & i = n, \\ 0, & i \neq n. \end{cases}$$

Then

$$F(f) = \mu\{\psi(f)\} = \int_{\pi D \times B_1} \sum_{i=1}^{\infty} a(i) \theta(i)(f(i), u(i)) d\nu(\theta, u).$$

But $F(f) = \langle f(1), x^*_1 \rangle$. It follows that

$$x^*_1 = a(1) \cdot \int_{\pi D \times B_1} \theta(1) \cdot u(1) d\nu(\theta, u),$$

where the integral here is the Pettis integral, [4]. Set

$$Z^*_1 = \int_{\pi D \times B_1} \theta(1) u(1) d\nu(\theta, u).$$

Hence $x^*_1 = a(1) \cdot Z^*_1$. Similarly $x^*_i = a(i) \cdot Z^*_i$, $i = 2, 3, \ldots$. It remains to show that the function $g : N \to E^*$, defined by $g(i) = Z^*_i$, is an element of $l^q'(E^*)$. To see that, consider

$$\langle g(i), x \rangle \leq \int_{\pi D \times B_1} |\theta(i)\langle u(i), x \rangle| d\mu(\theta, u) \quad (x \in E, \|x\| \leq 1)$$

$$= \int_{\pi D \times B_1} |\langle u(i), x \rangle| d\mu(\theta, u)$$

$$= \left( \int_{\pi D \times B_1} |\langle u(i), x \rangle|^{q'} d\mu(\theta, u) \right)^{1/q'}.$$

Hence

$$\sum_{i=1}^{\infty} |\langle g(i), x \rangle|^{q'} \leq \sum_{i=1}^{\infty} \int_{\pi D \times B_1} |\langle u(i), x \rangle|^{q'} d\mu(\theta, u).$$

The monotone convergence theorem implies that
\[ \sum_{i=1}^{\infty} |\langle g(i), x \rangle|^{q'} \leq \int_0^1 \sum_{i=1}^{\infty} |\langle u(i), x \rangle|^{q'} d\mu(\theta, u) \]
\[ \leq \sup_{\mu \in B_1} \sum_{i=1}^{\infty} |\langle u(i), x \rangle|^{q'} \cdot |\mu| , \]

where $|\mu|$ is the total variation of $\mu$. Thus $g \in l^{q'}(E^*)$. So $F = a \cdot g$, $a \in \ell^{p'}$, $g \in l^{q'}(E^*)$. This completes the proof of the theorem.

2. Schur multipliers

Let $p, q \geq 1$. A bounded function $\phi$ on $\mathbb{N} \times \mathbb{N}$ is called a Schur multiplier of $l^p \hat{\otimes} l^q$ if $\phi \cdot \psi \in l^p \hat{\otimes} l^q$ for all $\psi \in l^p \hat{\otimes} l^q$, where $\phi \cdot \psi$ denotes pointwise multiplication.

If $X$ and $Y$ are Banach spaces, then a bounded linear map $A : X \to Y$ is called $p$-summing operator if

\[ \sum_{i=1}^{n} \|Ax_i\|^{p} \leq \zeta \cdot \sup_{x^*} \sum_{i=1}^{n} |\langle x^*, x_i \rangle|^{p} , \]

for all $x_1, \ldots, x_n$ in $X$ and some constant $\zeta$ independent of $n$. The supremum is taken over all elements $x^*$ in the unit ball of $X^*$, [10]. Bennett [2] proved that a bounded function $\phi$ is a multiplier of $l^p \hat{\otimes} l^q$ if and only if $\phi \cdot u \otimes v : l^p \to l^\infty$ is $q^*$ summing operator for all $u \otimes v \in l^\infty \hat{\otimes} l^p$. For more about multipliers we refer to [2], [6], [7] and [3].

**Lemma 2.1.** Let $A : l^p \to l^\infty$ be a bounded operator. If $A$ is $q$-summing, then $A \in l^{q'}(l^p)$.

**Proof.** Let $f : \mathbb{N} \to l^{p'}$ be the function defined by $f(i) = A_i$, where $A_i(j) = A(i, j)$ (considering $A$ as an infinite matrix). If $g \in l^q(l^p)$, then
\[
\sum_{i=1}^{\infty} |\langle f(i), g(i) \rangle|^q = \sum_{i=1}^{\infty} |\langle A_i, g(i) \rangle|^q \\
= \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} A(i, j)g(i)(j) \right|^q \\
\leq \sum_{i=1}^{\infty} \sup_k \left| \sum_{j=1}^{\infty} A(k, j)g(i)(j) \right|^q \\
= \sum_{i=1}^{\infty} \|A(g(i))\|^q \\
\leq \xi \sup_h \sum_{i=1}^{\infty} |\langle g(i), h \rangle|^q \quad \text{(by assumption)},
\]
where \( h \) is the unit ball of \( \ell^p' \). Hence \( f \in \ell^q \ast \ell^q' \). Let \( \mathcal{M}(\ell^p \hat{\otimes} \ell^q) \) denote the space of all multipliers of \( \ell^p \hat{\otimes} \ell^q \). Then:

**THEOREM 2.2.** Let \( \phi \) be a bounded function on \( N \times N \). Then \( \phi \in \mathcal{M}(\ell^p \hat{\otimes} \ell^q) \) if and only if \( \phi \ast u \in \ell^q \ast \ell^q' \), for all \( u \in \ell^p \).

Proof. Let \( \phi \in \mathcal{M}(\ell^p \hat{\otimes} \ell^q) \). Then by Bennett's result [2], \( \phi \ast u : \ell^p' \to \ell^\infty \) is \( q' \)-summing for all \( u \in \ell^p \). Lemma 2.1 then implies that \( \phi \ast u \in \ell^q \ast \ell^q' \).

Conversely, let \( \phi \ast u \in \ell^q \ast \ell^p' \) for all \( u \in \ell^p \). It is enough to show that \( \phi \in \mathcal{M}(\ell^q \hat{\otimes} \ell^p) \). So let \( u \otimes v \in \ell^q \hat{\otimes} \ell^p \), and \( \psi \in \ell^q' \hat{\otimes} \ell^p' \). Then

\[
|\langle \phi \ast u \otimes v, \psi \rangle| = \left| \sum_{i,j=1}^{\infty} \phi(i, j)u(i)v(j)\psi(i, j) \right| \\
= \left| \sum_{i=1}^{\infty} u(i)\langle \phi_i \ast v, \psi_i \rangle \right|,
\]
where \( \phi_i(j) = \phi(i, j) \) and \( \psi_i(j) = \psi(i, j) \). Since \( \psi \in \ell^q' \hat{\otimes} \ell^p' \) it follows that \( g : \ell^2 \to \ell^p' \) defined by \( g(i) = \psi_i \) is an element of \( \ell^q' \hat{\otimes} \ell^p' \), [3]. Hence
This completes the proof of the theorem.

**Lemma 2.3.** If $A : l^p \to l^\infty$ is $q'$-summing, then $A \in \mathcal{M}(l^p \hat{\otimes} l^q)$.

**Proof.** It is enough to show that $A \cdot \hat{\otimes} v : l^p \to l^\infty$ is $q'$-summing operator for all $v \in l^p$, [2]. But

$$
\sum_{i=1}^\infty \| (A \cdot \hat{\otimes} v) f_i \|_{l^q}' = \sum_{i=1}^\infty \| A (v \cdot f_i) \|_{l^q}'
\leq \zeta \sup_h \sum_{i=1}^\infty |(v \cdot f_i, h)|^{q'}
\leq \zeta \sup_h \sum_{i=1}^\infty |(f_i, v \cdot h)|^{q'}
\leq \zeta \sup_k \sum_{i=1}^\infty |(f_i, k)|^{q'}
$$

where $h$ and $k$ are in the unit ball of $l^p$, and the lemma follows.

It follows from Lemmas 2.3 and 2.1 that the set of all $q'$-summing maps from $l^p$ into $l^\infty$ is contained in $\mathcal{M}(l^p \hat{\otimes} l^q) \cap l^{q',p}(l^p)$, where $\cap$ denotes the intersection of the two sets.

**References**


Department of Mathematics, University of Kuwait, PO Box 5969, Kuwait.