MORITA DUALITY AND FINITELY GROUP-GRADED RINGS

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We give the relation between the (rigid) graded Morita duality and the Morita duality on a finitely group-graded ring and the relation between a left Morita ring and some of its matrix rings.

0. INTRODUCTION

The characterisations of Morita dualities can be found in [6]. (Rigid) graded Morita dualities are characterised in [2]. We use freely the same terminologies and notations on the Morita duality as in [6] and on the (rigid) graded Morita duality as in [2].

Throughout this paper, all rings are associative and have identity, all modules are unitary, G is a finite group with an identity e, and \(|G| = m\). R is a graded ring of type G. \(R - \text{mod} (R - gr)\) denotes the category of all (graded) left R-modules (of type G).

Let \(M_G(R)\) denote the ring of m by m matrices over R with rows and columns indexed by the elements of G. If \(x \in M_G(R)\), we write \(x_{g,h}\) for the entry in \((g, h)\)-position of \(x\). Then if \(x, y \in M_G(R)\), the matrix product of \(xy\) is given by

\[
(xy)_{g,h} = \sum_{i \in G} x_{g,i} y_{i,h}
\]

Following [6], we call the ring

\[
RG = \{x \in M_G(R) \mid x_{g,h} \in R_{g,h^{-1}}\}
\]

is smash product of R with G.

In this paper, we first prove that a graded ring R has a (rigid) graded Morita duality on the left if and only if R has a left Morita duality. Secondly, we probe that a graded ring R has a left Morita duality if and only \(M_n(R)_\mathbb{C}(g)\) has a left Morita duality for every natural number n and every \(\bar{g} = (g_1, g_2, \ldots, g_n) \in G^n\), where

\[
M - n(R)_\mathbb{C}(\bar{g}) = \begin{pmatrix}
\tau_{g_1 g_1^{-1}} & \tau_{g_1 g_2^{-1}} & \cdots & \tau_{g_1 g_n^{-1}} \\
\tau_{g_2 g_1^{-1}} & \tau_{g_2 g_2^{-1}} & \cdots & \tau_{g_2 g_n^{-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{g_n g_1^{-1}} & \tau_{g_n g_2^{-1}} & \cdots & \tau_{g_n g_n^{-1}}
\end{pmatrix}
\]

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Finally, we prove that a graded ring $R$ has a left Morita duality if and only if $R\{H\}$ has a left Morita duality for any subgroup $H$ of $G$, where

$$R\{H\} = \left\{ \alpha \in M_G(R) \mid \alpha_{x,y} \in R_{xH^y-1} \right\} = \bigoplus_{g \in eH^y-1} R_g.$$

1. Graded Morita Duality and Morita Duality

Let $xe_g$ denote the column vector with $x$ in the $g$-position and zero in the other positions and $e(g, h)$ denote the $m$ by $m$ matrix with the identity of $R$ in the $(g, h)$-position and the zero of $R$ in the other positions. For every graded left $R$-module $M = \bigoplus_{g \in G} gM$, we denote $F(M) = \bigoplus_{g \in G} gMe_g$ and define $r \cdot \bar{m} = \sum_g \left( \sum_h r_{g,h} m \right) e_g$ for every $r \in R\#G$ and $\bar{m} = \sum_{g \in G} gme_g \in F(M)$. So $F(M)$ is a left $R\#G$-module with this scalar multiplication and the column vector addition. Conversely, for every left $R\#G$-module $N$, we denote $G(N) = \bigoplus_{g \in G} (e(g \cdot g)N)$ and define $r \cdot n = \sum_{g \in G} r_{g} e(gh, h)n$ for every $r \in R$ and $n \in N$. Let $gG(N) = e(g \cdot g)N$ for every $g \in G$, then $G(N)$ is a graded left $R$-module of type $G$ with this scalar multiplication and the original addition.

We view every graded left $R$-homomorphism $f: M \to N$ as left $R\#G$-homomorphism

$$F(f): F(M) \to F(N)$$

and view every left $R\#G$-homomorphism $g: U \to V$ as a graded left $R$-homomorphism $G(g): G(V) \to G(V)$. Following [9], we know that $F: R - gr \to R\#G - \text{mod}$ and $G: R\#G - \text{mod} \to R - gr$ are functors such that $FG = 1$ and $GF = 1$. It is clear that a graded left $R$-homomorphism $f: M \to N$ is monic (epic) if and only if $F(f): F(M) \to F(N)$ is monic (epic) and a left $R\#G$-homomorphism $g: U \to V$ is monic (epic) if and only if $G(g): G(U) \to G(V)$ is monic (epic). So the lattice of submodules of a left $R\#G$-module $U$ is isomorphic to the lattice of graded submodules of $G(U)$. Then we have the following.

**Lemma 1.1.** Let $M$ be a left $R\#G$-module, then

1. $M$ is injective if and only if $G(M)$ is gr-injective.
2. $M$ is finitely cogenerated if and only if $G(M)$ is finitely gr-cogenerated.
3. $M$ is cogenerator if and only if \{$(g)G(M) \mid g \in G$\} is a set of cogenerators in $R - gr$.

**Definition 1.2:** (1) Suppose $M$ is a left $R$-module $m_i \in M$, $M_i$ is a submodule of $M$, $i \in I$. A family $\{m_i, M_i\}_{i \in I}$ is called soluble in case there is an $m \in M$ such that $m - m_i \in M_i$ for all $i \in I$, it is called finitely soluble if $\{m_i, M_i\}_{i \in I}$ is soluble.
for any finite subset $F \subseteq I$, and the module $M$ is called linearly compact in case any finitely solvable family of $M$ is solvable.

(2) Let $M$ be a graded left $R$-module. A pair $(m, N)$ is called a homogeneous pair of degree $g$ if $N$ is a graded submodule of $M$ and $m \in_g M$. A homogeneous family of $M$ is a family of homogeneous pairs all of them with the same degree and the graded left $R$-module $M$ is called $gr$-linearly compact in case any finitely solvable homogeneous family of $M$ is solvable.

**Lemma 1.3.** (1) $RR$ is $gr$-linearly compact if $R#GR$ is linearly compact.

(2) $G(M)$ is $gr$-linearly compact if a left $R#G$-module $M$ is linearly compact.

**Proof:** (1) Suppose that $\{m_i, M_i\}_{i \in I}$ is a finitely solvable homogeneous family of $RR$ with the same degree $g$. Let $M_i^# = \{\alpha \in M_G(R) \mid \alpha_{gh} \in e_{gh^{-1}} M_i\}$ and $m_i^# = \sum_{h \in G} m_i e(gh, h), i \in I$. For every finite subset $F \subseteq I$, $\{m_i, M_i\}_{i \in F}$ is solvable, so there is a $m_F \in M$ such that $m_F - m_i \in M_i, i \in F$, so $g m_F - m_i \in e M_i, i \in F$. Let $m_F^# = \sum_{h \in G} g m_F e(gh, h)$, then $m_F^# \in R#G$ such that $m_F^# - m_i^# \in M_i^#$, $i \in F$.

So $\{m_i^#, M_i\}_{i \in I}$ is finitely solvable. Since $R#GR#G$ is linearly compact, there is an $r \in R#G$ such that $r - m_i^# \in M_i^#$, $i \in I$, so $r g_e - m_i \in e M_i \subseteq M_i, i \in I$, so $RR$ is $gr$-linearly compact.

(2) Suppose that $\{n_i, N_i\}_{i \in I}$ is a finitely solvable homogeneous family of $G(M)$ with the same degree $g (g \in G)$ and $n_i = e(g, g)m_i, m_i \in M, i \in I$. Let $M_i = F(N_i), i \in I$. For any finite subset $F \subseteq I$, $\{n_i, N_i\}_{i \in F}$ is solvable. So there is an $n_F \in G(M)$ such that

$$n_F - n_i \in N_i, i \in F, \text{ so } g n_F - n_i \in e N_i, i \in F.$$ Let $n_F = \sum_{h \in G} e(h, h)m^{(h)}$, then

$$e(g, g)m^{(g)} - e(g, g)m = e(g, g)\left(m^{(h)} - m_i\right) \in e N_i, i \in F$$

Since $N_i = G(M_i), i \in I, g N_i = e(g, g)M_i, i \in F, m^{(g)} - m_i \in M_i, i \in F$. Therefore, $\{m_i, M_i\}_{i \in I}$ is finitely solvable. $M$ is linearly compact, so there is $m \in M$ such that $m - m_i \in M_i, i \in I$. Let $n = e(g, g)m$, then $n \in G(M)$ such that $n - n_i \in e G(M_i) = N_i, i \in I$, so $G(M)$ is $gr$-linearly compact.

**Theorem 1.4.** A graded ring $R$ has a left Morita duality if and only if $R$ has a rigid graded Morita duality on the left.

**Proof:** If $R$ has a rigid graded Morita duality on the left, then $R$ has a left Morita duality by [3, Proposition 4.3].

Conversely, if $R$ has a left Morita duality then $R#G$ has a left Morita duality by [8] Theorem 3.9. Suppose that $R#G$ has a left Morita duality induced by a left
$R\#G$-module $W$, then $R\#G R\#G$ is linearly compact and $W$ is a linearly compact finitely cogenerated injective cogenerator by [7] Theorem 4.5. So $G(W)$ is a $gr$-finitely cogenerated $gr$-linearly compact left $R$-module such that $(g)G(W) \mid g \in G$ is a set of cogenerators of $R - gr$ by Lemma 1.1 and 1.3, and $R R$ is $gr$-linearly compact by Lemma 1.3. Therefore $R$ has a rigid graded. Morita duality on the left by [3] Theorem 5.19.

2. Morita Rings and Matrix Rings

For any natural number $n$ and every $\bar{g} = (g_1, g_2, \ldots, g_n) \in G^n$ and every $h \in G$, let

$$M_n(R)_h(\bar{g}) = \left\{ \begin{pmatrix} \tau_{g_1 h g_1^{-1}} & \tau_{g_2 h g_2^{-1}} & \cdots & \tau_{g_n h g_n^{-1}} \\ \tau_{g_2 h g_1^{-1}} & \tau_{g_2 h g_2^{-1}} & \cdots & \tau_{g_n h g_2^{-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{g_n h g_1^{-1}} & \tau_{g_n h g_2^{-1}} & \cdots & \tau_{g_n h g_n^{-1}} \end{pmatrix} : \tau_{g_i h g_j^{-1}} \in R_{g_i h g_j^{-1}} \right\}$$

and $M_n(R)(\bar{g}) = \bigoplus_{h \in G} M_n(R)_h(\bar{g})$, then $M_n(R)_e(\bar{g})$ a ring with the matrix multiplication and the matrix addition and $M_n(R)(\bar{g})$ is a graded ring of type $G$, we have

**Theorem 2.1.** If $R$ has a left Morita duality, then $M_n(R)_e(\bar{g})$ has a left Morita duality for every natural number $n$ and every $\bar{g} \in G^n$. Conversely, if $M_n(R)_e(\bar{g})$ has a left Morita duality for some natural number $n$ and every $\bar{g} \in G^n$, then $R$ has a left Morita duality.

**Proof:** If $R$ has a left Morita duality, and let $E$ be the minimal injective cogenerator of $R_e - \text{mod}$, then $R_e$ has a left Morita duality and $R_g$ and $\text{Hom}_{R_g}(R_e, E)$ are linearly compact for every $g \in G \setminus \{e\}$ by [2] Theorem 2.3. Following [4], we know the matrix ring $M_n(R)_e(\bar{g})$ has a left Morita duality for every natural number $n$ and every $\bar{g} \in G^n$.

Conversely, if the matrix ring $M_n(R)_e(\bar{g})$ has a left Morita duality for some natural number $n$ and every $\bar{g} \in G^n$, then, following [4], $R_e$ has a left Morita duality, and $R_{g_i g_j^{-1}}$ and $\text{Hom}_{R_e}(R_{g_i g_j^{-1}}, E)$ are linearly compact, $i, j = 1, 2, \ldots, n$, so $R_e$ has a left Morita duality and $R_g$ and $\text{Hom}_{R_g}(R_g, E)$ are linearly compact for every $g \in G \setminus \{e\}$. Following [2] Theorem 2.3, $R$ has a left Morita duality.

**Theorem 2.2.** If $R$ is a strongly graded ring and $M_n(R)(\bar{g})$ has a left Morita duality for some natural number $n$ and some $\bar{g} \in G^n$, then $R$ has a left Morita duality.

**Proof:** If $R$ is a strongly graded ring, then $M_n(R)(\bar{g})$ is a strongly graded ring by [5] Theorem 1.5.6. $M_n(R)_e(\bar{g})$ has a left Morita duality, so $M_n(R)(\bar{g})$ has a left
Morita duality by [2] Corollary 2.6. Since \( M_n(R)(g) \) is equivalent to \( R \), \( R \) has a left Morita duality by [7] Corollary 4.6.

If \( U \) is a nonempty subset of \( G \), let \( R(U) = \sum_{x \in U} R_x \). Suppose \( H \) is a subgroup of \( G \), we define \( R\{H\} \subseteq M_G(R) \) by

\[
R\{H\} = \{ \alpha \in M_G(R) \mid \alpha_{x,y} \in R_{xHy^{-1}} \}.
\]

**Theorem 2.3.** If \( R \) has a left Morita duality, then \( R\{H\} \) has a left Morita duality for any subgroup \( H \) of \( G \). Conversely, if \( R\{H\} \) has a left Morita duality for some subgroup \( H \) of \( G \), then \( R \) has a left Morita duality.

**Proof:** \( R \) has a left Morita duality, so \( R \# G \) has a left Morita duality by [8] Theorem 3.9, so, \( R\{H\} \) has a left Morita duality by [6] Lemma 1.2 and [2] Corollary 2.6 for every subgroup \( H \) of \( G \). Conversely, if \( R\{H\} \) has a left Morita duality for some subgroup \( H \) of \( G \). Since \( R\{H\} \) is a strongly graded ring by [6] Lemma 1.2, \( R \# G \) has a left Morita duality by [2] Corollary 2.6. So \( R \) has a left Morita duality by [8] Theorem 3.9.

**References**


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