

# Colimits of algebras revisited

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It has been open for some time whether, given an algebraic theory (triple, monad)  $\Pi$  in a cocomplete category  $K$ , also the category  $K^\Pi$  of  $\Pi$ -algebras must be cocomplete. We solve this in the negative by exhibiting a free algebraic theory  $\Pi$  in the category  $Gra$  of graphs such that  $Gra^\Pi$  is not cocomplete. Further, we improve somewhat the well-known colimit theorem of Barr and Linton by showing that the base category need not be complete.

## I. Categories of algebras ...

I.1. Is it true that an arbitrary theory of continuous (or ordered or compact, and so on) algebras allows the formation of sums? More generally: given an algebraic theory  $\Pi$  in a "decent" cocomplete category  $K$ , is it true that the category  $K^\Pi$  of  $\Pi$ -algebras is also cocomplete? While analogous questions about limits are elementary (the forgetful functor  $K^\Pi \rightarrow K$  always creates limits), colimits present an interesting problem. Various sufficient conditions (which cover all of the important cases, in fact) have been found. For example, Linton proved in [7]:

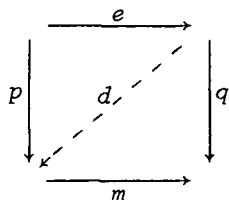
**THEOREM (Linton).** *If  $K$  has sums and  $K^\Pi$  has coequalizers then  $K^\Pi$  is cocomplete.*

I.2. Other conditions involve *factorization systems*. Let us recall (for example from [5] or [9]) that a factorization system  $(E, M)$  in a category  $K$  consists of classes  $E, M$  of morphisms subject to the following conditions:

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- (i) all  $M$ -morphisms are monos, all  $E$ -morphisms are epis;
- (ii)  $M$  and  $E$  are subcategories, that is, closed to composition, both containing all isomorphisms;
- (iii)  $K = M.E$ , that is every morphism  $f$  has a factorization  $f = m.e$  with  $e \in E$  and  $m \in M$ ;
- (iv) diagonal fill-in: for every commutative square



with  $e \in E$ ,  $m \in M$ , there exists a (diagonal) morphism  $d$ , making both triangles commute.

Factorization systems have a lot of natural properties, easy to verify, such as the following:

- (v)  $E$  contains all coequalizers (this is an exercise in [5]);
- (vi) opposite an  $E$ -morphism in a pushout there is an  $E$ -morphism (see [9]);
- (vii) a multiple pushout of  $E$ -morphisms consists of  $E$ -morphisms (this is proved, more generally, in IV.1 below).

I.3. The following important theorem has been proved by Linton [7] and, in a different way, by Barr [4].

**THEOREM** (Barr and Linton). *Let  $K$  be a category with a factorization system  $(E, M)$  which is*

- (a) complete,
- (b) cocomplete,
- (c)  $E$ -cowell-powered.

*Let  $\Pi = (T, \mu, \eta)$  be an algebraic theory which preserves  $E$ ; that is such that  $e \in E$  implies  $Te \in E$ . Then the category  $K^\Pi$  is cocomplete.*

(Neither Barr nor Linton used the above definition of a factorization system; but we show in Section IV that their definitions are equivalent to ours. Linton supposed that  $\Pi$  preserves also  $M$ .)

I.4. Two of the assumptions in the above colimit theorem can be felt as not entirely natural: completeness (cannot we do without it in a colimit theorem?) and preservation of  $E$  (is it necessary to assume things not only about  $K$  but also about  $\Pi$ ?). The aim of the present paper is to show that completeness is redundant (see Section II) while preservation of  $E$  is not (see Section III).

Let us remark that Barr exhibits in [4] another colimit theorem: if  $\Pi$  has rank then  $K^\Pi$  is cocomplete. This covers all "natural" theories  $\Pi$ . Thus, it is no surprise that the counterexample in Section III consists of an ugly algebraic theory  $\Pi$  (in a nice category  $K$ , though).

## II. ... are often cocomplete ...

II.1. We shall consider not only  $\Pi$ -algebras of an algebraic theory but, more generally,  $F$ -algebras of an arbitrary endofunctor  $F : K \rightarrow K$ . An  $F$ -algebra is a pair  $(A, \alpha)$ , consisting of an object  $A$  of  $K$  and a morphism  $\alpha : FA \rightarrow A$  (subject to no axioms). Given two  $F$ -algebras  $(A, \alpha)$  and  $(B, \beta)$ , by an  $F$ -homomorphism  $f : (A, \alpha) \rightarrow (B, \beta)$  is meant a  $K$ -morphism  $f : A \rightarrow B$  such that  $f \cdot \alpha = \beta \cdot Ff$ . We denote by  $K(F)$  the category of  $F$ -algebras and  $F$ -homomorphisms.

Thus, given an algebraic theory  $\Pi = (T, \mu, \eta)$  in  $K$  the category  $K^\Pi$  of  $\Pi$ -algebras is a full subcategory of the category  $K(T)$  of  $T$ -algebras.

II.2. Categories  $K(F)$  were used by Barr [4] for the study of free algebraic theories - this study was then applied by Arbib and Manes [3] to automata in categories. The latter call  $F$  an *input process* provided that the forgetful functor  $K(F) \rightarrow K$  has a left adjoint, in other words, provided that each object  $A$  in  $K$  generates a *free  $F$ -algebra*. Explicitly, this free  $F$ -algebra consists of an  $F$ -algebra  $(A^\#, \phi^A)$  and a morphism  $s^A : A \rightarrow A^\#$  in  $K$  which is universal in the following sense. Given an  $F$ -algebra  $(B, \beta)$ , for every morphism  $f : A \rightarrow B$  there is a unique  $F$ -homomorphism  $f^\# : (A^\#, \phi^A) \rightarrow (B, \beta)$  with  $f = f^\# \cdot s^A$ .

For each input process  $F$  there arises an algebraic theory  $\Pi$  (freely generated by  $F$ ) with

$$TA = A^\# ;$$

$\mu^A : A^{\#\#} \rightarrow A^\#$  is the unique  $F$ -homomorphism

$$(A^{\#\#}, \varphi^{A^\#}) \rightarrow (A^\#, \varphi^A) \quad \text{with} \quad \mu^A \cdot s^A = 1_A ;$$

$$\eta^A = s_A : A \rightarrow A^\# .$$

Barr [4] proves that, under additional assumptions on  $K$ , these are the only free algebraic theories in  $K$ .

**PROPOSITION (Barr).** *Let  $F$  be an input process and let  $\Pi$  be the corresponding free algebraic theory. Then the categories  $K(F)$  and  $K^\Pi$  are isomorphic.*

II.3. When aiming at a cocompleteness theorem for categories  $K^\Pi$ , we can restrict our attention to coequalizers in  $K^\Pi$  (I.1); it turns out that, sufficiently often, we can work with coequalizers in  $K(T)$ :

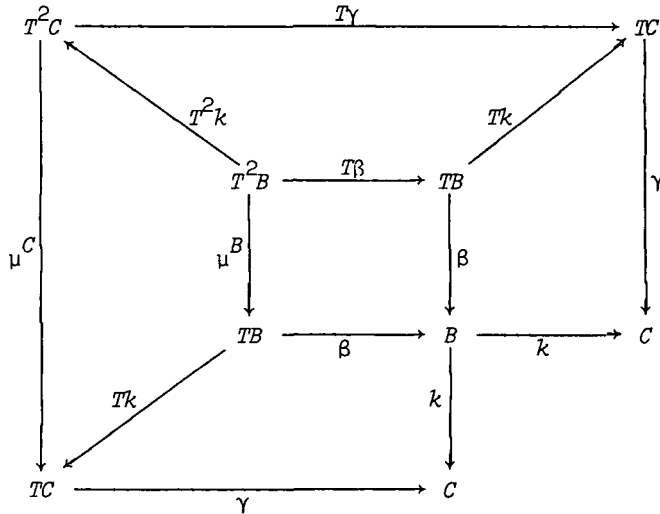
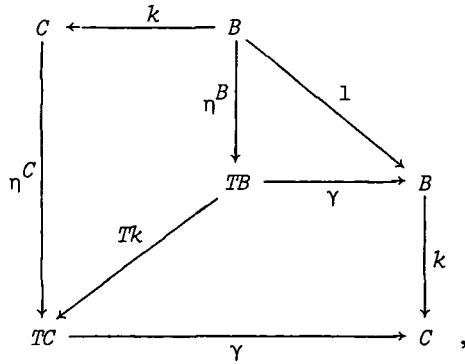
**LEMMA.** *Let  $K$  be a category with a factorization system  $(E, M)$ , let  $\Pi = (T, \mu, \eta)$  be an algebraic theory, preserving  $E$ . Then for every coequalizer in  $K(T)$ ,*

$$(A, \alpha) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} (B, \beta) \xrightarrow{k} (C, \gamma)$$

such that  $(B, \beta)$  is a  $\Pi$ -algebra, also  $(C, \gamma)$  is a  $\Pi$ -algebra.

**Proof.** Let  $E^T$  denote the class of all  $T$ -homomorphisms with underlying morphism in  $E$ ; analogously  $M^T$ . Then  $(E^T, M^T)$  is a factorization system in  $K(T)$ ; see [9], 3.4.17. Hence, by I.2 (v),  $k \in E$ . By hypothesis, also  $Tk \in E$ ,  $T^2k \in E$ , and so on.

To show that  $(C, \gamma)$  is indeed a  $\Pi$ -algebra, consider the following diagrams, which clearly commute:



By the first one,  $(\gamma \cdot \eta^C) \cdot k = k$ , hence  $\gamma \cdot \eta^C = 1$  ( $k$  is epi). By the second one,  $(\gamma \cdot \mu^C) \cdot T^2k = (\gamma \cdot T\gamma) \cdot T^2k$ ; hence  $\gamma \cdot \mu^C = \gamma T\gamma$  ( $T^2k$  is epi).

II.4. The following theorem is proved in [2] in a different manner, as a part of a more general study of colimits in  $K(F)$ . (An iterative colimit-construction is exhibited there, generalizing that used in universal algebra.) We present a straightforward proof. The help of Václav Koubek with this proof is gratefully acknowledged.

**THEOREM.** *Let  $K$  be a cocomplete category with a factorization system  $(E, M)$ ; let  $K$  be  $E$ -cowell-powered. Then for every functor  $F : K \rightarrow K$  which preserves  $E$ , the category  $K(F)$  has coequalizers.*

**Proof.** Let  $f, g : (A, \alpha) \rightarrow (B, \beta)$  be arbitrary  $F$ -homomorphisms.

Denote by  $\Omega$  the class of all  $E$ -epis  $t : B \rightarrow T$  in  $K$  with the following property:

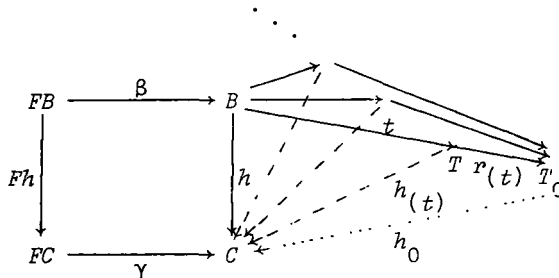
for every  $F$ -homomorphism  $h : (B, \beta) \rightarrow (C, \gamma)$  with  $h.f = h.g$  there exists  $h_{(t)} : T \rightarrow C$  in  $K$  such that  $h = h_{(t)} \cdot t$ .

Since  $K$  is cocomplete and  $E$ -cowell-powered, the diagram  $\Omega$  has a colimit (multiple pushout)

$$(1) \quad r_0 = r_{(t)} \cdot t : B \rightarrow T_0 \quad (r_{(t)} : T \rightarrow T_0 \text{ for each } t \in \Omega) .$$

Each  $t \in \Omega$  is in  $E$ , hence (by I.2 (vii)) each  $r_{(t)}$  is in  $E$ ; thus  $t_0 \in E$  and  $Ft_0 \in E$ .

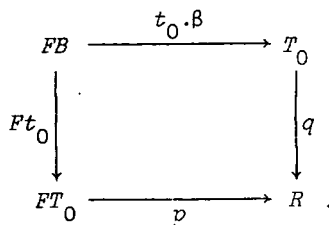
Fix a homomorphism  $h : (B, \beta) \rightarrow (C, \gamma)$  with  $h.f = h.g$ . Then we have a bound of  $\Omega : h_{(t)} : T \rightarrow C$  ( $t \in \Omega$ ). Thus there exists



a unique  $h_0 : T_0 \rightarrow C$  with

$$(2) \quad h_0 \cdot r_{(t)} = h_{(t)} \quad (t \in \Omega) \text{ and } h_0 \cdot t_0 = h .$$

Consider the pushout of  $Ft_0$  and  $t_0 \cdot \beta$ :



CLAIM.  $q$  is an isomorphism. It suffices to show that  $q \cdot t_0 \in \Omega$ ; then by (1),  $t_0 = r_{(q \cdot t_0)} \cdot q \cdot t_0$ , which implies  $1 = r_{(q \cdot t_0)} \cdot q$  since  $t_0$

is epi) and so  $q$  is a split mono as well as an  $E$ -epi (opposite  $Ft_0 \in E$  in a pushout, see I.2 (vi)) - thus,  $q$  is an isomorphism. To show  $q.t_0 \in \Omega$  we first remark that, since  $q \in E$  and  $t_0 \in E$  we have  $q.t_0 \in E$ . Secondly, consider any homomorphism  $h : (B, \beta) \rightarrow (C, \gamma)$  with  $h.f = h.g$ : we have  $h_0.t_0 = h$  by (2) and  $h.\beta = \gamma.Fh$ , hence

$$h_0.(t_0.\beta) = \gamma.Fh = (\gamma.Fh_0).Ft_0 .$$

This implies that the pair  $h_0; (\gamma.Fh_0)$  factorizes through  $q; p$  above; that is, there is a unique morphism, denoted by  $h_{(q.t_0)}$  from  $R$  to  $C$ , with  $h_0 = h_{(q.t_0)}.q$  and  $\gamma.Fh_0 = h_{(q.t_0)}.p$ . The first implies  $h = h_{(q.t_0)}.(q.t_0)$ , by (2). Thus,  $q.t_0 \in \Omega$  and  $q$  is an isomorphism.

Let us show that the  $F$ -homomorphism  $t_0 : (B, \beta) \rightarrow (T_0, q^{-1}.p)$  is a coequalizer of  $f$  and  $g$  in  $K(F)$ .

Firstly,  $t_0.f = t_0.g$ : indeed, consider the coequalizer  $c$  of  $f$  and  $g$  in  $K$ ;  $c \in E$  by I.2 (v), and clearly  $c \in \Omega$ . Hence  $t_0 = r_{(c)}.c$ , which proves  $t_0.f = t_0.g$ .

Secondly, for every homomorphism  $h : (B, \beta) \rightarrow (C, \gamma)$  with  $h.f = h.g$  we have  $h_0 : T_0 \rightarrow C$  with  $h = h_0.t_0$ , by (2). This  $h_0$  is unique, because  $t_0$  is epi. To conclude the proof we only have to show that  $h_0$  is a homomorphism; that is, that  $h_0.(q^{-1}.p) = \gamma.Fh_0$ . We use (2) and the fact that  $Ft_0 \in E$  is epi, and that  $p.Ft_0 = q.t_0.\beta$  (see the pushout above):

$$\begin{aligned} [h_0.(q^{-1}.p)].Ft_0 &= h_0.(q^{-1}.q).t_0.\beta \\ &= h.\beta \\ &= \gamma.Fh \\ &= [\gamma.Fh_0].Ft_0 . \end{aligned}$$

II.5. COROLLARY. *Let  $K$  and  $(E, M)$  be as in II.4. Then for every*

algebraic theory  $\Pi$  which preserves  $E$ , the category  $K^\Pi$  is cocomplete.

Proof. By II.4, the category  $K(T)$  has coequalizers, hence (by II.3) so does  $K^\Pi$ . By I.1 this implies the cocompleteness.

**COROLLARY.** *Let  $K$  be a cowell-powered, cocomplete category. Then for every algebraic theory  $\Pi$  preserving epis, also  $K^\Pi$  is cocomplete.*

Proof. It is proved in [5] (the dual to 34.1) that  $K$  has a factorization system  $(E, M)$  with  $E$  equal to all epis,  $M$  equal to all extremal monos.

II.6. The latter corollary is proved in [1] in the same way as in the present paper. The first corollary was first formulated by Reiterman. See [6], where a completely different method is used (related to that used in [2] to prove Theorem II.4 above).

### III. ... but not always!

III.1. We denote by  $Gra$  the category of graphs and compatible mappings. A graph is a pair  $A = \langle A, K \rangle$  consisting of a set  $A$  and a subset  $K$  of  $A \times A$ . A compatible mapping  $f : \langle A, K \rangle \rightarrow \langle B, L \rangle$  is a mapping  $f : A \rightarrow B$  for which  $(x, y) \in K$  implies  $(f(x), f(y)) \in L$ .

$Gra$  is a complete and cocomplete concrete category, with underlying functor  $Gra \rightarrow Set$  creating all limits and colimits; it is also a well-powered and cowell-powered category and is, in one word, decent.

III.2. We shall define an input process  $F$  in  $Gra$  such that the category  $Gra(F)$  of  $F$ -algebras is not cocomplete. Before doing this, we shall make a simple observation about  $P$ -algebras, where  $P : Set \rightarrow Set$  is the power-set functor (sending a set  $X$  to the power set  $PX = 2^X$  and a mapping  $f : X \rightarrow Y$  to the mapping

$$Pf : A \mapsto \{f(a); a \in A\}.$$

We recall that an object  $O$  of a category is *weak initial* if for every other object  $X$  there exists at least one morphism from  $O$  to  $X$ .

**LEMMA.** *The category  $Set(P)$  of  $P$ -algebras has no weak initial object.*

Proof. It is easy to see that  $Set(P)$  is a complete category (with



limits created by the forgetful functor  $Set(P) \rightarrow Set$  ). Thus the existence of a weak initial object would imply the existence of an initial object; see [8].

Now let  $(A, \alpha)$  be an initial  $P$ -algebra. Barr proves in [4] that  $\alpha$  is then an isomorphism. But there exists no isomorphism from a power set  $PA$  to  $A$ , of course; a contradiction.

III.3. We start by defining a functor  $F : Gra \rightarrow Gra$ . First, for every graph  $A = \langle A, K \rangle$ , define a set

$$A^{(3)} = \{(x, y, z) \in A \times A \times A; (x, y) \in K \text{ and } (y, z) \in K\} .$$

Given a compatible mapping  $f : A \rightarrow B$ , define a mapping

$$f^{(3)} : A^{(3)} \rightarrow B^{(3)} \text{ by}$$

$$f^{(3)} : (x, y, z) \mapsto (f(x), f(y), f(z)) .$$

Now define  $F$  as follows: for each graph  $A$  put

$$FA = \langle PA^{(3)}, M_A \rangle \text{ where } (X, Y) \in M \text{ iff } X = \emptyset \text{ and } Y \neq \emptyset \text{ } (X, Y \subset A^{(3)}) ;$$

for each compatible map  $f : A \rightarrow B$  put

$$Ff = Pf^{(3)} .$$

Clearly,  $Pf^{(3)} : FA \rightarrow FB$  is compatible and  $F$  is a correctly defined functor.

III.4. LEMMA. *F is an input process.*

Proof. For every graph  $A$  define a new graph

$$A^\# = A \vee FA$$

and notice that  $(FA)^{(3)} = \emptyset$ ; hence  $FA^\# = FA$ . Denote by  $\delta^A : A \rightarrow A^\#$ ,  $\varphi^A : FA^\# = FA \rightarrow A^\#$  the canonical injections. Then  $(A^\#, \varphi^A)$  is a free  $F$ -algebra generated by  $A$  with universal morphism  $\delta^A$ .

Indeed, let  $(B, \beta)$  be an  $F$ -algebra and let  $f : A \rightarrow B$  be a morphism (that is a compatible mapping). Define  $f^\# : A^\# \rightarrow B$  by

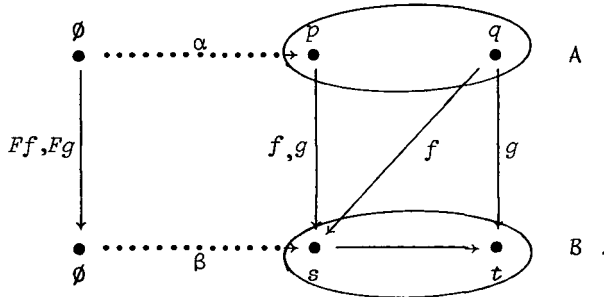
$$(3) \quad f^\# \cdot \varphi^A = \beta \cdot Ff ,$$

$$(4) \quad f^\# \cdot s^A = f .$$

Since  $(FA)^{(3)} = \emptyset$ , clearly  $Ff^\# = Ff$ , and so (3) means that  $f^\# : (A^\#, \varphi^A) \rightarrow (B, \beta)$  is an  $F$ -homomorphism; by (4),  $f^\#$  extends  $f$ . The uniqueness of  $f^\#$  follows from the fact that (3) and (4) are actually necessary.

III.5. We define a pair  $f, g : (A, \alpha) \rightarrow (B, \beta)$  of  $F$ -homomorphisms of which we shall prove that they do not have a coequalizer in  $\text{Gra}(F)$ .

Let  $A = \langle \{p, q\}, \emptyset \rangle$  and  $B = \langle \{s, t\}, \{(s, t)\} \rangle$ .



Clearly  $A^{(3)} = B^{(3)} = \emptyset$ , hence  $FA = FB = \langle \{\emptyset\}, \emptyset \rangle$ . Define  $\alpha : FA \rightarrow A$  by  $\alpha(\emptyset) = p$ ;  $\beta : FB \rightarrow B$  by  $\beta(\emptyset) = s$ .

Finally, define  $f, g : \{p, q\} \rightarrow \{s, t\}$  by

$$f(p) = g(p) = f(q) = s \quad \text{and} \quad g(q) = t .$$

Clearly,  $f$  is a homomorphism with  $f \cdot \alpha = \beta \cdot Ff : \emptyset \mapsto s$ , analogously  $g$ .

III.6. Assuming that  $f, g$  have a coequalizer  $c : (B, \beta) \rightarrow (C, \gamma)$  in  $\text{Gra}(F)$ , we shall find a weak initial object in  $\text{Set}(P)$  - a contradiction.

We have  $C = \langle C, K \rangle$ . Put  $\bar{s} = c(s)$  ( $= c(t)$ , because  $c \cdot f = c \cdot g$ ). Since  $c : B \rightarrow C$  is compatible, clearly  $(\bar{s}, \bar{s}) \in K$ .

Put

$$C_0 = \{x \in C; (\bar{s}, x) \in K\} .$$

For every subset  $X \subset C_0$  put

$$\hat{X} = \{(\bar{s}, \bar{s}, x); x \in X\} \in PC^{(3)} .$$

We have  $\gamma(\hat{X}) \in C$  - let us show that, in fact,  $\gamma(\hat{X}) \in C_0$ . If  $X = \emptyset$ , then  $\hat{X} = \emptyset$  and  $\gamma(\emptyset) = \bar{s} \in C_0$ , because  $\gamma.Fc = c.\beta$  and  $Fc(\emptyset) = \emptyset$ ; thus

$$\gamma(\emptyset) = c\{\beta(\emptyset)\} = c(s) = \bar{s} .$$

If  $X \neq \emptyset$ , then  $\hat{X} \neq \emptyset$  and so  $(\emptyset, \hat{X}) \in M_C$  (see III.3). Since  $\gamma : FC \rightarrow C$  is compatible, this yields  $(\gamma(\emptyset), \gamma(\hat{X})) \in K$ ; that is,

$$(\bar{s}, \gamma(\hat{X})) \in K ; \text{ thus } \gamma(\hat{X}) \in C_0 .$$

Now we define a  $P$ -algebra  $(C_0, \hat{\gamma})$  by

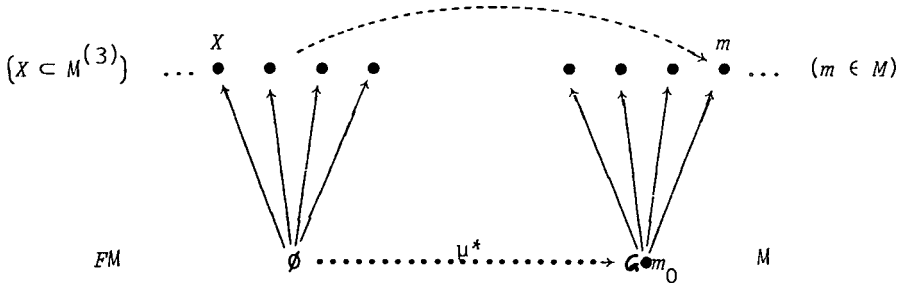
$$\hat{\gamma}(X) = \gamma(\hat{X}) \quad (X \subset C_0) .$$

This  $P$ -algebra is weak initial.

Proof. Let  $(M, \mu)$  be another  $P$ -algebra, that is, a set  $M$  and a mapping  $\mu : PM \rightarrow M$ ; put  $m_0 = \mu(\emptyset)$ . Define an  $F$ -algebra

$(M, \mu^*) : M = \langle M, \{(m_0, m); m \in M\} \rangle$  where  $\mu^* : FM \rightarrow M$  is defined by

$$\mu^*(X) = \mu\{m \in M; (m_0, m) \in X\} .$$



Particularly,  $\mu^*(\emptyset) = m_0$ . Thus  $h : (B, \beta) \rightarrow (M, \mu^*)$ , defined by  $h(s) = h(t) = m_0$ , is an  $F$ -homomorphism. Since  $h.f = h.g$ , there exists an  $F$ -homomorphism  $k : (C, \gamma) \rightarrow (M, \mu^*)$  such that  $k.c = h$  - particularly,  $k(\bar{s}) = m_0$ .

The proof will be concluded when we show that the restriction  $k_0 : C_0 \rightarrow M$  of  $k$  is a  $P$ -homomorphism; that is, that  $k_0.\hat{\gamma} = \mu.Pk_0$ . Given  $X \subset C_0$  we have

$$k_0.\hat{\gamma}(X) = k.\gamma(\hat{X}) = \mu^*.Fk(\hat{X}) .$$

Furthermore,  $\hat{X} = \{(\bar{s}, \bar{s}, x); x \in X\}$  implies

$$Fk(\hat{X}) = \{ \{m_0, m_0, k(x)\}; x \in X \}$$

and  $k(x) = k_0(x)$  for  $x \in X$  (since  $X \subset C_0$ ); thus

$$\mu^*.Fk(\hat{X}) = \mu(\{k_0(x); x \in X\}) = \mu.Pk_0(X) .$$

Thus  $(C_0, \hat{\gamma})$  is a weak initial  $P$ -algebra, in contradiction to Lemma III.2.

CONCLUSION. The free algebraic theory  $\Pi$  generated by the above input process  $F$  in  $Gna$  is such that  $Gna^\Pi$  is not complete. Explicitly,  $\Pi = (T, \mu, \eta)$  with

$$TA = A \vee FA \quad (\eta^A : A \rightarrow TA \text{ and } \varphi^A : FA \rightarrow TA \text{ canonical})$$

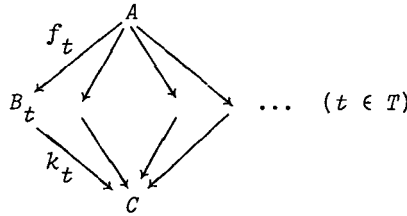
and  $\mu^A : T^2A = (A \vee FA) \vee FA \rightarrow TA$  is defined on  $A$  as  $\eta^A$  and on both copies of  $FA$  as  $\varphi^A$ .

#### IV. Appendix on factorizations

IV.1. Barr's right factorization systems. For the colimit theorem of I.3, Barr [4] uses a right factorization system, which is a pair  $(E, M)$  as in I.2, except that  $E$ -morphisms need not be epis. More precisely, a right factorization system consists of a class  $E$  of morphisms and a class  $M$  of monos such that conditions (ii)-(iv) of I.2 are fulfilled.

There always exists a simple right factorization system:  $E$  equals all morphisms,  $M$  equals all isomorphisms. In this case,  $K$  is seldom  $E$ -cowell-powered (as required in the colimit theorem I.3). We shall show that this is no coincidence.

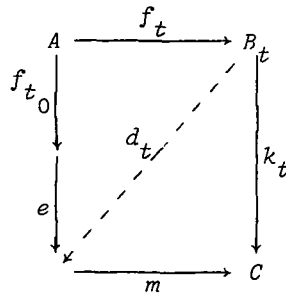
LEMMA. *For each right factorization system  $(E, M)$  and each multiple pushout*



with  $f_t \in E$  for  $t \in T$ , also  $k_t \in E$  for  $t \in T$ .

Proof. Choose  $t_0 \in T$  and let  $k_{t_0} = m.e$  be an  $E$ - $M$ -factorization.

For every  $t \in T$  use the diagonal fill-in:

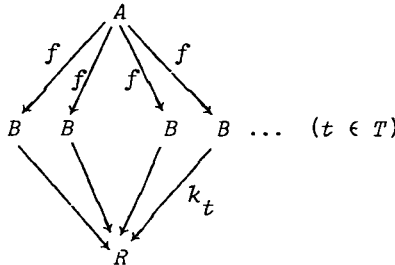


to obtain  $d_t$  with  $d_t.f_t = e.f_{t_0}$  (hence  $d_t$  is a bound of the pushout) and  $m.d_t = k_t$ . There exists a unique  $d$  with  $d_t = d.k_t$  ( $t \in T$ ). Then  $(m.d).k_t = k_t$  ( $t \in T$ ); hence  $m.d = 1$ . Since  $m \in M$ ,  $m$  is a mono as well as a split epi - thus  $m$  is an isomorphism. This shows that  $k_{t_0} = m.e$  is in  $E$ .

**PROPOSITION.** Every right factorization system  $(E, M)$  in a cocomplete,  $E$ -cowell-powered category is a factorization system (that is, all  $E$ -morphisms are epis).

Proof. Assume that  $K$  is a cocomplete category with a right factorization system  $(E, M)$ . Given  $f : A \rightarrow B$  in  $E$  which is not epi, we shall show that  $K$  is not  $E$ -cowell-powered. Indeed, if  $K$  is  $E$ -cowell-powered, there exist  $q_i : B \rightarrow Q_i$  ( $i \in I$ ,  $I$  a set) in  $E$  such that each  $E$ -morphism with domain  $B$  is isomorphic to some  $q_i$ . Choose a cardinal  $\lambda$  such that  $\text{card } \text{hom}(B, Q_i) < \lambda$  for each  $i \in I$ .

Let  $\{k_t\}_{t \in T}$  be the multiple



pushout of a  $T$ -indexed family of copies of  $f$ , where  $T$  is a set of power  $\lambda$ . By the above lemma,  $k_t \in E$  for each  $t \in T$ . To conclude the proof it suffices to show that the  $k_t$ 's are pairwise distinct: then  $\text{card hom}(B, R) \geq \lambda$ ; hence  $R$  is not isomorphic to any of  $Q_i$ .

Since  $f$  is not epi, there exist distinct morphisms  $g_1, g_2 : B \rightarrow C$  with  $g_1 \cdot f = g_2 \cdot f$ . Consider the following bound  $g_{n_t} : B \rightarrow C$  of the above pushout: for a given  $t_0 \in T$ ,  $n_{t_0} = 1$ ; else  $n_t = 2$ . There exists a unique  $h : R \rightarrow C$  with  $g_{n_t} = h \cdot k_t$  ( $t \in T$ ). Since  $g_{n_{t_0}} \neq g_{n_t}$ , we have  $k_{t_0} \neq k_t$  for each  $t_0 \neq t$ . This shows that the  $k_t$ 's are pairwise distinct.

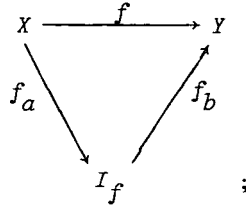
**IV.2. Linton's factorization functors.** Denote by  $K^2$  the morphism category of  $K$  and by  $K^3$  the triangle category of  $K$  (objects are triples  $(f; g, h)$  of  $K$ -morphisms with  $f = g \cdot h$ ; morphisms are triples  $(p, r, q) : (f; g, h) \rightarrow (f'; g', h')$  of  $K$ -morphisms with  $r \cdot h = h' \cdot p$  and  $q \cdot g = g' \cdot q$ ). There is a natural forgetful functor  $\gamma : K^3 \rightarrow K^2$  (sending  $(f; g, h)$  to  $f$ ).

Linton [7] uses a factorization functor, that is a functor

$$\Delta : K^2 \rightarrow K^3$$

such that

(1)  $\gamma \circ \Delta = 1$  ; for  $f : X \rightarrow Y$  in  $K^2$  ,  $\Delta(f)$  is denoted by



(2)  $f_a$  is epi,  $f_b$  is mono, for each  $f \in K^2$  ;

(3)  $(f_a)_b$  and  $(f_b)_a$  are isomorphisms, for each  $f \in K^2$  .

PROPOSITION. (A) Given a factorization functor  $\Delta$  , the pair  $(E_\Delta, M_\Delta)$  with

$$E_\Delta = \{f \in K^2; f_b \text{ is an isomorphism}\} ,$$

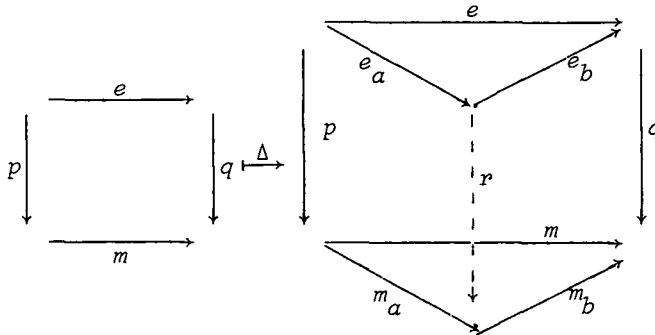
$$M_\Delta = \{f \in K^2; f_a \text{ is an isomorphism}\} ,$$

is a factorization system in  $K$  .

(B) If  $\Delta, \Delta'$  are naturally equivalent factorization functors then  $E_\Delta = E_{\Delta'}$  , and  $M_\Delta = M_{\Delta'}$  .

(C) For every factorization system  $(E, M)$  there exists a factorization functor  $\Delta$  , unique up to natural equivalence, with  $(E, M) = (E_\Delta, M_\Delta)$  .

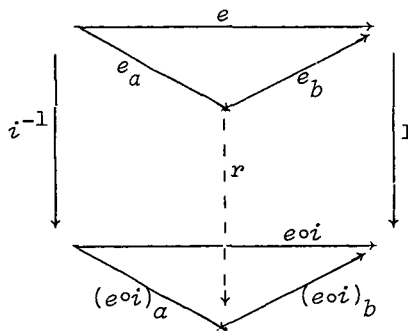
Proof. (A). Conditions (i), (iii) in I.2 are evident. Condition (iv) follows from the fact that  $(p, q) : e \rightarrow m$  is a morphism in  $K^2$  ; hence we have  $\Delta(p, q) : \Delta(e) \rightarrow \Delta(m)$  in  $K^3$  .



Since  $\gamma \circ \Delta = 1$ , clearly  $\Delta(p, q) = (p, r, q)$  for some morphism  $r : I_e \rightarrow I_m$ , making the above diagram commute. The diagonal morphism is then

$$d = m_a^{-1} \cdot r \cdot e_b^{-1} .$$

Finally, to verify condition (ii) it suffices to show that  $E$  and  $M$  are closed to composition with isomorphisms; see [5], 33.3. Let us verify, for example, that  $e \in E$  implies  $e \circ i \in E$  whenever  $i$  is an isomorphism; the rest is analogous. Consider  $(i^{-1}, 1) : e \rightarrow e \cdot i$  in  $K^2$ ; we have  $\Delta(i^{-1}, 1)$  of the form  $(i^{-1}, r, 1) : \Delta(e) \rightarrow \Delta(e \circ i)$  in  $K^3$ :



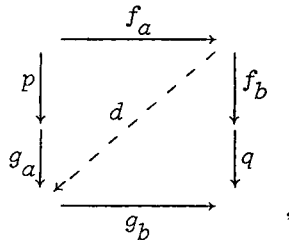
Since  $(e \circ i)_b \circ r \circ e_b^{-1} = 1$ , we see that  $(e \circ i)_b$  is a split epi as well as a mono by (2) above. Hence  $(e \circ i)_b$  is an isomorphism; thus  $e \circ i \in E$ .

(B) follows easily from the fact that both  $E$  and  $M$  are closed to



composition with isomorphisms.

(C). For every morphism  $f$  choose a fixed factorization  $f = f_b \cdot f_a$  with  $f_a \in E$ ,  $f_b \in M$ . Put  $\Delta(f) = (f; f_b, f_a)$ . Given a morphism  $(p, q) : f \rightarrow g$  in  $K^2$ ,



use the diagonal fill-in on the square above to find  $d$  and define

$$\Delta(p, q) = (p, d, q) : \Delta(f) \rightarrow \Delta(g).$$

This gives rise to a factorization functor  $\Delta : K^2 \rightarrow K^3$  with  $E = E_\Delta$  and  $M = M_\Delta$ . Uniqueness follows from (B).

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