BULL. AUSTRAL. MATH. SOC. 45D05
VOL. 18 (1978), 321-334.

On Stieltjes-Volterra integral equations

S.G. Pandit

A Stieltjes-Volterra integral equation system

$$x(t) = f(t) + \int_{t_0}^{t} K\big(t, s, x(s)\big)du(s)$$

is firstly considered. Pointwise estimates and boundedness of its solutions are obtained under various conditions on the function K . To do this, the well-known Gronwall-Bellman integral inequality is generalized. For a particular choice of u , it is shown that the integral equation reduces to a difference equation. The problem of existence (and non-existence), uniqueness (and non-uniqueness) of the difference equation is discussed. Gronwall-Bellman inequality is further generalized to n linear terms and is subsequently applied to obtain sufficient conditions in order that a certain stability of the unperturbed Volterra system

$$x(t) = f(t) + \int_{t_0}^{t} a(t, s)x(s)ds$$

implies the corresponding local stability of the (discontinuously) perturbed system

$$x(t) = f(t) + \int_{t_0}^{t} a(t, s)x(s)ds + \int_{t_0}^{t} b(t, s)F\big(s, x(s)\big)du(s) .$$

Received 11 January 1978.

https://doi.org/10.1017/S0004972700008200 Published online by Cambridge University Press

1.

In many problems of physics and engineering (optimal control theory in particular), one can not expect perturbations to be well-behaved and it is therefore important to consider the cases when the perturbations are impulsive [3, 7]. Such systems would be described by differential equations containing measures, which are equivalent to Volterra integral equations with perturbations involving Lebesgue-Stieltjes integrals. The purpose of this paper is to obtain pointwise estimates and boundedness of solutions of Stieltjes-Volterra integral equations and to study a stability property of Volterra integral equations with discontinuous perturbations. The tools used for the purpose are the generalized Gronwall-Bellman inequalities involving Lebesgue-Stieltjes integrals.

Let $J = [t_0, \infty)$, $t_0 \geq 0$, and $BV(J, R^n) = BV(J)$ denote the space of all functions of bounded variation which are defined on J and taking values in R^n . The norm of $x = x(t) \in BV(J)$ is defined by $\|x\| = V(x, J) + |x(t_0)|$ where $V(x, J)$ is the total variation of x on J and $|\cdot|$ is any norm in R^n . Let u be a scalar function which is right-continuous and of bounded variation on every compact subinterval of J . We consider the following Volterra integral equations

$$(1.1)\quad x(t) = f(t) + \int_{t_0}^{t} K(t, s, x(s))du(s) ,$$

$$(1.2)\quad x(t) = f(t) + \int_{t_0}^{t} a(t, s)x(s)ds ,$$

$$(1.3)\quad x(t) = f(t) + \int_{t_0}^{t} a(t, s)x(s)ds + \int_{t_0}^{t} b(t, s)F(s, x(s))du(s) ,$$

where $x, f \in BV(J)$, $K(t, s, \phi) : J \times J \times R^n \to R^n$, $F : J \times R^n \to R^n$, and $a(t, s), b(t, s)$ are $n \times n$ matrices defined for $t_0 \leq s \leq t < \infty$.

A special case of (1.1) is considered in [2] where the integrals are in the Riemann-Stieltjes sense. (1.2) and (1.3) have been dealt with in [9]. [6, 10] also treat these equations when $a(t, s) = b(t, s)$ and u is absolutely continuous on J.

In Section 2, we generalize the Gronwall-Bellman integral inequality and apply it to obtain pointwise estimates and boundedness of solutions of (1.1). Section 3 deals with a difference equation arising from (1.1) for a particular choice of u. The existence (or non-existence) and uniqueness (or non-uniqueness) of solutions of the difference equation are discussed. Finally, in Section 4, we further generalize the Gronwall-Bellman inequality and study a stability property of (1.3) in the light of (1.2). In the following discussion, it is assumed that (1.1)-(1.3) possess solutions on J.

2.

Let $t_1 < t_2 < \ldots$ denote the discontinuities of u on J (note that u is of bounded variation). We assume that the discontinuities are isolated. u may be decomposed as $u = u_1 + u_2$ where u_1 is an absolutely continuous function of bounded variation on J and u_2 is a sum of jump functions, the jumps being those of u. It follows that u' exists (and is equal to u_1' almost everywhere) on J. Let $\lambda_k = u(t_k) - u(t_k-)$ denote the jump of u at $t = t_k$, $k = 1, 2, \ldots$. In the following all functions of one variable are assumed to be defined, real-valued, and measurable on J. Such a function w is said to be *locally du-integrable on* J if, for each $t \in J$, the Lebesgue-Stieltjes integral $\int_{t_0}^{t} w(s)du(s)$ is finite.

THEOREM 2.1. *Suppose that*

$$x(t) \le f(t) + g(t) \int_{t_0}^{t} h(s)x(s)du(s) , \quad t \in J , \tag{2.1}$$

where

(i) x, f, g, *and* h *are non-negative and locally du-integrable on* J, *with* f *non-decreasing and* $g \ge 1$,

(ii) u *is such that* $u_1' \ge 0$ *on* J *and*

https://doi.org/10.1017/S0004972700008200 Published online by Cambridge University Press

(2.2) $$\lambda_k g(t_k) h(t_k) < 1 , \quad k = 1, 2, \ldots ,$$

(iii) the series

(2.3) $$\sum_{k=1}^{\infty} \lambda_k g(t_k) h(t_k)$$

converges absolutely.

Then

(2.4) $$x(t) \le P^{-1} f(t) g(t) \exp\left\{\int_{t_0}^{t} g(s) h(s) u_1'(s) ds\right\} , \quad t \in J ,$$

where

$$P = \prod_{k=1}^{\infty} \{1-\lambda_k g(t_k) h(t_k)\} .$$

Proof. Since f is non-decreasing and $g \ge 1$ on J, (2.1) may be written as

(2.5) $$\frac{x(t)}{f(t)} \le g(t)\left[1 + \int_{t_0}^{t} h(s) \frac{x(s)}{f(s)} du(s)\right] , \quad t \in J .$$

Denote the bracket on the right side of (2.5) by $r(t)$. Firstly suppose $t_0 \le t < t_1$. Since u is differentiable on $[t_0, t_1)$, by the classical Gronwall-Bellman inequality [1, p. 58], we obtain

(2.6) $$r(t) \le \exp\left\{\int_{t_0}^{t} g(s) h(s) u_1'(s) ds\right\} .$$

At $t = t_1$ we have

$$r(t_1) = r(t_1 - \varepsilon) + \int_{t_1-\varepsilon}^{t_1} h(s) \frac{x(s)}{f(s)} du(s) ,$$

where $\varepsilon > 0$. Taking the limit as $\varepsilon \to 0_+$ and using (2.6), we get

$$r(t_1) \le \exp\left\{\int_{t_0}^{t_1} g(s) h(s) u_1'(s) ds\right\} + \lambda_1 g(t_1) h(t_1) r(t_1) ,$$

https://doi.org/10.1017/S0004972700008200 Published online by Cambridge University Press

which, in view of (2.2), yields

$$r(t_1) \le P_1^{-1} \exp\left\{\int_{t_0}^{t_1} g(s)h(s)u_1'(s)ds\right\} ,$$

where

$$P_k = \prod_{n=1}^{k} \{1-\lambda_n g(t_n)h(t_n)\} , \quad k = 1, 2, \ldots .$$

By mathematical induction, it follows that

$$(2.7) \qquad r(t_m) \le P_m^{-1} \exp\left\{\int_{t_0}^{t_m} g(s)h(s)u_1'(s)ds\right\} , \quad m = 1, 2, \ldots .$$

Since $P_i \ge P_{i+1}$ for each $i \ge 1$ and $\lim_{i\to\infty} P_i = P$ (which exists in view of hypothesis *(iii)*), we may write (2.7) as

$$r(t_m) \le P^{-1} \exp\left\{\int_{t_0}^{t_m} g(s)h(s)u_1'(s)ds\right\} , \quad m = 1, 2, \ldots .$$

Now, given any $t \in J$, there is a unique integer $m \ge 0$ such that $t \in [t_m, t_{m+1})$. Therefore

$$r(t) = r(t_m) + \int_{(t_m,t]} h(s) \frac{x(s)}{f(s)} du(s)$$
$$\le r(t_m)\exp\left\{\int_{t_m}^{t} g(s)h(s)u_1'(s)ds\right\} .$$

Hence we conclude that

$$x(t) \le f(t)g(t)r(t)$$
$$\le P^{-1}f(t)g(t)\exp\left\{\int_{t_0}^{t} g(s)h(s)u_1'(s)ds\right\} , \quad t \in J .$$

This completes the proof.

As an illustration of Theorem 2.1, consider the inequality

https://doi.org/10.1017/S0004972700008200 Published online by Cambridge University Press

$$x(t) \le t^2 + e^t \int_1^t 2e^{-s}\left(s^2-s+1\right)^{-1} x(s)du(s) , \quad t \in [1, \infty) ,$$

where

$$u(t) = 2^{-1}\left(t^2-t\right) + \left\{\sum_{i=1}^{k-1} (i+1)^{-1}\right\} \chi_{[k-1,k)}(t) , \quad k = 2, 3, \ldots .$$

Here χ_A, the characteristic function of the set A, is defined as $\chi_A(t) = 1$ if $t \in A$ and equal to zero otherwise. It is easily seen that $u'(t) = 2^{-1}(2t-1)$ almost everywhere on $[1, \infty)$; $t_k = k$, $\lambda_k = (k+1)^{-1}$ for $k = 2, 3, \ldots$; $\lambda_k g\left(t_k\right) h\left(t_k\right) = 2\left(k^3+1\right)^{-1} < 1$ for all $k \ge 2$; the series

$$\sum_{k=2}^{\infty} 2\left(k^3+1\right)^{-1} < \infty$$

by comparison test and

$$P = \prod_{k=2}^{\infty} \left\{1-2\left(k^3+1\right)^{-1}\right\} = \frac{2}{3} .$$

Following the estimate in (2.4), we obtain

$$x(t) \le \frac{3}{2}\left(t^4-t^3+t^2\right)e^t , \text{ for all } t \ge 1 .$$

We apply Theorem 2.1, in the natural way, to Volterra integral equations of the form (1.1). To this end, we assume that there exist non-negative functions g and h which are defined and locally du-integrable on J and are such that

(2.8) $$\left|K\left(t, s, \phi_1\right)-K\left(t, s, \phi_2\right)\right| \le g(t)h(s)\left|\phi_1-\phi_2\right|$$

for all $\phi_1, \phi_2 \in R^n$.

THEOREM 2.2. *Suppose that*

(i) (1.1) *has a bounded solution* x *defined on* J,

(ii) $g \ge 1$ *is bounded on* J *and*

$$(2.9) \qquad \int_{t_0}^{\infty} g(s)h(s)v_1'(s)ds < \infty$$

where $v = v_1 + v_2$ *is the decomposition of* $v(t) = V(u(t), |t_0, t|)$ *, the total variation function of* $u(t)$ *on* $|t_0, t|$ *,*

(iii) $v_1' \geq 0$ *on* J *and* $\mu_k g(t_k)h(t_k) < 1$ *, where* $\mu_k = v(t_k) - v(t_k-)$ *,* $k = 1, 2, \ldots$ *;*

the series $\sum_{k=1}^{\infty} \mu_k g(t_k)h(t_k)$ *converges absolutely.*

If $f^* \in BV(J)$ *is locally du-integrable on* J *and* $\|f(t)-f^*(t)\|$ *is non-decreasing and bounded on* J *, then any solution of the equation*

$$(2.10) \qquad y(t) = f^*(t) + \int_{t_0}^{t} K(t, s, y(s))du(s) , \quad t \in J ,$$

is bounded.

Proof. From (1.1), (2.8), and (2.10), we obtain

$$\|x(t)-y(t)\| \leq \|f(t)-f^*(t)\| + g(t) \int_{t_0}^{t} h(s)\|x(s)-y(s)\|dv(s) , \quad t \in J .$$

Since v is a right-continuous function of bounded variation and has discontinuities where u has, a suitable application of Theorem 2.1 gives

$$(2.11) \qquad \|x(t)-y(t)\| \leq P^{-1}\|f(t)-f^*(t)\|g(t)\exp\left\{\int_{t_0}^{t} g(s)h(s)v_1'(s)ds\right\} , \quad t \in J .$$

As x, g , and $\|f-f^*\|$ are all bounded on J , the conclusion follows from (2.9), (2.11), and the fact that $\|y(t)\| \leq \|y(t)-x(t)\| + \|x(t)\|$.

REMARK 2.1. A result similar to Theorem 2.2 is proved in [5, Theorem 3] where the integrals are in the *Riemann-Stieltjes* sense. Hence it is necessary that the integrand and the integrator should not have the same discontinuities. In our case, x and u have the same discontinuities and therefore the methods of [5] are not applicable.

3.

In this section, we consider a special case of (1.1), namely

$$x(t) = f(t) + \int_{t_0}^{t} A(t, s)x(s)du(s) \ , \quad t \in J \ , \tag{3.1}$$

where $A(t, s)$ is an $n \times n$ matrix defined for $t_0 \le s \le t < \infty$. We show, under certain conditions, that (3.1) reduces to a difference equation. Choose u to be a step function (that is $u_1 \equiv 0$) of the form

$$u(t) = \left\{\sum_{i=0}^{k-1} a_i\right\} \chi_{|t_{k-1}, t_k)}(t) \ , \quad k = 1, 2, \ldots \ ,$$

where the a_i's are constants. Let $J_{t_0} = \{t_k\}$, $k = 0, 1, \ldots$. Denote by B_k the matrix $I - a_k A(t_k, t_k)$, $k = 1, 2, \ldots$, where I is the identity $n \times n$ matrix.

THEOREM 3.1. *On* J_{t_0}, (3.1) *reduces to the difference equation*

$$\nabla x(t_k) = \nabla f(t_k) + a_k A(t_k, t_k)x(t_k) \ , \quad x(t_0) = f(t_0) \ , \tag{3.2}$$

where ∇ *is the operator such that* $\nabla x(t_k) = x(t_k) - x(t_{k-1})$. *Furthermore, if* B_k *is non-singular for each* $k = 1, 2, \ldots$ *then the unique solution of* (3.2) *is given by the recurrence formula*

$$x(t_k) = B_k^{-1}\{x(t_{k-1}) + \nabla f(t_k)\} \ , \quad k = 1, 2, \ldots \ . \tag{3.3}$$

Proof. It is clear that $x(t_0) = f(t_0)$. For $t_1 \in J_{t_0}$, we have from (3.1),

$$\begin{aligned} x(t_1) &= f(t_1) + \int_{t_0}^{t_1} A(t_1, s)x(s)du(s) \\ &= f(t_1) + a_1 A(t_1, t_1)x(t_1) \ . \end{aligned}$$

Similarly

https://doi.org/10.1017/S0004972700008200 Published online by Cambridge University Press

$$x(t_2) = f(t_2) + \int_{t_0}^{t_1} A(t_1, s)x(s)du(s) + \int_{t_1}^{t_2} A(t_2, s)x(s)du(s)$$
$$= x(t_1) + \nabla f(t_2) + a_2 A(t_2, t_2)x(t_2) .$$

In general, by induction,

(3.4) $x(t_k) = x(t_{k-1}) + \nabla f(t_k) + a_k A(t_k, t_k)x(t_k)$, $k = 1, 2, \ldots$,

which is the same as (3.2). If B_k is invertible, it follows from (3.4) that $x(t_k)$ exists uniquely and is given by (3.3).

REMARK 3.1. *If, for some* k , a_k *is zero, then* B_k $(= I)$ is clearly invertible. If $A(t, s) = A$ is a constant matrix and if $a_k \neq 0$, then a sufficient condition for B_k to be invertible is that a_k^{-1} is not an eigenvalue of A .

REMARK 3.2. Suppose B_k is not invertible for some k . Then it follows from (3.4) that, in general, $x(t_k)$ does not exist. On the other hand, if $x(t_{k-1}) + \nabla f(t_k) = 0$, then $x(t_k)$ is arbitrarily determined, which means that there are infinitely many solutions at t_k . It is to be noted that if $f \equiv 0$, then $x(t_k) = 0$ for each $k = 0, 1, \ldots$.

EXAMPLE 3.1. Let $f(t) = t$ and $A(t, s) = (e^t+t) \sin \frac{\pi s}{2}$ be scalar scalar functions on $[0, \infty)$. Choose

$$u(t) = \left\{\sum_{i=0}^{k-1} i^{-1}\right\} \chi_{[k-1,k)}(t) , \quad k = 1, 2, \ldots .$$

Then u is discontinuous at isolated points $t_k = k$ and $a_k = k^{-1}$ for $k = 1, 2, \ldots$. The difference equation corresponding to (3.1) is

$$x(0) = 0 ,$$

$$x(k) = x(k-1) + 1 + k^{-1}(e^k+k) \sin \frac{k\pi}{2} x(k) , \quad k = 1, 2, \ldots .$$

Since $(e^k+k) \sin \frac{k\pi}{2} \neq k$ for any $k \geq 1$, the condition of Theorem 3.1 is satisfied. $x(k)$ can now be determined from (3.3).

https://doi.org/10.1017/S0004972700008200 Published online by Cambridge University Press

EXAMPLE 3.2. Let $A(t, s) = A$ be the constant matrix

$$\begin{bmatrix} 2 & -1/3 \\ -6 & 1 \end{bmatrix}$$

and

$$u(t) = \frac{2}{3}\chi_{[0,1)}(t) + \left\{\sum_{i=1}^{k-1} i^2\right\}\chi_{[k-1,k)}(t) , \quad k = 2, 3, \ldots .$$

Here $a_1^{-1} = 3$ is an eigenvalue of A, the corresponding eigenvector being $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. If $f(1) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x(1)$ does not exist. Moreover, $c\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is also an eigenvector where c is any constant. Therefore, if $f(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then $x(1) = c\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, meaning thereby that there are infinitely many solutions.

4.

In this section, we obtain sufficient conditions in order that a certain stability of the system (1.2) implies the corresponding local stability of the system (1.3). The solutions $y(t)$ and $x(t)$ of (1.2) and (1.3) are respectively given by (the variation of constants formula)

$$y(t) = f(t) + \int_{t_0}^{t} R(t, s)f(s)ds , \quad t \geq t_0 , \tag{4.1}$$

and

$$x(t) = y(t) + \int_{t_0}^{t} R^*(t, s)F\{s, x(s)\}du(s) , \quad t \geq t_0 , \tag{4.2}$$

where $R(t, s)$ and $R^*(t, s)$ satisfy

$$R(t, s) = a(t, s) + \int_{s}^{t} R(t, \tau)a(\tau, s)d\tau \tag{4.3}$$

and

(4.4) $$R^*(t, s) = b(t, s) + \int_s^t R^*(t, \tau)b(\tau, s)d\tau \ .$$

The main result (Theorem 4.1) of this section depends on the following lemma, which is interesting in itself.

LEMMA 4.1. *Assume*

(i) x, f, *and* u *are as in Theorem* 2.1,

(ii) g_i, h_i *are non-negative functions, locally du-integrable on* J, *and* $g_i \geq 1$ *for* $i = 1, 2, \ldots, n$,

(iii) $\lambda_k g_i(t_k) h_i(t_k) < 1$ *for* $k \geq 1$ *and the* n *series*

$$\sum_{k=1}^{\infty} \lambda_k g_i(t_k) h_i(t_k)$$

converge absolutely for $i = 1, 2, \ldots, n$.

Then the inequality

(4.5) $$x(t) \leq f(t) + \sum_{i=1}^{n} g_i(t) \int_{t_0}^{t} h_i(s)x(s)du(s) \ , \quad t \in J \ ,$$

implies

(4.6) $$x(t) \leq P^{-1}E^n f$$

where

$$E^0 f = f \ ,$$

(4.7) $$E^r f = f\left(E^{r-1}g_r\right)\exp\left(\int_{t_0}^{t} h_r\left(E^{r-1}g_r\right)u_1'(s)ds\right) \ , \quad r = 1, 2, \ldots, n \ ,$$

$$P_i = \prod_{k=1}^{\infty} \{1-\lambda_k g_i(t_k)h_i(t_k)\} \ , \quad i = 1, 2, \ldots, n \ ,$$

and

$$P = P_1 P_2 \ldots P_n \ .$$

The proof can be obtained by applying Theorem 2.1 and the method of Theorem 1 in [4]. We omit the details.

https://doi.org/10.1017/S0004972700008200 Published online by Cambridge University Press

Now consider equations (1.2) and (1.3) whose solutions are given by (4.1) and (4.2) respectively. Assume that

H_1. There exists $r > 0$ such that

$$|F(t, x)| \leq f(t)\|x\| \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad \|x\| < r ,$$

where $f(t)$ is non-negative and du-integrable on J.

H_2. R^* satisfies

$$|R^*(t, s)| \leq \sum_{i=1}^{n} g_i(t)h_i(s) \quad \text{for} \quad t_0 \leq s \leq t < \infty ,$$

where, for $i = 1, 2, \ldots, n$, g_i, h_i are non-negative functions, du-integrable on J, and $g_i \geq 1$; $\mu_k f(t_k)g_i(t_k)h_i(t_k) < 1$ for $k \geq 1$, and the n series

$$\sum_{k=1}^{\infty} \mu_k f(t_k)g_i(t_k)h_i(t_k)$$

converge absolutely where μ_k is as defined in Theorem 2.2.

THEOREM 4.1. *Under the hypotheses* H_1 *and* H_2, *any solution* x *of* (1.3) *satisfies*

$$\|x(t)\| \leq P^{-1}E^n\|y\|$$

where y *is any solution of* (1.2); E^n *is as defined in Lemma* 4.1 *except that* u_1' *is replaced by* v_1';

$$P_i = \prod_{k=1}^{\infty} \{1-\mu_k f(t_k)g_i(t_k)h_i(t_k)\} , \quad i = 1, 2, \ldots, n \quad \textit{and} \quad P = P_1P_2 \ldots P_n .$$

Proof. We have

$$\|x(t)\| \leq \|y(t)\| + \sum_{i=1}^{n} f(t)g_i(t) \int_{t_0}^{t} h_i(s)\|x(s)\|dv(s) , \quad t \in J .$$

Since $\|y(t)\|$ is non-decreasing on J, an application of Lemma 4.1 gives the desired conclusion.

REMARK 4.1. Theorem 4.1 may be regarded as a result on local stability of the system (1.3) with respect to the system (1.2) in the following sense: given $\delta > 0$ and sufficiently small, the solution x of (1.3) satisfies $\|x(t)\| < c\delta$, $c > 0$, $t \geq t_0$, whenever $\|y(t)\| < \delta$.

As an illustration of Lemma 4.1, consider the inequality

$$x(t) \leq e^t + t\int_1^t s^{-2}x(s)du(s) + t^2\int_1^t \left(4s^3\right)^{-1}x(s)du(s) , \tag{4.8}$$

where

$$u(t) = t + \left\{\sum_{i=1}^{k-1} i^{-1}\right\}\chi_{[k-1,k)}(t) , \quad k = 2, 3, \ldots .$$

Here $t_k = k$, $\lambda_k = k^{-1}$ for $k = 2, 3, \ldots$;

$$P_1 = \prod_{k=2}^{\infty} \left(1-k^{-2}\right) = 1/2 ;$$

$$P_2 = \prod_{k=2}^{\infty} \left(1 - \frac{1}{4k^2}\right) = \frac{8}{3\pi} .$$

In view of (4.6), we obtain

$$x(t) \leq P^{-1}E^2f = \frac{3\pi}{4} t^4 \exp\left(\frac{t^2+8t-1}{8}\right) , \text{ for all } t \geq 1 .$$

REMARK 4.1. Lemma 4.1 has a distinct advantage over Theorem 2.1. To see this, consider the inequality (4.8). Since $t \geq s \geq 1$, we may write it as

$$x(t) \leq e^t + 2t^2\int_1^t s^{-2}x(s)du(s) , \quad t \geq 1 ,$$

which is of the form (2.1). In the notation of Theorem 2.1, we see that $\lambda_k g\left(t_k\right)h\left(t_k\right) = 2k^{-1}$, $k = 2, 3, \ldots$. However, Theorem 2.1 is not applicable here for two reasons; firstly because $\lambda_2 g\left(t_2\right)h\left(t_2\right) \nless 1$, and secondly because the series $\sum_{k=2}^{\infty} 2k^{-1}$ diverges.

https://doi.org/10.1017/S0004972700008200 Published online by Cambridge University Press

References

[1] Richard E. Bellman and Kenneth L. Cooke, *Differential-difference equations* (Academic Press, New York, London, 1963).

[2] Constantin Corduneanu, *Integral equations and stability of feedback systems* (Mathematics in Science and Engineering, 104. Academic Press [Harcourt Brace Jovanovich], New York and London, 1973).

[3] P.C. Das and R.R. Sharma, "Existence and stability of measure differential equations", *Czechoslovak Math. J.* 22 (97) (1972), 145-158.

[4] U.D. Dhongade and S.G. Deo, "Pointwise estimates of solutions of some Volterra integral equations", *J. Math. Anal. Appl.* 45 (1974), 615-628.

[5] G. Stephen Jones, "Fundamental inequalities for discrete and discontinuous functional equations", *J. Soc. Indust. Appl. Math.* 12 (1964), 43-57.

[6] R.K. Miller, J.A. Nohel, and J.S.W. Wong, "Perturbations of Volterra integral equations", *J. Math. Anal. Appl.* 25 (1969), 676-691.

[7] S.G. Pandit, "On the stability of impulsively perturbed differential systems", *Bull. Austral. Math. Soc.* 17 (1977), 423-432.

[8] V. Raghavendra and M. Rama Mohana Rao, "Integral equations of Volterra type and admissibility theory", *Rev. Roum. Math. Pure Appl.* 18 (1973), 571-580.

[9] V. Raghavendra and M. Rama Mohana Rao, "Volterra integral equations with discontinuous perturbations", *Mathematica (Cluj)* (to appear).

[10] Aaron Strauss, "On a perturbed Volterra integral equation", *J. Math. Anal. Appl.* 30 (1970), 564-575.

Department of Mathematics,
Centre of Post-Graduate Instruction and Research,
University of Bombay,
Panaji, Goa,
India.