

## NOTE ON TOTAL CATEGORIES

WALTER THOLEN

It is shown that, for a semi-topological functor  $T : A \rightarrow X$ , the category  $\hat{A}$  is total, that is, the Yoneda embedding of  $\hat{A}$  has a left adjoint, if  $X$  is total. In particular, monadic categories over  $Set$  (possibly without rank) are total, and full reflective subcategories of total categories are total.

### 1. Total and compact categories

A category  $A$  with small hom-classes is called *total* [6], if the Yoneda-embedding

$$Y_A : A \rightarrow \hat{A} = [A^{OP}, Set], \quad A \mapsto A(-, A),$$

has a left adjoint. It is known [6] that any full reflective subcategory of a functor category  $[D, Set]$  with  $D$  being small is total. In particular, monadic categories over  $Set$  with rank and their full reflective subcategories are total.

A total category  $A$  is *compact* [3], that is,  $A$  has small hom-classes and any functor  $U : A \rightarrow B$  preserving all existing colimits in  $A$  has a right adjoint, provided  $U$  is admissable [6] (that is, the hom-classes  $B(UA, B)$  are small for all  $A \in \text{Ob } A$ ,  $B \in \text{Ob } B$ ). The reverse implication is false: it is proved in [2] that Adámek's [1] non-cocomplete (hence non-total) monadic category is compact.

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However, a cocompact category  $A$  (for example, any category satisfying the sufficient conditions of Freyd's Special Adjoint Functor Theorem) is total, provided  $A$  contains a generating set of objects. Namely, this last condition implies that  $Y_A$  is co-admissible whence,  $Y_A$ , preserving trivially all limits, has a left adjoint. Therefore, for categories which contain a generating set and a cogenerating set all notions total, cototal, compact, and cocompact coincide.

### 2. The general lifting technique

It is proved in [2] for a semi-topological [7] functor  $T : A \rightarrow X$ , that  $A$  is compact if  $X$  is. In the following we shall prove that a corresponding result holds for total categories. For this we consider a left adjoint  $F : X \rightarrow A$  of  $T$  and a natural equivalence

$$\varphi : \hat{F} \circ Y_A \rightarrow Y_X \circ T \quad (\text{with } \hat{F} = [F^{OP}, \text{End}] ) :$$

$$(1) \quad \begin{array}{ccc} A & \xrightarrow{T} & X \\ Y_A \downarrow & \xrightarrow{\varphi} & \downarrow Y_X \\ \hat{A} & \xrightarrow{\hat{F}} & \hat{X} \end{array} .$$

Let  $Y_X$  have a left adjoint. According to the General Lifting Theorem 2.27 of [5], in order to prove right adjointness of  $Y_A$  it suffices to prove that *semi-initial factorizations of  $T$ -sources are locally respected* by the above diagram. This means: if the commutative diagram (2),

$$(2) \quad \begin{array}{ccc} & TA & \\ e \nearrow & & \searrow Tm_i \\ X & \xrightarrow{x_i} & TB_i \end{array}$$

has the property that for any  $z : TC \rightarrow X$  and all  $b_i : C \rightarrow B_i$  with  $x_i z = Tb_i$  there is an  $a : C \rightarrow A$  with  $ez = Ta$  and, therefore,  $m_i a = b_i$  ("diagram (2) is  $T$ -semi-initial"), then diagram (3),

$$(3) \quad \begin{array}{ccccc} & & Y_X TA & \xleftarrow{\varphi^A} & \hat{F}Y_A A \\ & \nearrow^{Y_X e} & & & \searrow^{\hat{F}Y_A m_i} \\ & & Y_X TB_i & \xleftarrow{\varphi_{B_i}} & \hat{F}Y_A B_i \\ & \searrow_{Y_X x_i} & & & \\ Y_X X & \xrightarrow{Y_X x_i} & & & \end{array}$$

has the following property: for any  $\zeta : \hat{F}H \rightarrow Y_X X$  ( $H \in \text{Ob } \hat{A}$ ) and all  $\beta_i : H \rightarrow Y_A B_i$  with  $(Y_X x_i)\zeta = (\varphi_{B_i})(\hat{F}\beta_i)$  there is an  $\alpha : H \rightarrow Y_A A$  with  $(Y_X e)\zeta = (\varphi^A)(\hat{F}\alpha)$  and  $(Y_A m_i)\alpha = \beta_i$  ("diagram (3) is  $\hat{F}$ -semi-initial"). Usually  $i$  ranges over a (proper) index class  $I$ . But the definition of semi-topological functors and the proof of 2.27 of [5] show that it does not matter, if  $I$  belongs to any higher universe. In the present situation, one takes  $I$  to be legitimate with respect to some universe for which  $\hat{A}$  is legitimate.

### 3. The lifting theorem

**THEOREM.** *Let  $T : A \rightarrow X$  be a semi-topological functor. Then  $A$  is total, if  $X$  is total.*

*Proof.* Let the  $T$ -semi-initial diagram (2) be given. In order to prove  $\hat{F}$ -semi-initiality of diagram (3) let  $\zeta$  and  $\beta_i$  be as above. For each  $C \in \text{Ob } A$  one then obtains the commutative diagram (4):

$$(4) \quad \begin{array}{ccccc} & & X(TC, TA) & \xleftarrow{(\varphi^A)(TC)} & A(FTC, A) \\ & \nearrow^{X(TC, e)} & & & \searrow^{A(FTC, m_i)} \\ & & X(TC, TB_i) & \xleftarrow{(\varphi_{B_i})(TC)} & A(FTC, B_i) \\ & \nearrow^{X(TC, X)} & & & \searrow^{A(\varepsilon_C, B_i)} \\ & & X(TC, X) & \xrightarrow{X(TC, x_i)} & X(TC, TB_i) \\ & \nearrow^{\zeta TC} & & & \nearrow^{T_{C, B_i}} \\ HC & \xrightarrow{H \in C} & HFC & & A(C, B_i) \\ & \searrow^{\beta_i C} & & & \end{array}$$

(\*)

Here  $\varepsilon$  denotes the co-unit of the adjoint pair  $(F, T)$  such that the triangle (\*) commutes for  $T_{C, B_i}$  being the respective restriction of  $T$ .

For each  $s \in HC$  one now has the commutative diagram (5):

(5)

$$\begin{array}{ccccc}
 & & & TA & \\
 & & & \nearrow e & \searrow Tm_i \\
 & & X & \xrightarrow{x_i} & TB_i \\
 & \nearrow (\zeta TC)(H \in C)(s) & & & \\
 TC & & & & \\
 & \searrow T([\beta_i C](s)) & & & 
 \end{array}$$

From the  $T$ -semi-initiality of (2) one therefore obtains a morphism  $(\alpha C)(s) : C \rightarrow A$  with  $T([\alpha C](s)) = e([\zeta TC)(H \in C)(s)]$ . It is easily checked that, in this way, a natural transformation  $\alpha : H \rightarrow Y_A^A$  satisfying the needed equations is defined.

COROLLARY 1. Any monadic category over the category of sets is total.

COROLLARY 2. A full reflective subcategory of a total category is total.

#### 4. Final remark

One is not able to prove a corresponding result as in the theorem for arbitrary monadic (instead of semi-topological) functors as Rattray [4] did in the case of compactness. To see this consider again the category of graphs over which Adámek [1] has constructed his non-total but monadic category: by the theorem, his base category, being semi-topological over sets, is total.

#### References

[1] Jiří Adámek, "Colimits of algebras revisited", *Bull. Austral. Math. Soc.* 17 (1977), 433-450.  
 [2] R. Börger, W. Tholen, M.B. Wischnewsky, H. Wolff, "Compactness and hypercompleteness", preprint.

- [3] John R. Isbell, "Small subcategories and completeness", *Math. Systems Theory* 2 (1968), 27-50.
- [4] B.A. Rattray, "Adjoint functors from categories of algebras", *Comm. Algebra* 3 (1975), 563-569.
- [5] R. Street, W. Tholen, M.B. Wischnewsky, H. Wolff, "Semitopological functors III: lifting of monads and adjoint functors", *J. Pure Appl. Algebra* (to appear).
- [6] Ross Street and Robert Walters, "Yoneda structures on 2-categories", *J. Algebra* 50 (1978), 350-379.
- [7] Walter Tholen, "Semi-topological functors I", *J. Pure Appl. Algebra* 15 (1979), 53-73.

Fernuniversität,  
Postfach 940,  
D-5800 Hagen,  
Federal Republic of Germany.