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NOTE ON TOTAL CATEGORIES

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It is shown that, for a semi-topological functor $T : A \to X$, the category A is total, that is, the Yoneda embedding of A has a left adjoint, if X is total. In particular, monadic categories over $\mathcal{Set}$ (possibly without rank) are total, and full reflective subcategories of total categories are total.

1. Total and compact categories

A category A with small hom-classes is called *total* [6], if the Yoneda-embedding

$$Y_A : A \to \hat{A} = [A^{op}, \mathcal{Set}] , \quad A \mapsto A(-, A) ,$$

has a left adjoint. It is known [6] that any full reflective subcategory of a functor category $[\mathcal{D}, \mathcal{Set}]$ with $\mathcal{D}$ being small is total. In particular, monadic categories over $\mathcal{Set}$ with rank and their full reflective subcategories are total.

A total category A is *compact* [3], that is, A has small hom-classes and any functor $U : A \to B$ preserving all existing colimits in A has a right adjoint, provided U is admissable [6] (that is, the hom-classes $B(UA, B)$ are small for all $A \in \text{Ob } A$, $B \in \text{Ob } B$). The reverse implication is false: it is proved in [2] that Adámek's [1] non-cocomplete (hence non-total) monadic category is compact.

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However, a cocompact category A (for example, any category satisfying the sufficient conditions of Freyd's Special Adjoint Functor Theorem) is total, provided A contains a generating set of objects. Namely, this last condition implies that Y_A is co-admissible whence, Y_A, preserving trivially all limits, has a left adjoint. Therefore, for categories which contain a generating set and a cogenerating set all notions total, cototal, compact, and cocompact coincide.

2. The general lifting technique

It is proved in [2] for a semi-topological [7] functor $T : A \to X$, that A is compact if X is. In the following we shall prove that a corresponding result holds for total categories. For this we consider a left adjoint $F : X \to A$ of T and a natural equivalence $\varphi : \hat{F} \circ Y_A \to Y_X \circ T$ (with $\hat{F} = [F^{op}, \mathcal{E}ns]$):

(1)
$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "Y_A"'] \arrow[dr, phantom, "\overset{\varphi}{\Longrightarrow}"] & X \arrow[d, "Y_X"] \\ \hat{A} \arrow[r, "\hat{F}"] & \hat{X} \end{tikzcd} .$$

Let Y_X have a left adjoint. According to the General Lifting Theorem 2.27 of [5], in order to prove right adjointness of Y_A it suffices to prove that *semi-initial factorizations of T-sources are locally respected* by the above diagram. This means: if the commutative diagram (2),

(2)
$$\begin{tikzcd} & TA \arrow[dr, "Tm_i"] & \\ X \arrow[ur, "e"] \arrow[rr, "x_i"'] & & TB_i \end{tikzcd}$$

has the property that for any $z : TC \to X$ and all $b_i : C \to B_i$ with $x_i z = Tb_i$ there is an $a : C \to A$ with $ez = Ta$ and, therefore, $m_i a = b_i$ ("*diagram* (2) *is T-semi-initial*"), then diagram (3),

(3)

$$\begin{tikzcd}
 & Y_X TA \arrow[dr, "Y_X Tm_i"] & & \hat{F}Y_A A \arrow[ll, "\varphi A"'] \arrow[dr, "\hat{F}Y_A m_i"] & \\
Y_X X \arrow[ur, "Y_X e"] \arrow[rr, "Y_X x_i"'] & & Y_X TB_i & & \hat{F}Y_A B_i \arrow[ll, "\varphi B_i"]
\end{tikzcd}$$

has the following property: for any $\zeta : \hat{F}H \to Y_X X$ $(H \in \mathrm{Ob}\,\hat{A})$ and all $\beta_i : H \to Y_A B_i$ with $(Y_X x_i)\zeta = (\varphi B_i)(\hat{F}\beta_i)$ there is an $\alpha : H \to Y_A A$ with $(Y_X e)\zeta = (\varphi A)(\hat{F}\alpha)$ and $(Y_A m_i)\alpha = \beta_i$ ("*diagram* (3) *is* $\hat{F}$-*semi-initial*"). Usually i ranges over a (proper) index class I. But the definition of semi-topological functors and the proof of 2.27 of [5] show that it does not matter, if I belongs to any higher universe. In the present situation, one takes I to be legitimate with respect to some universe for which $\hat{A}$ is legitimate.

3. The lifting theorem

THEOREM. *Let* $T : A \to X$ *be a semi-topological functor. Then* A *is total, if* X *is total.*

Proof. Let the T-semi-initial diagram (2) be given. In order to prove $\hat{F}$-semi-initiality of diagram (3) let ζ and β_i be as above. For each $C \in \mathrm{Ob}\,A$ one then obtains the commutative diagram (4):

(4)

$$\begin{tikzcd}
 & & & X(TC,TA) \arrow[dr, "X(TC{,}Tm_i)"] & & A(FTC,A) \arrow[ll, "(\varphi A)(TC)"'] \arrow[dr, "A(FTC{,}m_i)"] & \\
 & & X(TC,X) \arrow[ur, "X(TC{,}e)"] \arrow[rr, "X(TC{,}x_i)"'] & & X(TC,TB_i) & & A(FTC,B_i) \arrow[ll, "(\varphi B_i)(TC)"] \\
 & HFTC \arrow[ur, "\zeta TC"] & & & & & \\
HC \arrow[ur, "H\varepsilon C"] \arrow[rrrr, "\beta_i C"] & & & & A(C,B_i) \arrow[uu, "T_{C,B_i}"] \arrow[uurr, "A(\varepsilon C{,}B_i)"'] \arrow[ur, phantom, "(*)"] & &
\end{tikzcd}$$

Here ε denotes the co-unit of the adjoint pair (F, T) such that the triangle (*) commutes for T_{C,B_i} being the respective restriction of T.

For each $s \in HC$ one now has the commutative diagram (5):

$$\begin{tikzcd} & & TA \arrow[dr, "Tm_i"] & \\ & X \arrow[ur, "e"] \arrow[rr, "x_i"'] & & TB_i \\ TC \arrow[ur, "(\zeta TC)(H\varepsilon C)(s)"] \arrow[urrr, "T\left((\beta_i C)(s)\right)"'] & & & \end{tikzcd} \tag{5}$$

From the T-semi-initiality of (2) one therefore obtains a morphism $(\alpha C)(s) : C \to A$ with $T\big((\alpha C)(s)\big) = e\big((\zeta TC)(H\varepsilon C)(s)\big)$. It is easily checked that, in this way, a natural transformation $\alpha : H \to Y_{A}A$ satisfying the needed equations is defined.

COROLLARY 1. *Any monadic category over the category of sets is total.*

COROLLARY 2. *A full reflective subcategory of a total category is total.*

4. Final remark

One is not able to prove a corresponding result as in the theorem for arbitrary monadic (instead of semi-topological) functors as Rattray [4] did in the case of compactness. To see this consider again the category of graphs over which Adámek [1] has constructed his non-total but monadic category: by the theorem, his base category, being semi-topological over sets, is total.

References

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