BULL. AUSTRAL. MATH. SOC.   18C15
VOL. 17 (1977), 433-450.   (18A20, 18A25, 18A30)

Colimits of algebras revisited

Jiří Adámek

It has been open for some time whether, given an algebraic theory (triple, monad) $\mathbb{T}$ in a cocomplete category K, also the category $K^{\mathbb{T}}$ of $\mathbb{T}$-algebras must be cocomplete. We solve this in the negative by exhibiting a free algebraic theory $\mathbb{T}$ in the category $\mathcal{Gra}$ of graphs such that $\mathcal{Gra}^{\mathbb{T}}$ is not cocomplete. Further, we improve somewhat the well-known colimit theorem of Barr and Linton by showing that the base category need not be complete.

I. Categories of algebras ...

I.1. Is it true that an arbitrary theory of continuous (or ordered or compact, and so on) algebras allows the formation of sums? More generally: given an algebraic theory $\mathbb{T}$ in a "decent" cocomplete category K, is it true that the category $K^{\mathbb{T}}$ of $\mathbb{T}$-algebras is also cocomplete? While analogous questions about limits are elementary (the forgetful functor $K^{\mathbb{T}} \to K$ always creates limits), colimits present an interesting problem. Various sufficient conditions (which cover all of the important cases, in fact) have been found. For example, Linton proved in [7]:

THEOREM (Linton). *If* K *has sums and* $K^{\mathbb{T}}$ *has coequalizers then* $K^{\mathbb{T}}$ *is cocomplete.*

I.2. Other conditions involve *factorization systems*. Let us recall (for example from [5] or [9]) that a factorization system (E, M) in a category K consists of classes E, M of morphisms subject to the following conditions:

Received 26 July 1977.

(i) all M-morphisms are monos, all E-morphisms are epis;

(ii) M and E are subcategories, that is, closed to composition, both containing all isomorphisms;

(iii) $K = M.E$, that is every morphism f has a factorization $f = m.e$ with $e \in E$ and $m \in M$;

(iv) diagonal fill-in: for every commutative square

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "p"'] & \cdot \arrow[d, "q"] \arrow[dl, dashed, "d"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

with $e \in E$, $m \in M$, there exists a (diagonal) morphism d , making both triangles commute.

Factorization systems have a lot of natural properties, easy to verify, such as the following:

(v) E contains all coequalizers (this is an exercise in [5]);

(vi) opposite an E-morphism in a pushout there is an E-morphism (see [9]);

(vii) a multiple pushout of E-morphisms consists of E-morphisms (this is proved, more generally, in IV.1 below).

I.3. The following important theorem has been proved by Linton [7] and, in a different way, by Barr [4].

THEOREM (Barr and Linton). *Let* K *be a category with a factorization system* (E, M) *which is*

(a) complete,

(b) cocomplete,

(c) E-cowell-powered.

Let $\mathbb{T} = (T, \mu, \eta)$ *be an algebraic theory which preserves* E *; that is such that* $e \in E$ *implies* $Te \in E$ *. Then the category* $K^{\mathbb{T}}$ *is cocomplete.*

(Neither Barr nor Linton used the above definition of a factorization system; but we show in Section IV that their definitions are equivalent to ours. Linton supposed that $\mathbb{T}$ preserves also M .)

I.4. Two of the assumptions in the above colimit theorem can be felt as not entirely natural: completeness (cannot we do without it in a colimit theorem?) and preservation of E (is it necessary to assume things not only about K but also about $\mathbb{T}$?). The aim of the present paper is to show that completeness is redundant (see Section II) while preservation of E is not (see Section III).

Let us remark that Barr exhibits in [4] another colimit theorem: if $\mathbb{T}$ has rank then $K^{\mathbb{T}}$ is cocomplete. This covers all "natural" theories $\mathbb{T}$. Thus, it is no surprise that the counterexample in Section III consists of an ugly algebraic theorem $\mathbb{T}$ (in a nice category K , though).

II. ... are often cocomplete ...

II.1. We shall consider not only $\mathbb{T}$-algebras of an algebraic theory but, more generally, F-algebras of an arbitrary endofunctor $F : K \to K$. An *F-algebra* is a pair (A, α) , consisting of an object A of K and a morphism $\alpha : FA \to A$ (subject to no axioms). Given two F-algebras (A, α) and (B, β) , by an *F-homomorphism* $f : (A, \alpha) \to (B, \beta)$ is meant a K-morphism $f : A \to B$ such that $f.\alpha = \beta.Ff$. We denote by $K(F)$ the category of F-algebras and F-homomorphisms.

Thus, given an algebraic theory $\mathbb{T} = (T, \mu, \eta)$ in K the category $K^{\mathbb{T}}$ of $\mathbb{T}$-algebras is a full subcategory of the category $K(T)$ of T-algebras.

II.2. Categories $K(F)$ were used by Barr [4] for the study of free algebraic theories - this study was then applied by Arbib and Manes [3] to automata in categories. The latter call F an *input process* provided that the forgetful functor $K(F) \to K$ has a left adjoint, in other words, provided that each object A in K generates a *free F-algebra*. Explicitly, this free F-algebra consists of an F-algebra $\left(A^{\#}, \varphi^{A}\right)$ and a morphism $s^{A} : A \to A^{\#}$ in K which is universal in the following sense. Given an F-algebra (B, β) , for every morphism $f : A \to B$ there is a unique F-homomorphism $f^{\#} : \left(A^{\#}, \varphi^{A}\right) \to (B, \beta)$ with $f = f^{\#}.s^{A}$.

For each input process F there arises an algebraic theory Π (freely generated by F) with

$TA = A^{\#}$;

$\mu^A : A^{\#\#} \to A^{\#}$ is the unique F-homomorphism

$$(A^{\#\#}, \varphi^{A^{\#}}) \to (A^{\#}, \varphi^A) \text{ with } \mu^A.s^A = 1_A \text{ ;}$$

$\eta^A = s_A : A \to A^{\#}$.

Barr [4] proves that, under additional assumptions on K , these are the only free algebraic theories in K .

PROPOSITION (Barr). *Let* F *be an input process and let* Π *be the corresponding free algebraic theory. Then the categories* $K(F)$ *and* K^{Π} *are isomorphic.*

II.3. When aiming at a cocompleteness theorem for categories K^{Π} , we can restrict our attention to coequalizers in K^{Π} (I.1); it turns out that, sufficiently often, we can work with coequalizers in $K(T)$:

LEMMA. *Let* K *be a category with a factorization system* (E, M) *, let* $\Pi = (T, \mu, \eta)$ *be an algebraic theory, preserving* E *. Then for every coequalizer in* $K(T)$ *,*

$$(A, \alpha) \overset{f}{\underset{g}{\rightrightarrows}} (B, \beta) \xrightarrow{k} (C, \gamma)$$

such that (B, β) *is a* Π*-algebra, also* (C, γ) *is a* Π*-algebra.*

Proof. Let E^T denote the class of all T-homomorphisms with underlying morphism in E ; analogously M^T . Then (E^T, M^T) is a factorization system in $K(T)$; see [9], 3.4.17. Hence, by I.2 (v), $k \in E$. By hypothesis, also $Tk \in E$, $T^2k \in E$, and so on.

To show that (C, γ) is indeed a Π-algebra, consider the following diagrams, which clearly commute:

$$\begin{tikzcd}
C \arrow[dd, "\eta^C"'] & B \arrow[l, "k"'] \arrow[d, "\eta^B"'] \arrow[dr, "1"] & \\
& TB \arrow[r, "\gamma"'] \arrow[dl, "Tk"'] & B \arrow[d, "k"] \\
TC \arrow[rr, "\gamma"'] & & C
\end{tikzcd},$$

$$\begin{tikzcd}
T^2C \arrow[rrr, "T\gamma"] \arrow[ddd, "\mu^C"'] & & & TC \arrow[dd, "\gamma"] \\
& T^2B \arrow[ul, "T^2k"'] \arrow[r, "T\beta"] \arrow[d, "\mu^B"'] & TB \arrow[ur, "Tk"'] \arrow[d, "\beta"] & \\
& TB \arrow[r, "\beta"'] \arrow[dl, "Tk"'] & B \arrow[r, "k"'] \arrow[d, "k"'] & C \\
TC \arrow[rr, "\gamma"'] & & C &
\end{tikzcd}$$

By the first one, $(\gamma.\eta^C).k = k$, hence $\gamma.\eta^C = 1$ (k is epi). By the second one, $(\gamma.\mu^C).T^2k = (\gamma.T\gamma).T^2k$; hence $\gamma.\mu^C = \gamma\ T\gamma$ (T^2k is epi).

II.4. The following theorem is proved in [2] in a different manner, as a part of a more general study of colimits in $K(F)$. (An iterative colimit-construction is exhibited there, generalizing that used in universal algebra.) We present a straightforward proof. The help of Václav Koubek with this proof is gratefully acknowledged.

THEOREM. *Let* K *be a cocomplete category with a factorization system* (E, M) ; *let* K *be* E*-cowell-powered. Then for every functor* $F : K \to K$ *which preserves* E , *the category* $K(F)$ *has coequalizers.*

Proof. Let $f, g : (A, \alpha) \to (B, \beta)$ be arbitrary F-homomorphisms.

Denote by Ω the class of all E-epis $t : B \to T$ in K with the following property:

> for every F-homomorphism $h : (B, \beta) \to (C, \gamma)$ with $h.f = h.g$ there exists $h_{(t)} : T \to C$ in K such that $h = h_{(t)}.t$.

Since K is cocomplete and E-cowell-powered, the diagram Ω has a colimit (multiple pushout)

(1) $$r_0 = r_{(t)}.t : B \to T_0 \quad \left(r_{(t)} : T \to T_0 \text{ for each } t \in \Omega\right) .$$

Each $t \in \Omega$ is in E , hence (by I.2 (vii)) each $r_{(t)}$ is in E ; thus $t_0 \in E$ and $Ft_0 \in E$.

Fix a homomorphism $h : (B, \beta) \to (C, \gamma)$ with $h.f = h.g$. Then we have a bound of $\Omega : h_{(t)} : T \to C$ $(t \in \Omega)$. Thus there exists

$$\begin{tikzcd}
FB \arrow[r, "\beta"] \arrow[d, "Fh"'] & B \arrow[r, "t"] \arrow[d, "h"] \arrow[rr, bend left] & T \arrow[r, "r_{(t)}"] \arrow[dl, dashed, "h_{(t)}"] & T_0 \arrow[dll, dotted, "h_0"] \\
FC \arrow[r, "\gamma"'] & C & &
\end{tikzcd}$$

a unique $h_0 : T_0 \to C$ with

(2) $$h_0 . r_{(t)} = h_{(t)} \quad (t \in \Omega) \quad \text{and} \quad h_0 . t_0 = h .$$

Consider the pushout of Ft_0 and $t_0.\beta$:

$$\begin{tikzcd}
FB \arrow[r, "t_0.\beta"] \arrow[d, "Ft_0"'] & T_0 \arrow[d, "q"] \\
FT_0 \arrow[r, "p"'] & R
\end{tikzcd} .$$

CLAIM. q is an isomorphism. It suffices to show that $q.t_0 \in \Omega$; then by (1), $t_0 = r_{(q.t_0)}.q.t_0$, which implies $1 = r_{(q.t_0)}.q$ since t_0

is epi) and so q is a split mono as well as an E-epi (opposite $Ft_0 \in E$ in a pushout, see I.2 (vi)) - thus, q is an isomorphism. To show $q.t_0 \in \Omega$ we first remark that, since $q \in E$ and $t_0 \in E$ we have $q.t_0 \in E$. Secondly, consider any homomorphism $h : (B, \beta) \to (C, \gamma)$ with $h.f = h.g$: we have $h_0.t_0 = h$ by (2) and $h.\beta = \gamma.Fh$, hence

$$h_0.(t_0.\beta) = \gamma.Fh = (\gamma.Fh_0).Ft_0 .$$

This implies that the pair $h_0; (\gamma.Fh_0)$ factorizes through $q; p$ above; that is, there is a unique morphism, denoted by $h_{(q.t_0)}$ from R to C, with $h_0 = h_{(q.t_0)}.q$ and $\gamma.Fh_0 = h_{(q.t_0)}.p$. The first implies $h = h_{(q.t_0)}.(q.t_0)$, by (2). Thus, $q.t_0 \in \Omega$ and q is an isomorphism.

Let us show that the F-homomorphism $t_0 : (B, \beta) \to (T_0, q^{-1}.p)$ is a coequalizer of f and g in $K(F)$.

Firstly, $t_0.f = t_0.g$: indeed, consider the coequalizer c of f and g in K; $c \in E$ by I.2 (v), and clearly $c \in \Omega$. Hence $t_0 = r_{(c)}.c$, which proves $t_0.f = t_0.g$.

Secondly, for every homomorphism $h : (B, \beta) \to (C, \gamma)$ with $h.f = h.g$ we have $h_0 : T_0 \to C$ with $h = h_0.t_0$, by (2). This h_0 is unique, because t_0 is epi. To conclude the proof we only have to show that h_0 is a homomorphism; that is, that $h_0.(q^{-1}.p) = \gamma.Fh_0$. We use (2) and the fact that $Ft_0 \in E$ is epi, and that $p.Ft_0 = q.t_0.\beta$ (see the pushout above):

$$\begin{aligned} \left[h_0.(q^{-1}.p)\right].Ft_0 &= h_0.(q^{-1}.q).t_0.\beta \\ &= h.\beta \\ &= \gamma.Fh \\ &= [\gamma.Fh_0].Ft_0 . \end{aligned}$$

II.5. **COROLLARY.** *Let* K *and* (E, M) *be as in* II.4. *Then for every* *algebraic theory* Π *which preserves* E *, the category* K^{Π} *is cocomplete.*

Proof. By II.4, the category $K(T)$ has coequalizers, hence (by II.3) so does K^{Π} . By I.1 this implies the cocompleteness.

COROLLARY. *Let* K *be a cowell-powered, cocomplete category. Then for every algebraic theory* Π *preserving epis, also* K^{Π} *is cocomplete.*

Proof. It is proved in [5] (the dual to 34.1) that K has a factorization system (E, M) with E equal to all epis, M equal to all extremal monos.

II.6. The latter corollary is proved in [1] in the same way as in the present paper. The first corollary was first formulated by Reiterman. See [6], where a completely different method is used (related to that used in [2] to prove Theorem II.4 above).

III. ... but not always!

III.1. We denote by $\mathcal{Gra}$ the category of graphs and compatible mappings. A graph is a pair $A = \langle A, K\rangle$ consisting of a set A and a subset K of $A \times A$. A compatible mapping $f : \langle A, K\rangle \to \langle B, L\rangle$ is a mapping $f : A \to B$ for which $(x, y) \in K$ implies $\big(f(x), f(y)\big) \in L$.

$\mathcal{Gra}$ is a complete and cocomplete concrete category, with underlying functor $\mathcal{Gra} \to \mathcal{Set}$ creating all limits and colimits; it is also a well-powered and cowell-powered category and is, in one word, decent.

III.2. We shall define an input process F in $\mathcal{Gra}$ such that the category $\mathcal{Gra}(F)$ of F-algebras is not cocomplete. Before doing this, we shall make a simple observation about P-algebras, where $P : \mathcal{Set} \to \mathcal{Set}$ is the power-set functor (sending a set X to the power set $PX = 2^X$ and a mapping $f : X \to Y$ to the mapping

$$Pf : A \mapsto \{f(a);\ a \in A\}\ ).$$

We recall that an object O of a category is *weak initial* if for every other object X there exists at least one morphism from O to X .

LEMMA. *The category* $\mathcal{Set}(P)$ *of* P*-algebras has no weak initial object.*

Proof. It is easy to see that $\mathcal{Set}(P)$ is a complete category (with limits created by the forgetful functor $\mathcal{S}et(P) \to \mathcal{S}et$). Thus the existence of a weak initial object would imply the existence of an initial object; see [8].

Now let (A, α) be an initial P-algebra. Barr proves in [4] that α is then an isomorphism. But there exists no isomorphism from a power set PA to A, of course; a contradiction.

III.3. We start by defining a functor $F : \mathcal{G}ra \to \mathcal{G}ra$. First, for every graph $\mathsf{A} = \langle A, K \rangle$, define a set

$$\mathsf{A}^{(3)} = \{(x, y, z) \in A \times A \times A;\ (x, y) \in K \text{ and } (y, z) \in K\} .$$

Given a compatible mapping $f : \mathsf{A} \to \mathsf{B}$, define a mapping $f^{(3)} : \mathsf{A}^{(3)} \to \mathsf{B}^{(3)}$ by

$$f^{(3)} : (x, y, z) \mapsto \big(f(x), f(y), f(z)\big) .$$

Now define F as follows: for each graph A put

$$F\mathsf{A} = \left\langle P\mathsf{A}^{(3)}, M_{\mathsf{A}} \right\rangle \text{ where } (X, Y) \in M \text{ iff } X = \emptyset \text{ and } Y \neq \emptyset \quad \big(X, Y \subset \mathsf{A}^{(3)}\big) ;$$

for each compatible map $f : \mathsf{A} \to \mathsf{B}$ put

$$Ff = Pf^{(3)} .$$

Clearly, $Pf^{(3)} : F\mathsf{A} \to F\mathsf{B}$ is compatible and F is a correctly defined functor.

III.4. LEMMA. *F is an input process.*

Proof. For every graph A define a new graph

$$\mathsf{A}^{\#} = \mathsf{A} \vee F\mathsf{A}$$

and notice that $(F\mathsf{A})^{(3)} = \emptyset$; hence $F\mathsf{A}^{\#} = F\mathsf{A}$. Denote by $\delta^{\mathsf{A}} : \mathsf{A} \to \mathsf{A}^{\#}$, $\varphi^{\mathsf{A}} : F\mathsf{A}^{\#} = F\mathsf{A} \to \mathsf{A}^{\#}$ the canonical injections. Then $\big(\mathsf{A}^{\#}, \varphi^{\mathsf{A}}\big)$ is a free F-algebra generated by A with universal morphism δ^{A}.

Indeed, let (B, β) be an F-algebra and let $f : \mathsf{A} \to \mathsf{B}$ be a morphism (that is a compatible mapping). Define $f^{\#} : \mathsf{A}^{\#} \to \mathsf{B}$ by

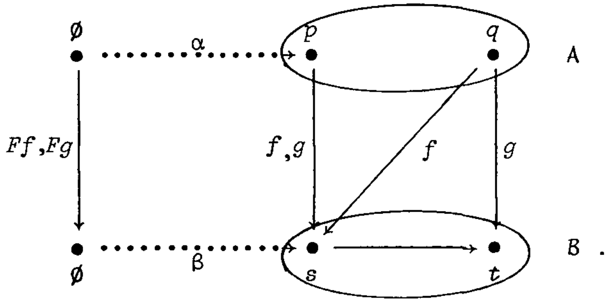
$$f^{\#}.\varphi^{\mathsf{A}} = \beta.Ff , \tag{3}$$

(4) $$f^{\#}.s^{A} = f\ .$$

Since $(FA)^{(3)} = \emptyset$, clearly $Ff^{\#} = Ff$, and so (3) means that $f^{\#} : (A^{\#}, \varphi^{A}) \to (B, \beta)$ is an F-homomorphism; by (4), $f^{\#}$ extends f . The uniqueness of $f^{\#}$ follows from the fact that (3) and (4) are actually necessary.

III.5. We define a pair $f, g : (A, \alpha) \to (B, \beta)$ of F-homomorphisms of which we shall prove that they do not have a coequalizer in $\mathcal{G}\!\mathit{ra}(F)$.

Let $A = \langle \{p, q\}, \emptyset \rangle$ and $B = \langle \{s, t\}, \{(s, t)\} \rangle$.



Clearly $A^{(3)} = B^{(3)} = \emptyset$, hence $FA = FB = \langle \{\emptyset\}, \emptyset \rangle$. Define $\alpha : FA \to A$ by $\alpha(\emptyset) = p$; $\beta : FB \to B$ by $\beta(\emptyset) = s$.

Finally, define $f, g : \{p, q\} \to \{s, t\}$ by

$$f(p) = g(p) = f(q) = s \quad \text{and} \quad g(q) = t\ .$$

Clearly, f is a homomorphism with $f.\alpha = \beta.Ff : \emptyset \mapsto s$, analogously g .

III.6. Assuming that f, g have a coequalizer $c : (B, \beta) \to (C, \gamma)$ in $\mathcal{G}\!\mathit{ra}(F)$, we shall find a weak initial object in $\mathcal{S}\!\mathit{et}(P)$ – a contradiction.

We have $C = \langle C, K \rangle$. Put $\bar{s} = c(s)$ $(= c(t)$, because $c.f = c.g)$. Since $c : B \to C$ is compatible, clearly $(\bar{s}, \bar{s}) \in K$.

Put

$$C_0 = \{x \in C;\ (\bar{s}, x) \in K\}\ .$$

For every subset $X \subset C_0$ put

$$\hat{X} = \{(\bar{s}, \bar{s}, x);\ x \in X\} \in PC^{(3)}\ .$$

We have $\gamma(\hat{X}) \in C$ – let us show that, in fact, $\gamma(\hat{X}) \in C_0$. If $X = \emptyset$, then $\hat{X} = \emptyset$ and $\gamma(\emptyset) = \bar{s} \in C_0$, because $\gamma . Fc = c . \beta$ and $Fc(\emptyset) = \emptyset$; thus

$$\gamma(\emptyset) = c\big(\beta(\emptyset)\big) = c(s) = \bar{s} .$$

If $X \neq \emptyset$, then $\hat{X} \neq \emptyset$ and so $(\emptyset, \hat{X}) \in M_C$ (see III.3). Since $\gamma : FC \to C$ is compatible, this yields $\big(\gamma(\emptyset), \gamma(\hat{X})\big) \in K$; that is,

$$\big(\bar{s}, \gamma(\hat{X})\big) \in K \text{ ; thus } \gamma(\hat{X}) \in C_0 .$$

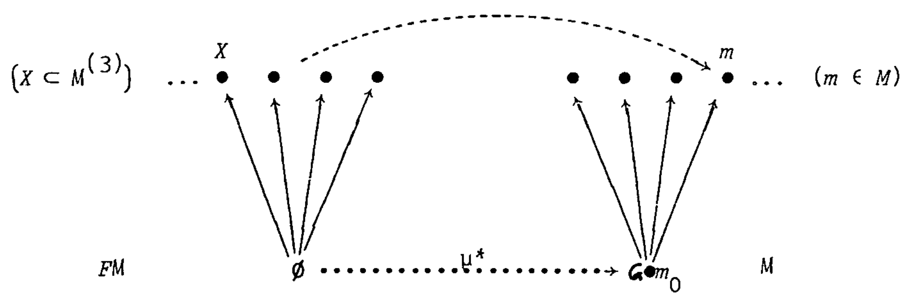
Now we define a P-algebra $\big(C_0, \hat{\gamma}\big)$ by

$$\hat{\gamma}(X) = \gamma(\hat{X}) \quad \big(X \subset C_0\big) .$$

This P-algebra is weak initial.

Proof. Let (M, μ) be another P-algebra, that is, a set M and a mapping $\mu : PM \to M$; put $m_0 = \mu(\emptyset)$. Define an F-algebra $(M, \mu^*) : M = \langle M, \{(m_0, m) ; m \in M\}\rangle$ where $\mu^* : FM \to M$ is defined by

$$\mu^*(X) = \mu\big(\{m \in M ; (m_0, m_0, m) \in X\}\big) .$$



Particularly, $\mu^*(\emptyset) = m_0$. Thus $h : (B, \beta) \to (M, \mu^*)$, defined by $h(s) = h(t) = m_0$, is an F-homomorphism. Since $h.f = h.g$, there exists an F-homomorphism $k : (C, \gamma) \to (M, \mu^*)$ such that $k.c = h$ – particularly, $k(\bar{s}) = m_0$.

The proof will be concluded when we show that the restriction $k_0 : C_0 \to M$ of k is a P-homomorphism; that is, that $k_0 . \hat{\gamma} = \mu . Pk_0$. Given $X \subset C_0$ we have

$$k_0.\hat{\gamma}(X) = k.\gamma(\hat{X}) = \mu^*.Fk(\hat{X}) .$$

Furthermore, $\hat{X} = \{(\bar{s}, \bar{s}, x);\ x \in X\}$ implies

$$Fk(\hat{X}) = \{(m_0, m_0, k(x));\ x \in X\}$$

and $k(x) = k_0(x)$ for $x \in X$ (since $X \subset C_0$); thus

$$\mu^*.Fk(\hat{X}) = \mu(\{k_0(x);\ x \in X\}) = \mu.Pk_0(X) .$$

Thus $(C_0, \hat{\gamma})$ is a weak initial P-algebra, in contradiction to Lemma III.2.

CONCLUSION. The free algebraic theory $\mathbb{T}$ generated by the above input process F in $\mathcal{G}ra$ is such that $\mathcal{G}ra^{\mathbb{T}}$ is not complete. Explicitly, $\mathbb{T} = (T, \mu, \eta)$ with

$$TA = A \vee FA \quad (\eta^A : A \to TA \text{ and } \varphi^A : FA \to TA \text{ canonical})$$

and $\mu^A : T^2A = (A \vee FA) \vee FA \to TA$ is defined on A as η^A and on both copies of FA as φ^A.

IV. Appendix on factorizations

IV.1. **Barr's right factorization systems.** For the colimit theorem of I.3, Barr [4] uses a right factorization system, which is a pair (E, M) as in I.2, except that E-morphisms need not be epis. More precisely, a right factorization system consists of a class E of morphisms and a class M of monos such that conditions (ii)-(iv) of I.2 are fulfilled.

There always exists a simple right factorization system: E equals all morphisms, M equals all isomorphisms. In this case, K is seldom E-cowell-powered (as required in the colimit theorem I.3). We shall show that this is no coincidence.

LEMMA. *For each right factorization system* (E, M) *and each multiple pushout*

$$\begin{tikzcd}
 & & A \arrow[dll, "f_t"'] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
B_t \arrow[drr, "k_t"'] & {} \arrow[dr] & & {} \arrow[dl] & {} \arrow[dll] \quad \cdots \quad (t \in T) \\
 & & C & &
\end{tikzcd}$$

with $f_t \in E$ *for* $t \in T$, *also* $k_t \in E$ *for* $t \in T$.

Proof. Choose $t_0 \in T$ and let $k_{t_0} = m.e$ be an E-M-factorization. For every $t \in T$ use the diagonal fill-in:

$$\begin{tikzcd}
A \arrow[r, "f_t"] \arrow[d, "f_{t_0}"'] & B_t \arrow[dd, "k_t"] \arrow[ddl, dashed, "d_t"'] \\
{} \arrow[d, "e"'] & \\
{} \arrow[r, "m"'] & C
\end{tikzcd}$$

to obtain d_t with $d_t.f_t = e.f_{t_0}$ (hence d_t is a bound of the pushout) and $m.d_t = k_t$. There exists a unique d with $d_t = d.k_t$ $(t \in T)$. Then $(m.d).k_t = k_t$ $(t \in T)$; hence $m.d = 1$. Since $m \in M$, m is a mono as well as a split epi - thus m is an isomorphism. This shows that $k_{t_0} = m.e$ is in E .

PROPOSITION. *Every right factorization system* (E, M) *in a cocomplete,* E*-cowell-powered category is a factorization system (that is, all* E*-morphisms are epis).*

Proof. Assume that K is a cocomplete category with a right factorization system (E, M) . Given $f : A \to B$ in E which is not epi, we shall show that K is not E-cowell-powered. Indeed, if K is E-cowell-powered, there exist $q_i : B \to Q_i$ $(i \in I$, I a set) in E such that each E-morphism with domain B is isomorphic to some q_i . Choose a cardinal λ such that card $\hom(B, Q_i) < \lambda$ for each $i \in I$.

Let $\{k_t\}_{t\in T}$ be the multiple

$$\begin{tikzcd} & & A \arrow[dll, "f"'] \arrow[dl, "f"] \arrow[d, "f"] \arrow[dr, "f"] & & \\ B \arrow[drr] & B \arrow[dr] & B \arrow[d] & B \arrow[dl, "k_t"] & \cdots \ (t \in T) \\ & & R & & \end{tikzcd}$$

pushout of a T-indexed family of copies of f, where T is a set of power λ. By the above lemma, $k_t \in E$ for each $t \in T$. To conclude the proof it suffices to show that the k_t's are pairwise distinct: then card $\hom(B, R) \geq \lambda$; hence R is not isomorphic to any of Q_i.

Since f is not epi, there exist distinct morphisms $g_1, g_2 : B \to C$ with $g_1 \cdot f = g_2 \cdot f$. Consider the following bound $g_{n_t} : B \to C$ of the above pushout: for a given $t_0 \in T$, $n_{t_0} = 1$; else $n_t = 2$. There exists a unique $h : R \to C$ with $g_{n_t} = h . k_t$ $(t \in T)$. Since $g_{n_{t_0}} \neq g_{n_t}$, we have $k_{t_0} \neq k_t$ for each $t_0 \neq t$. This shows that the k_t's are pairwise distinct.

IV.2. Linton's factorization functors. Denote by K^2 the morphism category of K and by K^3 the triangle category of K (objects are triples $(f; g, h)$ of K-morphisms with $f = g . h$; morphisms are triples $(p, r, q) : (f; g, h) \to (f'; g', h')$ of K-morphisms with $r . h = h' . p$ and $q \cdot g = g' . q$). There is a natural forgetful functor $\gamma : K^3 \to K^2$ (sending $(f; g, h)$ to f).

Linton [7] uses a factorization functor, that is a functor

$$\Delta : K^2 \to K^3$$

such that

(1) $\gamma \circ \Delta = 1$; for $f : X \to Y$ in K^2 , $\Delta(f)$ is denoted by

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f_a"'] & & Y \\ & I_f \arrow[ur, "f_b"'] & \end{tikzcd} ;$$

(2) f_a is epi, f_b is mono, for each $f \in K^2$;

(3) $(f_a)_b$ and $(f_b)_a$ are isomorphisms, for each $f \in K^2$.

PROPOSITION. *(A) Given a factorization functor* Δ *, the pair* (E_Δ, M_Δ) *with*

$$E_\Delta = \{f \in K^2;\ f_b \text{ is an isomorphism}\} ,$$

$$M_\Delta = \{f \in K^2;\ f_a \text{ is an isomorphism}\} ,$$

is a factorization system in K .

(B) If Δ, Δ' *are naturally equivalent factorization functors then* $E_\Delta = E_{\Delta'}$ *, and* $M_\Delta = M_{\Delta'}$.

(C) For every factorization system (E, M) *there exists a factorization functor* Δ *, unique up to natural equivalence, with* $(E, M) = (E_\Delta, M_\Delta)$.

Proof. *(A)*. Conditions (i), (iii) in I.2 are evident. Condition (iv) follows from the fact that $(p, q) : e \to m$ is a morphism in K^2 ; hence we have $\Delta(p, q) : \Delta(e) \to \Delta(m)$ in K^3 .

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "p"'] & \cdot \arrow[d, "q"] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd} \quad \overset{\Delta}{\longmapsto} \quad \begin{tikzcd} \cdot \arrow[rr, "e"] \arrow[dr, "e_a"'] \arrow[dd, "p"'] & & \cdot \arrow[dd, "q"] \\ & \cdot \arrow[ur, "e_b"'] \arrow[dd, dashed, "r"] & \\ \cdot \arrow[rr, "m"] \arrow[dr, "m_a"'] & & \cdot \\ & \cdot \arrow[ur, "m_b"'] & \end{tikzcd}$$

Since $\gamma \circ \Delta = 1$, clearly $\Delta(p, q) = (p, r, q)$ for some morphism $r : I_e \to I_m$, making the above diagram commute. The diagonal morphism is then

$$d = m_a^{-1}.r.e_b^{-1} .$$

Finally, to verify condition (ii) it suffices to show that E and M are closed to composition with isomorphisms; see [5], 33.3. Let us verify, for example, that $e \in E$ implies $e \circ i \in E$ whenever i is an isomorphism; the rest is analogous. Consider $(i^{-1}, 1) : e \to e.i$ in K^2; we have $\Delta(i^{-1}, 1)$ of the form $(i^{-1}, r, 1) : \Delta(e) \to \Delta(e \circ i)$ in K^3:

$$\begin{tikzcd} \cdot \arrow[rr, "e"] \arrow[dr, "e_a"'] \arrow[dd, "i^{-1}"'] & & \cdot \arrow[dd, "1"] \\ & \cdot \arrow[ur, "e_b"'] \arrow[dd, dashed, "r"] & \\ \cdot \arrow[rr, "e \circ i"] \arrow[dr, "(e \circ i)_a"'] & & \cdot \\ & \cdot \arrow[ur, "(e \circ i)_b"'] & \end{tikzcd}$$

Since $(e \circ i)_b \circ r \circ e_b^{-1} = 1$, we see that $(e \circ i)_b$ is a split epi as well as a mono by (2) above. Hence $(e \circ i)_b$ is an isomorphism; thus $e \circ i \in E$.

(B) follows easily from the fact that both E and M are closed to

composition with isomorphisms.

(C). For every morphism f choose a fixed factorization $f = f_b \cdot f_a$ with $f_a \in E$, $f_b \in M$. Put $\Delta(f) = (f; f_b, f_a)$. Given a morphism $(p, q) : f \to g$ in K^2,

$$\begin{tikzcd}
\cdot \arrow[r, "f_a"] \arrow[d, "p"'] & \cdot \arrow[d, "f_b"] \arrow[ddl, dashed, "d"'] \\
\cdot \arrow[d, "g_a"'] & \cdot \arrow[d, "q"] \\
\cdot \arrow[r, "g_b"'] & \cdot
\end{tikzcd} ,$$

use the diagonal fill-in on the square above to find d and define

$$\Delta(p, q) = (p, d, q) : \Delta(f) \to \Delta(g) .$$

This gives rise to a factorization functor $\Delta : K^2 \to K^3$ with $E = E_\Delta$ and $M = M_\Delta$. Uniqueness follows from *(B)*.

References

[1] Jiří Adámek, "Kategoriální teorie automatů a universální algebra" [Categorical theory of automata and universal algebra] (Doctoral Dissertation, Charles University, Prague, 1974).

[2] J. Adámek, V. Koubek, "Algebras and automata over a functor", *Kibernetika* (to appear).

[3] Michael A. Arbib and Ernest G. Manes, "Machines in a category: an expository introduction", *SIAM Rev.* 16 (1974), 163-192.

[4] Michael Barr, "Coequalizers and free triples", *Math. Z.* 116 (1970), 307-322.

[5] Horst Herrlich, George E. Strecker, *Category theory: an introduction* (Allyn and Bacon, Boston, Massachusetts, 1973).

[6] Václav Koubek, Jan Reiterman, "Categorical constructions of free algebras, colimits and completions of partial algebras", submitted.

[7] F.E.J. Linton, "Coequalizers in categories of algebras", *Seminar on triples and categorical homology theory*, 75-90 (Lecture Notes in Mathematics, 80. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[8] S. Mac Lane, *Categories for the working mathematician* (Graduate Texts in Mathematics, 5. Springer-Verlag, New York, Heidelberg, Berlin, 1971).

[9] Ernest G. Manes, *Algebraic theories* (Graduate Texts in Mathematics, 26. Springer-Verlag, New York, Heidelberg, Berlin, 1976).

Faculty of Electrical Engineering,
České Vysoké Učení Technické v Praze,
Czechoslovakia.