ON VECTOR SPACES OF CERTAIN MODULAR FORMS 
OF GIVEN WEIGHTS: ADDENDUM 

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The statement $f = \lim_{t \to \infty} g_t$ used in proving Theorem 2 of [1] needs explanation. This was pointed out to us by Professor S. Raghwan of Tata Institute of Fundamental Research, Bombay, and we gave the explanation of this in [2]. For the sake of completeness we give here the full proof of the theorem; filling the gap in the proof. We use the same notations and definitions as those of [1]. Also for simplicity of notation we write $k_m, g_m$ and $a_m$ to mean $k_{m,n}, g_{m,n}$ and $a_{m,n}$ respectively. We need the following lemma.

**Lemma.** Let $p \geq 5$ be a prime number and $u$ an even integer such that $0 \leq u < p-1$. Further let $\{g_m\}$ be a family of all modular forms over $\text{SL}_2(\mathbb{Z})$ such that

$$k_m \equiv u \mod (p-1)$$

for all $m$, where $k_m$ denotes the weight of the modular form $g_m$. Then for each $n = 0, 1, 2, \ldots$, there exist non-negative integers $a(n)$ and $b(n)$, satisfying

(i) $4a(n) + 6b(n) + 12n \equiv u \mod (p-1)$, and

(ii) $k_m = [k_m - (4a(n) + 6b(n) + 12n)]/(p-1)$ are non-negative integers for $0 \leq n < d_m$, where $d_m$ denotes the dimension of vector space of modular forms over $\text{SL}_2(\mathbb{Z})$ of weight $k_m$.

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Proof. Arrange \( \{d_m^i\} \) in ascending order of magnitude. Let \( k_{m_i}^i \) be the least among the weights of modular forms of dimension \( d_i \) in the family \( \{g_m\} \). Then \( \{k_{m_i}^i\} \) are also in ascending order.

For \( 0 \leq n < d_0 \), we have \( k_{m_0}^m - 12n \geq 0 \) and not equal to 2. Therefore for these \( n \), we can choose non-negative integers \( a(n) \) and \( b(n) \) such that 
\[
4a(n) + 6b(n) = k_{m_0}^m - 12n .
\]
Then
\[
4a(n) + 6b(n) + 12n = k_{m_0}^m \equiv u \mod (p-1) .
\]
As \( k_{m_0}^m \leq k_m \) for all \( m \), therefore \( k_{m_1}^m \) are non-negative integers for \( 0 \leq n < d_0 \) and for all \( m \). Proceeding as above we can find non-negative integers \( a(n) \) and \( b(n) \) for \( d_i \leq n < d_{i+1} \), \( i = 0, 1, 2, \ldots \), satisfying the required conditions.

REMARK. Let \( S \) be any subfamily of the family of all modular forms \( \{g_m\} \) with weights \( k_m \) satisfying \( k_m \equiv u \mod (p-1) \). Then the same choice of \( a(n) \) and \( b(n) \), as is done for this family of all modular forms in the lemma, will work for the subfamily \( S \) also.

For a given prime \( p \geq 5 \) and \( k = (s, u) \), \( u \) as above, construct \( a(n) \) and \( b(n) \) for each \( n \). Consider
\[
f_m^s = \frac{q^{a(n)^s}b(n)^s}{\Delta_{E,s}^n / (p-1)} ,
\]
where \( s_n = \{s-(4a(n)+6b(n)+12n)\}/(p-1) \). It follows from Theorem 1 of [1] that \( f_n^s \) is a \( p \)-adic modular form of weight \( k \). Writing 'q' series expansion for \( f_n^s \), that is,
\[
f_n^s = \sum_{m=0}^{\infty} a_m^{(n)} q^m , \text{ with } a_m^{(n)} \text{ in } \mathbb{Q}_p ,
\]
we see that \( a_m^{(n)} = 0 \) for \( 0 \leq m < n \) and \( a_n^{(n)} = 1 \). We now give a complete proof of Theorem 2 of [1].
THEOREM. \( f \) is a \( p \)-adic modular form of weight \( k = (s, u) \) if and only if \( f = \sum_{n=0}^{\infty} a_n f_n \) with \( \nu_p(a_n) \to \infty \) as \( n \to \infty \).

Proof. Suppose first that \( f \) is a \( p \)-adic modular form of weight \( k \). Write

\[
f = \sum_{m=0}^{\infty} b_m(0) q^m, \quad b_m(0) \text{ in } \mathbb{Q}_p.
\]

The Fourier series expansion of \( f - b_0^{(0)} f_0 \) has no constant term, therefore we can write

\[
f - b_0^{(0)} f_0 = \sum_{m=1}^{\infty} b_m^{(1)} q^m \text{ with } b_m^{(1)} \text{ from } \mathbb{Q}_p.
\]

Similarly, we can write

\[
f - b_0^{(0)} f_0 - b_1^{(1)} f_1 = \sum_{m=2}^{\infty} b_m^{(2)} q^m \text{ with } b_m^{(2)} \text{ from } \mathbb{Q}_p.
\]

Repeating this process \( t \) times, we can write

\[
f - \sum_{n=0}^{t} b_n^{(n)} f_n = \sum_{m=t+1}^{\infty} b_m^{(t+1)} q^m \text{ with } b_m^{(t+1)} \text{ from } \mathbb{Q}_p.
\]

This means that we can find \( b_n^{(n)} \) in \( \mathbb{Q}_p \), \( n = 0, 1, 2, \ldots \), such that as a formal series in \( q \), we have

\[
f = \sum_{n=0}^{\infty} b_n^{(n)} f_n.
\]

Writing \( a_n \) for \( b_n^{(n)} \), we see that

\[
f = \sum_{n=0}^{\infty} a_n f_n, \text{ with } a_n \text{ in } \mathbb{Q}_p.
\]

Now we shall prove that \( \nu_p(a_n) \to \infty \) as \( n \to \infty \). Choose a sequence \( \{g_m\} \) of modular forms converging to \( f \). Then the sequence \( \{k_m\} \) of their weights converges to \( k \) in \( X \). Therefore \( \{k_m\} \) converges to \( s \) in
For the family \( \{g_m\} \) we construct \( a(n), b(n) \) and \( k_m \) as in the lemma. For each \( m \), define

\[
g_{mn} = q^{a(n)} R^{b(n)} \Delta E^{k_m}_{p-1} \quad \text{for} \quad n = 0, 1, \ldots, d_m - 1,
\]

where \( d_m \) is the dimension of the vector space \( M(k_m) \) of modular forms of weight \( k_m \). Then \( g_{mn} \) are modular forms of weight \( k_m \) and constitute a basis for \( M(k_m) \). Therefore for each \( m \), we can write

\[
g_m = \sum_{n=0}^{\infty} a_{mn} g_{mn}, \quad \text{with} \quad a_{mn} = 0 \quad \text{for} \quad n \geq d_m.
\]

Now \( k_m = \frac{[k_m - h(a(n) - 6b(n) - 12n)]}{(p-1)} \) are integers (may be negative for large \( n \)) for all \( m \) and all \( n \). Define, for each \( n \),

\[
s_n = \frac{[s - (h(a(n) + 6b(n) + 12n))]}{(p-1)}.
\]

Then \( s_n \in \mathbb{Z}_p \), as \( k_m + s \) in \( \mathbb{Z}_p \). Therefore \( k_{mn} \to s_n \) for each \( n \)

and this convergence is uniform in \( n \). Therefore \( E^{k_{mn}}_{p-n} \to E^{s_n}_{p-1} \) uniformly in \( n \). Hence \( g_{mn} \to f_n \) uniformly in \( n \).

Now

\[
(*) \quad f - g_m = \sum_{n=0}^{\infty} \left( a_n f_n - a_{mn} g_{mn} \right)
\]

\[
= \sum_{n=0}^{\infty} a_n \left( f_n - g_{mn} \right) + \left( a_n - a_{mn} \right) g_{mn}.
\]

As \( g_m \to f \) and \( g_{mn} \to f_n \) uniformly in \( n \), therefore given a positive integer \( N \), we can find a positive integer \( m_0 \) such that for each \( m \geq m_0 \), we have

\[
\nu_p \left( f - g_m \right) > N,
\]

and

\[
\nu_p \left( f_n - g_{mn} \right) > N \quad \text{for all} \quad n \geq 0.
\]
Therefore, for \( m \geq m_0 \) and \( n \geq 0 \), it follows from (*), that

\[
\nu_p \left( a_n - a_{mn} \right) > N.
\]

In particular,

\[
\nu_p \left( a_n - a_{m_0 n} \right) > N \quad \text{for all} \quad n \geq 0.
\]

But \( a_{m_0 n} = 0 \) for \( n \geq d_{m_0} \). Hence \( \nu_p(a_n) > N \) for \( n \geq d_{m_0} \). This shows that \( \nu_p(a_n) \to \infty \) as \( n \to \infty \).

Conversely suppose now that \( f = \sum_{n=0}^{\infty} a_n \frac{f^n}{n!} \), with \( \nu_p(a_n) \to \infty \) as \( n \to \infty \). Take \( g_t = \sum_{n=0}^{t} a_n \frac{f^n}{n!} \). Then \( g_t \) is a \( p \)-adic modular form of weight \( k \). Since \( \nu_p(a_n) \to \infty \) as \( n \to \infty \) and \( \nu_p(f^n) = 0 \), so \( g_t \) is a convergent sequence with its limit equal to \( f \). Hence \( f \) is a \( p \)-adic modular form of weight \( k \).

References


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