PRINCIPAL INDECOMPOSABLE MODULES FOR SOME THREE-DIMENSIONAL SPECIAL LINEAR GROUPS

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Let \( k \) be a finite field of characteristic 2, and let \( G \) be the three dimensional special linear group over \( k \). The principal indecomposable modules of \( G \) over \( k \) are constructed from tensor products of the irreducible modules, and formulae for their dimensions are given.

Let \( G \) be a finite group and \( k \) a field, and let \( kG \) be the group algebra. The right regular module, also denoted by \( kG \), has a finite number of isomorphism types of indecomposable direct summands \( P_1, \ldots, P_n \). These are the Principal Indecomposable Modules: they are precisely the indecomposable projective \( kG \)-modules and the projective covers of the irreducible \( kG \)-modules.

The aim of this paper is to construct the principal indecomposables over a field of characteristic 2 for the 3-dimensional special linear groups \( SL(3, 2^n) \). This is a natural sequel to work of Alperin [2] on the principal indecomposables for \( SL(2, 2^n) \), and uses similar methods. The idea is to start with the Steinberg module \( S \). Since \( S \) is projective, so is \( S \otimes U \) for any irreducible \( U \) (indeed for any module \( U \) [5, Example 62.2]) and each principal indecomposable occurs as a direct summand of one of these [1]. Finding these direct summands is reduced to counting

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how many times $S$ appears in the tensor product of two irreducibles. In Theorem 1 the principal indecomposables are displayed as tensor products or differences of tensor products of certain modules. The dimensions of the principal indecomposables can then be calculated and are given in Theorem 2.

This paper is based on the author's Oxford University D.Phil Thesis (1979).

1. Preliminaries

Assume $k$ is a splitting field for $G$ and let $V_1, \ldots, V_r$ be all the different isomorphism types of irreducible $kG$-modules. The principal indecomposables are in one-to-one correspondence $P_i \leftrightarrow V_i$ with the irreducibles and there is the decomposition

$$kG \simeq \sum_{i=1}^{r \Theta} (\dim V_i)P_i$$

(where for $n \in \mathbb{Z}$, $nM$ denotes $n$ copies of the module $M$). $P_i$ is called the projective cover of $V_i$ and has $V_i$ both as its unique bottom factor (minimal submodule) and unique top factor (irreducible quotient) [5, §§54, 58]. We also denote the projective cover of an irreducible $U$ by $P(U)$.

For two $kG$-modules $V$ and $W$ let $(V, W)$ be the dimension of the $k$-vector space $\text{Hom}_{kG}(V, W)$. By Schur's lemma

$$\langle P_i, V_j \rangle = \langle V_i, V_j \rangle = \delta_{ij}.$$ 

The next two easy lemmas give a method for finding the direct summands of a projective module.

**Lemma 1.1.** If $k$ is a splitting field for $G$ and $V_1, \ldots, V_r$ are the non-isomorphic irreducible $kG$-modules then for any projective $kG$-module $P$,

$$P \simeq \sum_{i=1}^{r \Theta} (P, V_i)P(V_i).$$
Modules for special linear groups

Proof. Since any projective module is a direct sum of principal indecomposables, this is immediate from the remarks above. //

Also by general properties of projective modules, for any module $W$, $(P_i, W)$ is the number of composition factors of $W$ isomorphic to $V_i$.

For a module $M$, $M^*$ denotes the dual module $\text{Hom}_K(M, 1)$ where 1 is the trivial module.

**LEMMA 1.2.** For any three $kG$-modules, $A$, $B$ and $C$,

$$(A \otimes B, C) = (A, B^* \otimes C).$$

Proof. We have the usual isomorphism of vector spaces

$$(*) \quad \iota : \text{Hom}_K(A \otimes B, C) \rightarrow \text{Hom}_K(A, \text{Hom}_K(B, C)).$$

$\text{Hom}_K(B, C)$ is made into a $kG$-module by defining

$$(fg)(b) = f(bg^{-1})g \quad \text{for } f \in \text{Hom}_K(B, C), \ b \in B, \ g \in G.$$ If both sides of (*) are made into $kG$-modules in the same way, then $\iota$ is a $kG$-isomorphism. Since $\text{Hom}_K(B, C) \cong B^* \otimes C$ the result follows. //

2. The irreducible modules

Henceforth $k$ is the field of $2^n$ elements and $G$ is $\text{SL}(3, k)$. Let $V_0$ be the standard 3-dimensional module for $G$; that is, the row vectors of length 3. If $\sigma$ is the Galois automorphism of $k$, then $\sigma$ acts on $g \in G$ by acting on its entries. $V_i$, the $i$th Galois conjugate of $V_0$, is defined as follows: $V_i$ consists of the row vectors of length 3 and for $v \in V_i$, $g \in G$, $v \cdot g = v(g^i)$

taking the usual $V_0$ action on the right. This gives $n - 1$ new modules $V_1', \ldots, V_{n-1}'$. Let $N = \{0, 1, \ldots, n-1\}$. Where necessary regard $N$ as $\mathbb{Z}/n\mathbb{Z}$ so that for $i \in N$, $i + 1$ always makes sense. For a subset $K$ of $N$ we let $\overline{K}$ denote the complement of $K$ in $N$. For $I \subseteq N$ we define
We also have the duals \( V^* \) and \( V^*_I \approx \bigoplus_{i \in I} V_i^* \).

Since \( V_0^* \) is afforded by the column vectors \( a \) of length 3 on which \( g \in G \) acts by \( a + g^{-1}a \), \( V_0^* \otimes V_0 \) is afforded by the \( 3 \times 3 \) matrices \( X \) on which \( g \) acts by \( X + g^{-1}Xg \), and so \( V_0^* \otimes V_0 \cong T \otimes W_0 \) where \( T \) consists of the scalar matrices and \( W_0 \) those of trace zero. \( T \) is the trivial module while \( W_0 \) is easily seen to be irreducible of dimension 8 and self-dual. Let \( W_i \) be the \( i \)th Galois conjugate of \( W_0 \) and let \( W_i^* \) be the \( i \)th tensor factor of the irreducible. Note that since all the absolute irreducibles can be written over \( k \), \( k \) is a splitting field so Lemma 1.1 applies.

We shall also use a more convenient notation. For an ordered triple \((I, J, K)\) of pairwise disjoint subsets of \( N \), let
\[
V(I, J, K) = V_I^* \otimes V_J^* \otimes W_K^* .
\]
Then the irreducibles are indexed by all such triples. Let \( \mathcal{P}(I, J, K) \) be the projective cover of \( V(I, J, K) \). The module \( W_N^* \) of dimension \( 8^N \) is the Steinberg module [5] and henceforth we denote it by \( S \). It is the reduction modulo 2 of an ordinary irreducible and so by a theorem of Brauer and Nesbitt [3, 86.3] it is a principal indecomposable and is projective.

We now obtain some information about the tensor product of two irreducibles.
LEMMA 2.1. $V_0 \otimes V_0$ has a unique composition series

$$V_0 \otimes V_0 \supseteq X \supseteq Y \supseteq \{0\}$$

where $\left( V_0 \otimes V_0 \right) / X \cong V_0^* \cong Y$ and $X / Y \cong V_1$.

Proof. $V_0 \otimes V_0$ is afforded by the $3 \times 3$ matrices $M$ over $k$, and $g \in G$ acts by $M \rightarrow g^t Mg$, where $g^t$ is the transpose of $g$. $X$ consists of the symmetric matrices and $Y$ those with zeros on the leading diagonal.

LEMMA 2.2. Assume $n \geq 2$. $V_0 \otimes W_0$ is indecomposable with composition factors (counting multiplicities):

$$V_0, V_1^*, V_0, V_0 \otimes V_1, V_1^*, V_0.$$

$V_0$ is the unique top and the unique bottom factor. $V_0 \otimes W_0$ has a submodule and a quotient isomorphic to $V_0^* \otimes V_0^*$.

Proof. $V_0 \otimes V_0$ has composition factors $V_1^*, V_1, V_1^*$ so $V_0^* \otimes V_0^*$ has composition factors $V_0, V_0^*, V_0^*$, and $V_0^* \otimes V_0 \otimes V_0$ has composition factors

$$V_0, V_1^*, V_0, V_0 \otimes V_1, V_1^*, V_0^*, V_0, V_1^*, V_0.$$

But $V_0^* \otimes V_0 \otimes V_0 \cong V_0 \oplus \left( V_0 \otimes W_0 \right)$ so the composition factors are as claimed. To find the top factor it is easily checked using Lemma 1.2, that $\left( V_0 \otimes W_0, U \right)$ is zero for each composition factor $U$, except that it is 1 for $U = V_0$. Similarly check $\left( U, V_0 \otimes W_0 \right)$ to find the bottom factor.

Now $V_0 \otimes V_0$ has submodules $X$ and $Y$ with $X \supset Y$ and

$$\left( V_0 \otimes V_0 \right) / X \cong V_0^* \cong Y.$$

So $V_0 \otimes V_0 \otimes V_0^*$ has submodules $X \otimes V_0^*$ and $Y \otimes V_0^*$ such that

$$\left( V_0 \otimes V_0 \otimes V_0^* \right) / \left( X \otimes V_0^* \right) \cong V_0^* \otimes V_0^* \cong Y \otimes V_0^*.$$

Also

$$V_0 \otimes V_0 \otimes V_0^* = C \oplus D.$$
where
\[ C \simeq V_0, \quad D \simeq V_0 \otimes W_0. \]
Let \( \pi \) be the projection onto \( D \). Since \( Y \otimes V_0^* \) is indecomposable, \( C = \ker \pi \) is not a submodule of \( Y \otimes V_0^* \), and so \( \pi \) maps \( Y \otimes V_0^* \) monomorphically into \( V_0 \otimes W_0 \) and \( V_0 \otimes W_0 \) has a submodule isomorphic to \( V_0^* \otimes V_0^* \).

Again
\[ (C \oplus D)/(X \otimes V_0^*) \simeq V_0^* \otimes V_0^* \]
is indecomposable. Since \( C \) is irreducible, \( (C+X \otimes V_0^*)/(X \otimes V_0^*) \) is either 0 or irreducible, and so in either case must be contained in \( (D+X \otimes V_0^*)/(X \otimes V_0^*) \). Hence
\[ V_0^* \otimes V_0^* \simeq (D+X \otimes V_0^*)/(X \otimes V_0^*) \simeq D/(D \cap (X \otimes V_0^*)) \]
and \( V_0^* \otimes V_0^* \) is a quotient of \( D \simeq V_0 \otimes W_0 \). \( // \)

**COROLLARY 2.3.** For \( n \geq 2 \), \( \{V_0 \otimes W_0, V_0 \otimes W_0\} \geq 3 \).

Proof. Let \( \alpha, \beta \) be the endomorphisms mapping \( V_0 \otimes W_0 \) to its submodules \( V_0^* \) and \( V_0^* \otimes W_0^* \) respectively, which exist by Lemma 2.2. Then \( \alpha, \beta \) and the identity are linearly independent. \( // \)

**LEMMA 2.4.** Assume \( n \geq 2 \). Then
\[ W_0 \otimes W_0 \simeq 2W_0 \oplus A_0 \]
where \( A_0 \) is a self-dual indecomposable module with composition factors (counting multiplicities)
\[ 1, 1, 1, 1, V_0 \otimes V_1, V_0 \otimes V_1, V_0^* \otimes V_1^*, V_0 \otimes V_1, W_1. \]
\( 1 \) is the unique top factor and the unique bottom factor of \( A_0 \).

Proof. The composition factors of \( V_0 \otimes V_0 \otimes V_0^* \otimes V_0^* \) are found from those of \( V_0 \otimes V_0 \) and \( V_0^* \otimes V_0^* \). But this module is \( 1 \oplus 2W_0 \oplus W_0 \otimes W_0 \),
giving the composition factors of $W_0 \otimes W_0$. Now, by Lemma 1.2,

$$
(W_0 \otimes W_0, 1 \oplus W_0) = (W_0 \otimes W_0, V_0 \otimes V_0^*)
\quad = (W_0 \otimes V_0, W_0 \otimes V_0^*) \geq 3
$$

by Corollary 2.3. Since $(W_0 \otimes W_0, 1) = 1$, $(W_0 \otimes W_0, W_0^*) \geq 2$. Thus $W_0 \otimes W_0$ has a submodule $R$ such that $(W_0 \otimes W_0)/R \simeq 2W_0$. By self-duality $W_0 \otimes W_0$ also has a submodule $Q$ isomorphic to $2W_0$. Since $W_0$ only appears twice as a composition factor of $W_0 \otimes W_0$, it follows that $Q \cap R = \{0\}$, so $W_0 \otimes W_0 \simeq Q \oplus R$ and we have the decomposition claimed.

Clearly $A_0$ is self-dual since $W_0 \otimes W_0$, and $W_0$ are. Finally the top and bottom factors of $A_0$ are found using Lemma 1.2. \hfill //

Lemmas 2.2–2.4 fail for $n = 1$ because the modules given as composition factors are no longer irreducible. However one can immediately deduce the corresponding result.

**Lemma 2.5.** For $n = 1$,

$$
V_0 \otimes W_0 = W_0 \oplus P(V_0), \quad V_0^* \otimes W_0 = W_0 \oplus P(V_0^*)
$$

and

$$
W_0 \otimes W_0 = 3W_0 \oplus P(V_0) \oplus P(V_0^*) \oplus P(1).
$$

**Proof.** From Lemma 2.2 we deduce that the composition factors of $V_0 \otimes W_0$ are

$$
V_0, V_0^*, V_0, 1, W_0, V_0^*.
$$

For $n = 1$, $W_0$ is projective and so $V_0 \otimes W_0 = W_0 \oplus Q$. Using Lemma 1.2, $(Q, V_0) = 1$, while $(Q, U) = 0$ for all other irreducibles. Hence $Q = P(V_0)$. The other equations follow similarly. \hfill //

3. The principal indecomposable modules

In this section we decompose the projective modules $S \otimes U$, for $U$ irreducible, to find the principal indecomposables. By Lemma 1.1,
$S \otimes U \simeq \sum \Phi (S \otimes U, V) P(V)$

summing over the irreducibles $V$. But, by Lemma 1.2,

$\langle S \otimes U, V \rangle = \langle S, U^{\dagger} \otimes V \rangle$.

The next few lemmas are devoted to calculating this.

**DEFINITION 3.1.** Let $U = U_0 \otimes U_1 \otimes \ldots \otimes U_{n-1}$ and $X = X_0 \otimes X_1 \otimes \ldots \otimes X_{n-1}$ be irreducibles where each $U_i$ and $X_i$ is $1$, $V_i$, or $W_i$. Then $Y = Y_0 \otimes Y_1 \otimes \ldots \otimes Y_{n-1}$ is a cross-section of $U \otimes X$ if each $Y_i$ is a composition factor of $U_i \otimes X_i$.

**LEMMA 3.2.** Let $U$ and $X$ be irreducibles and let $Y$ be a cross-section of $U \otimes X$. If $S$ is a composition factor of $Y$ then $Y$ is isomorphic to $S$ or to $V_i \otimes V_i^\dagger$.

Proof. By Lemmas 2.1-2.1, each $Y_i$ is isomorphic to one of $1$, $V_i$, $W_i$, $V_i \otimes V_{i+1}$, $V_i^\dagger \otimes V_{i+1}$, $V_{i+1}$, $W_{i+1}$ or to the dual of one of these. Thus $\dim Y = 2^{2(n-b)}$ and since $Y$ has $n$ tensor factors $a \leq 2(n-b)$. If $Y$ is reducible it has a reducible pair of tensor factors

$V_i \otimes V_i$, $V_i^\dagger \otimes V_i$, $V_i \otimes W_i$, $W_i \otimes W_i$

or the dual of one of these. Call $Y'$ a standard reduction of $Y$ if $Y'$ is obtained from $Y$ by replacing a reducible pair by one of its composition factors. Call it type A if $V_i \otimes V_i$ is replaced by $W_i$.

If $S$ is a composition factor of $Y$, it is obtained from $Y$ by a sequence of standard reductions. The only one which increases the power of 3 in the dimension is type A, which decreases the power of 3 by two. If a reduction increases the power of 3, then the power of 3 is reduced by the same amount. Therefore to obtain $S$, of dimension $3^a$, by standard reductions, we must have $a \geq 2(n-b)$. But then $a = 2(n-b)$ and

$\dim Y = 3^{2(n-b)}$, and there must be $n - b$ reductions of type A. If $b \neq n$ some $Y_i$ has dimension 9, and then so must $Y_{i-1}$ and $Y_{i+1}$ and hence...
all the $Y_i$'s, and so $Y \simeq V_N \otimes V_N^*$. Otherwise all $Y_i$ have dimension 8 and $Y \simeq S$. //

**DEFINITION 3.3.** Let $I$, $J$ and $K$ be subsets of $N$. The ordered triple $(I, J, K)$ is a trio if

1. \{I, J, K\} is a partition of $N$,
2. $|K|$ is even,
3. if $i, j \in I \cup J$ then both belong to $I$ or both belong to $J$ if and only if $|\{k \in K \mid i < k < j\}|$ is even.

Note that (ii) ensures the consistency of (iii) under the obvious circular ordering of $N$. Also given $K$ and $i \in N \setminus K$, $i$ may be assigned either to $I$ or to $J$ and then $I$ and $J$ are determined by (iii), so each $K \neq N$ with $|K|$ even determines a unique pair of trios $(I, J, K)$ and $(J, I, K)$. $(\emptyset, \emptyset, N)$ is a trio if and only if $n$ is even.

**LEMMA 3.4.** Let $U$ and $V(I, J, K)$ be irreducibles such that $U \otimes V(I, J, K)$ has a cross-section $Y$ isomorphic to $S$ or to $V_N \otimes V_N^*$.

(i) If $I \cup J \cup K \neq N$ then $Y \simeq S$ and $U \simeq V(J, I, L)$ where $I \cup J \cup K \cup L = N$.

(ii) Let $I \cup J \cup K = N$. If $Y \simeq S$ then $U \simeq V(J, I, L)$ where $L \subseteq K$; if $Y \simeq V_N \otimes V_N^*$ then $U \simeq V(K, Y, I \cup J \cup 2)$ where \{X, Y, Z\} is a partition of $K$ and $(I \cup X, J \cup Y, Z)$ is a trio.

**Proof.** Let the $i$th tensor factor of $V(I, J, K)$ be $X_i$.

In case (i) we have $Y_i \simeq U_i$ for some $i \notin I \cup J \cup K$ and so for this $i$, $Y_i \simeq W_i$, and consequently $Y \simeq S$. Then

$$U_i \otimes X_i \simeq \text{one of } V_i \otimes V_i^*, W_i \text{ or } W_i \otimes W_i$$

for each $i$, so $U$ has the form claimed.

In case (ii) first suppose $Y \simeq S$. If $Y_i \simeq W_i$ for all $i$, then $U \simeq V(J, I, L)$ with $L \subseteq K$. Otherwise $Y_i \simeq W_{i+1}$ for all $i$, and then both irreducibles must be isomorphic to $S$.
Now suppose $Y \simeq V_i \otimes V^*_i$. Each $Y_i$ is isomorphic to $V_i \otimes V_{i+1}$ or $V^*_i \otimes V^*_{i+1}$ or the dual of one of these. Define

$$E = \{ i \in N \mid Y_i \simeq V^*_i \otimes V^*_{i+1} \} ,$$
$$F = \{ i \in N \mid Y_i \simeq V_i \otimes V_{i+1} \} ,$$

and

$$H = \{ i \in N \mid Y_i \simeq V_i \otimes V^*_{i+1} \text{ or } V^*_i \otimes V_{i+1} \} .$$

Then $(E, F, H)$ is a trio. Also

$$i \in E \text{ implies } U_i \otimes X_i \simeq V_i \otimes W_i ,$$
$$i \in F \text{ implies } U_i \otimes X_i \simeq V^*_i \otimes W_i ,$$
$$i \in H \text{ implies } U_i \otimes X_i \simeq W_i \otimes W_i ,$$

by Lemmas 2.2 and 2.4. Thus $U_i$ is never trivial and $U_i \simeq W_i$ for $i \in I \cup J$.

Define a partition $\{X, Y, Z\}$ of $K$ by

$$V_i \text{ for } i \in X ,$$
$$U_i \simeq V^*_i \text{ for } i \in Y ,$$
$$W_i \text{ for } i \in Z .$$

Then $U \simeq V(X, Y, I \cup J \cup Z)$. Also

$$E = I \cup X , \quad F = J \cup Y , \quad H = Z .$$

So $(I \cup X, J \cup Y, Z)$ is a trio, and the proof is complete. //

**Proposition 3.5.** Let $U$ and $V(I, J, K)$ be irreducibles and suppose the dimension

$$[S, U \otimes V(I, J, K)] \neq 0 .$$

(i) If $I \cup J \cup K \neq N$ then $U \simeq V(J, I, L)$ where $I \cup J \cup K \cup L = N$ and the dimension is $2^{[K \cap L]}$.

(ii) If $I \cup J \cup K = N$ then either $U \simeq V(J, I, L)$ for some $L \subseteq K$ and the dimension is $2^{[L]}$, or $U \simeq V(X, Y, I \cup J \cup Z)$ where $\{X, Y, Z\}$
is a partition of $K$ and $(I \cup X, J \cup Y, Z)$ is a trio, and the dimension is $2^{|Z|}$, except that

$$(S, S \otimes S) = \begin{cases} 2^n + 1 & \text{if } n \text{ is odd}, \\ 3.2^n + 1 & \text{if } n \text{ is even}. \end{cases}$$

Proof. This dimension is the number of times that $S$ appears as a composition factor of $U \otimes V(I, J, K)$. Since each composition factor occurs in some cross-section, by Lemma 3.2 this is the number of cross-sections $Y$ isomorphic to $S$ or to $V_N \otimes V_N^*$. By Lemma 3.4, $U$ has one of the stated forms.

In case (i), $Y \cong S$ by Lemma 3.4 (i). There is a unique choice for each $Y_i$ except for $i \in K \cap L$ when there are two, since $W_i \otimes W_i$ has two composition factors $W_i$ and so the dimension is $2^{|K \cap L|}$.

In case (ii) first note that by Lemma 3.4 (ii) it is impossible to both choose $Y \cong S$ and $Y \cong V_N \otimes V_N^*$ except when $I \cup J = \emptyset$, both irreducibles are $S$, and $(\emptyset, \emptyset, N)$ is a trio so $n$ is even. There are $2^n$ cross-sections $Y$ of $S \otimes S$ with each $Y_i \cong W_i$, and one with each $Y_i \cong W_{i+1}$. When $n$ is even there are also $2.2^n$ with each $Y_i \cong V_i \otimes V_{i+1}$ or $V_i^* \otimes V_{i+1}^*$. This gives the stated dimensions for $(S, S \otimes S)$.

Otherwise at least one of $U$ and $V(I, J, K)$ is not isomorphic to $S$. If $U \cong V(J, I, L)$ with $L \subseteq K$ then (by Lemma 3.4 (ii)) $Y_i$ must be chosen isomorphic to $W_i$ for all $i$. For $i \in L$ there are 2 choices for $Y_i$ (as a composition factor of $U_i \otimes W_i \cong W_i \otimes W_i$), and otherwise only one choice. This gives $2^{|L|}$ choices for $Y$.

If $U \cong V(X, Y, I \cup J \cup Z)$ then $Y$ must be chosen isomorphic to $V_N \otimes V_N^*$. For $i \in X, Y, I$ or $J$ there is a unique choice for $Y_i$, isomorphic to one of $V_{i+1}^* \otimes V_i$ and $V_i \otimes V_{i+1}^*$. For $i \in Z, Y$ must be chosen compatibly as $V_i \otimes V_{i+1}$ or $V_i^* \otimes V_{i+1}^*$ from $W_i \otimes W_i$. This gives
choices for $Y$. The proof is complete. //

We can now decompose the modules $S \otimes U$. For a subset $K$ of $N$ let $\overline{K}$ be the complement of $K$ in $N$. Then from Lemma 2.4,

\[(3.6) \quad S \otimes W_L = \sum_{K \cup L = N} 2^{|K \cap L|} W_K \otimes A_{\overline{K}} \]

where $A_R = \prod_{r \in R} A_r$. On the other hand from Lemmas 1.1 and 1.2 and Proposition 3.5, for $L \neq N$ we have

\[(3.7) \quad S \otimes W_L = \sum_{K \cup L = N} 2^{|K \cap L|} P(W_K) .\]

So, for $K \neq \emptyset$,

\[(3.8) \quad P(W_R) = W_K \otimes A_{\overline{K}} .\]

NOTATION 3.9. Let $B_R$ denote $V_R \otimes W_R$, for $R \subseteq N$.

PROPOSITION 3.10. Let $I$, $J$ and $K$ be pairwise disjoint subsets of $N$ with $K \neq \emptyset$. Let $L = I \cup J \cup K$. Then

\[P(I, J, K) = B_I \otimes B_J \otimes W_K \otimes A_L .\]

Proof. Since

\[B_I \otimes B_J \otimes W_K \otimes A_L = V_I \otimes V_J \otimes P(W_{I \cup J \cup K}) \]

it is projective. We show that $V(I, J, K)$ is the only top factor. Now because $W_{I \cup J \cup K} \otimes A_L$ is a direct summand of $S \otimes W_L$ it follows that $B_I \otimes B_J \otimes W_K \otimes A_L$ is a direct summand of $S \otimes V(I, J, L)$. Consider, for an irreducible $U$,

\[(S \otimes V(I, J, L), U) = (S, V(J, I, L) \otimes U) .\]

Since $K \neq \emptyset$, $I \cup J \cup L \neq N$ so by Proposition 3.5 if this space is non-zero then $U \sim V(I, J, K \cup Z)$ with $Z \subseteq L$. Hence these $U$'s are the only candidates for top factors of $B_I \otimes B_J \otimes W_K \otimes A_L$. However

\[\{B_I \otimes B_J \otimes W_K \otimes A_L, V(I, J, K \cup Z)\} = \{W_L \otimes A_L, V_I \otimes V_J \otimes V(I, J, K \cup Z)\} ,\]

and
Which is a direct sum of irreducibles. Since, by (3.8),
\[ \mathcal{W}_L^* \otimes A_L = P\left(\mathcal{W}_L^*\right) = P\left(\mathcal{W}_{I\cup J\cup K}\right) \]
the dimension is non-zero only if \( Z = \emptyset \) and then it is one. This proves the proposition. //

An expression for \( P(1) \) comes from decomposing \( S \otimes S \). Let \( T \) be the set of trios not including \((\emptyset, \emptyset, N)\). Using \( \subset \) to mean proper inclusion, from Lemma 1.1 and Proposition 3.5 we have
\[ S \otimes S = \sum_{L \subset N} 2^{\left|L\right|} \mathcal{P}_L^* \otimes \sum_{(I, J, K) \in T} 2^{\left|K\right|} \mathcal{P}(I, J, K) \otimes m'S \]
where
\[ m' = \begin{cases} 2^n + 1 & \text{if } n \text{ is odd, and} \\ 3.2^n + 1 & \text{if } n \text{ is even.} \end{cases} \]

From (3.6) on the other hand
\[ S \otimes S = \sum_{K \subset N} 2^{\left|K\right|} \mathcal{W}_K^* \otimes A_{\neg K} \]
\[ = A_{N} \oplus \sum_{\emptyset \neq K \subset N} 2^{\left|K\right|} \mathcal{W}_K^* \otimes A_{\neg K} \oplus 2^nS \text{.} \]

Using (3.8) then we have
\[ A_N = P(1) \oplus \sum_{(I, J, K) \in T} 2^{\left|K\right|} \mathcal{P}(I, J, K) \oplus \left(m'-2^n\right)S \text{.} \]

Recall that \((\emptyset, \emptyset, N)\) is a trio if and only if \( n \) is even. Hence we may rewrite this as
\[ (3.11) \quad P(1) = A_N - \sum_{(I, J, K) \in T} 2^{\left|K\right|} \mathcal{P}(I, J, K) - mS \text{,} \]
where
\[ m = \begin{cases} 1 & \text{if } n \text{ is odd, and} \\ 2^n + 1 & \text{if } n \text{ is even,} \end{cases} \]
and the sum is now over all trios \((I, J, K)\).

**PROPOSITION 3.12.** If \(I \cup J \neq \emptyset\) and \(K = I \cup J\) then

\[
B_I \otimes B_J^* \otimes A_K = P(I, J, \emptyset) \oplus \sum_Z 2^{|Z|} P(X, Y, I \cup J \cup Z)
\]

where the sum is over partitions \((X, Y, Z)\) of \(K\) such that \((I \cup X, J \cup Y, Z)\) is a trio.

**Proof.** Since

\[
\left[B_I \otimes B_J^* \otimes W_K, U\right] = \left(S, V(J, I, K) \otimes U\right),
\]

using Proposition 3.5 and Lemma 1.1 as before in the proof of Proposition 3.10, we have

\[
B_I \otimes B_J^* \otimes W_K \otimes W_K = \sum_{L \subseteq K} 2^{|L|} P(I, J, L) \oplus \sum_Z 2^{|Z|} P(X, Y, I \cup J \cup Z)
\]

where the second sum is over the partitions stated. On the other hand from Lemma 2.5 we have

\[
B_I \otimes B_J^* \otimes W_L \otimes W_L = \sum_{L \subseteq K} 2^{|L|} B_I \otimes B_J^* \otimes W_L \otimes A_{K-L}.
\]

Since by Proposition 3.10,

\[
P(I, J, L) = B_I \otimes B_J^* \otimes W_L \otimes A_{K-L}
\]

for \(L \neq \emptyset\), the result follows. //

In stating Theorem 1 we use the notation that \(A = B - C\) means \(B \simeq A \oplus C\).

**THEOREM 1.** For an irreducible \(V(I, J, K)\) let \(L = I \cup J \cup K\). Let \(A_{R, I}, B_{R, J}\) be the modules defined in (3.6) and Notation 3.9 respectively.

(i) If \(K \neq \emptyset\) then

\[
P(I, J, K) = B_I \otimes B_J^* \otimes W_K \otimes A_{L}.
\]

(ii) If \(K = \emptyset, I \cup J \neq \emptyset\) then

\[
P(I, J, \emptyset) = B_I \otimes B_J^* \otimes A_L - \sum_Z 2^{|Z|} B_X \otimes B_Y^* \otimes W_{I \cup J \cup Z},
\]

where the sum runs over partitions \((X, Y, Z)\) of \(L\) such that
(i) $\{I \cup X, J \cup Y, Z\}$ is a trio.

(iii) $P(1) = A_N - \sum_{\Theta} 2^{\mid Z \mid} B_X \otimes B_Y \otimes B_Z - dS$, where

$$d = \begin{cases} -1 & \text{if } n \text{ is odd, and} \\ 2^n - 1 & \text{if } n \text{ is even,} \end{cases}$$

and the sum runs over all trios $(X, Y, Z)$.

Proof. (i) and (ii) are from Propositions 3.10 and 3.12. (iii) follows from (3.11) and Proposition 3.10, with the observation that by (ii), $P(N, \emptyset, \emptyset) = B_N - S$, and similarly for its dual. //

To give the dimensions of the principal indecomposables we need the following notion.

**Definition.** Let $I$ and $J$ be disjoint subsets of $N$ and let the elements of $I \cup J$ be $i_1 < i_2 < \ldots < i_r$. Let

$$k_j = \mid \{k \in N \mid i_j < k < i_{j+1}\} \mid$$

under the circular ordering of $N$, and for $j = 1, \ldots, r$ let

$$e_j = \begin{cases} 1 & \text{if both } i_j \text{ and } i_{j+1} \text{ belong to } I \text{ or both belong to } J, \\ -1 & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, r$. Then the type of $(I, J)$ is the sequence $(e_1, k_1, e_2, k_2, \ldots, e_r, k_r)$.

**Theorem 2.** (i) If $K \neq \emptyset$ then

$$\dim P(I, J, K) = 2^n \cdot 3^{n-\mid K \mid} \cdot 2^n \cdot \mid I \cup J \cup K \mid.$$

(ii) If $I \cup J \neq \emptyset$ let the type of $(I, J)$ be $(e_1, k_1, \ldots, e_r, k_r)$. Then

$$\dim P(I, J, \emptyset) = 2^n \cdot 3^{n-\mid I \cup J \mid} \cdot \prod_{j=1}^r \left( 5^{k_j} + e_j \right).$$

(iii) $\dim P(1) = 2^n (6^n - 5^n)$.

Proof. (i) is immediate from Theorem 1 (i). For (ii) let the
elements of \( I \cup J \) be \( i_1 < \ldots < i_r \) and let
\[
K_j = \{ k \in N \mid i_j < k < i_{j+1} \}
\]
so that \( k_j = |K_j| \). From Theorem 1 (iii) we have to sum the dimensions of the summands \( B_X \otimes B_Y^* \otimes W_{I \cup J} \otimes Z \) of \( B_I \otimes B_J^* \otimes A_L \). Suppose we have subsets \( L_j \subseteq K_j, \ j = 1, \ldots, r \), such that if \( Z = U L_j \) then there is a trio \((I \cup X, J \cup Y, Z)\). By the definition of a trio \(|L_j|\) must be even if \( e_j = 1 \) and odd if \( e_j = -1 \), and provided these conditions are fulfilled a trio exists. Now let \( \ell_j = |L_j| \). The corresponding principal indecomposable \( B_X \otimes B_Y^* \otimes W_{I \cup J} \otimes Z \) has dimension
\[
g^n \cdot 3^{k_1 - 1} \cdot 2^{k_2 - 2} \cdot \ldots \cdot 3^{k_r - r}
\]
and occurs \( 2^{\ell_j} = 2^{\ell_1 + \ell_2 + \ldots + \ell_r} \) times, so the total dimension of these summands is
\[
g^n \cdot 3^{\ell_1 - 1} \cdot 2^{\ell_2 - 1} \cdot \ldots \cdot 3^{\ell_r - r}.
\]
The sum of the dimensions of all the summands corresponding to trios \((I \cup X, J \cup Y, Z)\) is
\[
g^n \left( \sum_{\ell_1}^{k_1} \cdot 3^{\ell_1 - 1} \cdot 2^{\ell_2 - 1} \cdot \ldots \cdot 3^{\ell_r - r} \right),
\]
where the sum is over \( \ell_j \) odd or \( \ell_j \) even according as \( e_j \) is \(-1\) or \(1\). By the binomial theorem this is
\[
g^n \left( \frac{k_1}{6} + e_1 \right) / 2 \cdot \ldots \cdot \left( \frac{k_r}{6} + e_r \right) / 2.
\]
Since \( |I \cup J| = r \),
\[
\dim B_I \otimes B_J^* \otimes A_L = g^n \cdot 3^r \cdot 6^{n-r}
\]
giving the result.

For (iii) note that \( \dim A_N = g^n \cdot 6^n \) and
For each $Z \subset N$ with $|Z|$ even there are two trios so
\[
\dim P(1) = 8^n \left(6^{n-2} \cdot \sum_{Z \in \mathcal{F}} 2^{|Z|} \cdot 3^{n-|Z|} \cdot 2^{|Z|} \right)
\]
\[
= 8^n \left(6^{n+1-2} \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot 2 \cdot 3^{n-k} \right)
\]
\[
= 8^n \left(6^{n+1-2} \cdot \left(\frac{5^n+1}{2}\right)\right)
\]
\[
= 8^n \left(6^{n-5^n}\right). \quad //
\]

References


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