MORITA DUALITY AND FINITELY GROUP-GRADED RINGS

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We give the relation between the (rigid) graded Morita duality and the Morita duality on a finitely group-graded ring and the relation between a left Morita ring and some of its matrix rings.

0. INTRODUCTION

The characterisations of Morita dualities can be found in [6]. (Rigid) graded Morita dualities are characterised in [2]. We use freely the same terminologies and notations on the Morita duality as in [6] and on the (rigid) graded Morita duality as in [2].

Throughout this paper, all rings are associative and have identity, all modules are unitary, $G$ is a finite group with an identity $e$, and $|G| = m$. $R$ is a graded ring of type $G$. $R$-mod $(R - gr)$ denotes the category of all (graded) left $R$-modules (of type $G$).

Let $M_{G}(R)$ denote the ring of $m$ by $m$ matrices over $R$ with rows and columns indexed by the elements of $G$. If $x \in M_{G}(R)$, we write $x_{g,h}$ for the entry in $(g, h)$-position of $x$. Then if $x,y \in M_{G}(R)$, the matrix product of $xy$ is given by

$$(xy)_{g,h} = \sum_{t \in G} x_{g,t}y_{t,h}$$

Following [6], we call the ring

$$RG = \{x \in M_{G}(R) | x_{g,h} \in R_{g^{-1}h} \}$$

is smash product of $R$ with $G$.

In this paper, we first prove that a graded ring $R$ has a (rigid) graded Morita duality on the left if and only if $R$ has a left Morita duality. Secondly, we prove that a graded ring $R$ has a left Morita duality if and only $M_{n}(R)_{\bar{g}}(g)$ has a left Morita duality for every natural number $n$ and every $\bar{g} = (g_{1}, g_{2}, \ldots, g_{n}) \in G^{n}$, where

$$M - n(R)_{\bar{g}}(\bar{g}) = \left\{ \begin{pmatrix} \tau_{g_{1}g_{1}}^{-1} & \tau_{g_{1}g_{2}}^{-1} & \cdots & \tau_{g_{1}g_{n}}^{-1} \\ \tau_{g_{2}g_{1}}^{-1} & \tau_{g_{2}g_{2}}^{-1} & \cdots & \tau_{g_{2}g_{n}}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{g_{n}g_{1}}^{-1} & \tau_{g_{n}g_{2}}^{-1} & \cdots & \tau_{g_{n}g_{n}}^{-1} \end{pmatrix} \mid \tau_{g_{i}g_{j}}^{-1} \in \bar{R}_{g_{i}g_{j}^{-1}} \right\}$$
Finally, we prove that a graded ring \( R \) has a left Morita duality if and only if \( R\{H\} \) has a left Morita duality for any subgroup \( H \) of \( G \), where

\[
R\{H\} = \left\{ \alpha \in M_G(R) \mid \alpha_{x,y} \in R_{xHy}^{-1} = \bigoplus_{g \in eHy^{-1}} R_g \right\}.
\]

1. Graded Morita Duality and Morita Duality

Let \( xe_g \) denote the column vector with \( x \) in the \( g \)-position and zero in the other positions and \( e(g, h) \) denote the \( m \) by \( m \) matrix with the identity of \( R \) in the \((g, h)\)-position and the zero of \( R \) in the other positions. For every graded left \( R \)-module \( M = \bigoplus_{g \in G} gM \), we denote \( F(M) = \bigoplus_{g \in G} gMe_g \) and define \( r \cdot \widetilde{m} = \sum_{g} \left( \sum_{h} r_{g,h} m_{h} \right) e_g \) for every \( r \in R\#G \) and \( \widetilde{m} = \sum_{g \in G} gme_g \in F(M) \). So \( F(M) \) is a left \( R\#G \)-module with this scalar multiplication and the column vector addition. Conversely, for every left \( R\#G \)-module \( N \), we denote \( G(N) = \bigoplus_{g \in G} (e(g \cdot g)N) \) and define \( r \cdot n = \sum_{g \in G} r_g e(gh, h)n \) for every \( r \in R \) and \( n \in N \). Let \( gG(N) = e(g \cdot g)N \) for every \( g \in G \), then \( G(N) \) is a graded left \( R \)-module of type \( G \) with this scalar multiplication and the original addition.

We view every graded left \( R \)-homomorphism \( f: M \to N \) as left \( R\#G \)-homomorphism

\[
F(f): F(M) \to F(N)
\]

and view every left \( R\#G \)-homomorphism \( g: U \to V \) as a graded left \( R \)-homomorphism \( G(g): G(V) \to G(V) \). Following [9], we know that \( F: R - gr \to R\#G - \) mod and \( G: R\#G - \) mod \( \to R - gr \) are functors such that \( FG = 1 \) and \( GF = 1 \). It is clear that a graded left \( R \)-homomorphism \( f: M \to N \) is monic (epic) if and only if \( F(f): F(M) \to F(N) \) is monic (epic) and a left \( R\#G \)-homomorphism \( g: U \to V \) is monic (epic) if and only if \( G(g): G(U) \to G(V) \) is monic (epic). So the lattice of submodules of a left \( R\#G \)-module \( U \) is isomorphic to the lattice of graded submodules of \( G(U) \). Then we have the following.

**Lemma 1.1.** Let \( M \) be a left \( R\#G \)-module, then

1. \( M \) is injective if and only if \( G(M) \) is gr-injective.
2. \( M \) is finitely cogenerated if and only if \( G(M) \) is finitely gr-cogenerated.
3. \( M \) is cogenerator if and only if \( \{ (g)G(M) \mid g \in G \} \) is a set of cogenerators in \( R - gr \).

**Definition 1.2:** (1) Suppose \( M \) is a left \( R \)-module \( m_i \in M, M_i \) is a submodule of \( M, i \in I \). A family \( \{m_i, M_i\}_{i \in I} \) is called solvable in case there is an \( m \in M \) such that \( m - m_i \in M_i \) for all \( i \in I \), it is called finitely solvable if \( \{m_i, M_i\}_{i \in F} \) is solvable
for any finite subset \( F \subseteq I \), and the module \( M \) is called linearly compact in case any finitely solvable family of \( M \) is solvable.

(2) Let \( M \) be a graded left \( R \)-module. A pair \((m, N)\) is called a homogeneous pair of degree \( g \) if \( N \) is a graded submodule of \( M \) and \( m \in g M \). A homogeneous family of \( M \) is a family of homogeneous pairs all of them with the same degree and the graded left \( R \)-module \( M \) is called \( gr \)-linearly compact in case any finitely solvable homogeneous family of \( M \) is solvable.

**Lemma 1.3.** (1) \( RR \) is \( gr \)-linearly compact if \( R#G \) is linearly compact.

(2) \( G(M) \) is \( gr \)-linearly compact if \( R\#G \)-module \( M \) is linearly compact.

**Proof:** (1) Suppose that \( \{m_i, M_i\}_{i \in I} \) is a finitely solvable homogeneous family of \( RR \) with the same degree \( g \). Let \( M_i^g = \{\alpha \in M_G(R) \mid \alpha_{g,h} \in g^{-1} M_i\} \) and \( m_i^g = \sum_{h \in G} m_i e(g h, h) \), \( i \in I \). For every finite subset \( F \subseteq I \), \( \{m_i, M_i\}_{i \in F} \) is solvable, so there is a \( m_F \in M \) such that \( m_F - m_i \in M_i, i \in F \), \( g m_F - m_i \in g M_i, i \in F \). Let \( m_F^g = \sum_{h \in G} g m_F e(g h, h) \), then \( m_F^g \in R\#G \) such that \( m_F^g - m_i^g \in M_i^g, i \in F \).

So \( \{m_i^g, M_i^g\}_{i \in I} \) is finitely solvable. Since \( R\#G \) is linearly compact, there is an \( r \in R\#G \) such that \( r - m_i^g \in M_i^g, i \in I \), so \( r_{g,h} - m_i \in g M_i \subseteq M_i, i \in I \), so \( RR \) is \( gr \)-lineary compact.

(2) Suppose that \( \{n_i, N_i\}_{i \in I} \) is a finitely solvable homogeneous family of \( G(M) \) with the same degree \( g \) and \( n_i = e(g, g) m_i, m_i \in M, i \in I \). Let \( M_i = F(N_i), i \in I \). For any finite subset \( F \subseteq I \), \( \{n_i, N_i\}_{i \in F} \) is solvable. So there is an \( n_F \in G(M) \) such that

\[
n_F - n_i \in N_i, i \in F, \text{ so } g n_F - n_i \in g N_i, i \in F.\]

Let \( n_F = \sum_{h \in G} e(h, h) m^{(h)} \), then

\[
e(g, g) m^{(g)} - e(g, g) m_i = e(g, g) \left( m^{(h)} - m_i \right) \in N_i, i \in F.
\]

Since \( N_i = G(M_i), i \in F, g N_i = e(g, g) M_i, i \in F, m^{(g)} - m_i \in M_i, i \in F \). Therefore, \( \{m_i, M_i\}_{i \in F} \) is finitely solvable. \( M \) is linearly compact, so there is \( m \in M \) such that \( m - m_i \in M_i, i \in I \). Let \( n = e(g, g) m \), then \( n \in G(M) \) such that \( n - n_i \in G(M_i) = N_i, i \in I \), so \( G(M) \) is \( gr \)-linearly compact.

**Theorem 1.4.** A graded ring \( R \) has a left Morita duality if and only if \( R \) has a rigid graded Morita duality on the left.

**Proof:** If \( R \) has a rigid graded Morita duality on the left, then \( R \) has a left Morita duality by [3, Proposition 4.3].

Conversely, if \( R \) has a left Morita duality then \( R\#G \) has a left Morita duality by [8] Theorem 3.9. Suppose that \( R\#G \) has a left Morita duality induced by a left
$R \# G$-module $W$, then $R \# G R \# G$ is linearly compact and $W$ is a linearly compact finitely cogenerated injective cogenerator by [7] Theorem 4.5. So $G(W)$ is a $gr$-finitely cogenerated $gr$-linearly compact left $R$-module such that $\{(g)G(W) \mid g \in G\}$ is a set of cogenerators of $R - gr$ by Lemma 1.1 and 1.3, and $R R$ is $gr$-linearly compact by Lemma 1.3. Therefore $R$ has a rigid graded. Morita duality on the left by [3] Theorem 5.19.

2. Morita rings and matrix rings

For any natural number $n$ and every $\bar{g} = (g_1, g_2, \ldots, g_n) \in G^n$ and every $h \in G$, let

$$M_n(R)_h(\bar{g}) = \begin{pmatrix}
T_{g_1 h g_1^{-1}} & T_{g_1 h g_2^{-1}} & \cdots & T_{g_1 h g_n^{-1}} \\
T_{g_2 h g_1^{-1}} & T_{g_2 h g_2^{-1}} & \cdots & T_{g_2 h g_n^{-1}} \\
\vdots & \vdots & \ddots & \vdots \\
T_{g_n h g_1^{-1}} & T_{g_n h g_2^{-1}} & \cdots & T_{g_n h g_n^{-1}}
\end{pmatrix} \in R_{g_1 h g_j^{-1}}$$

and $M_n(R)(\bar{g}) = \bigoplus_{h \in G} M_n(R)_h(\bar{g})$, then $M_n(R)_e(\bar{g})$ a ring with the matrix multiplication and the matrix addition and $M_n(R)(\bar{g})$ is a graded ring of type $G$, we have

**Theorem 2.1.** If $R$ has a left Morita duality, then $M_n(R)_e(\bar{g})$ has a left Morita duality for every natural number $n$ and every $\bar{g} \in G^n$. Conversely, if $M_n(R)_e(\bar{g})$ has a left Morita duality for some natural number $n$ and every $\bar{g} \in G^n$, then $R$ has a left Morita duality.

**Proof:** If $R$ has a left Morita duality, and let $E$ be the minimal injective cogenerator of $R_e - \text{mod}$, then $R_e$ has a left Morita duality and $R_g$ and $\text{Hom}_{R_g}(R_e, E)$ are linearly compact for every $g \in G \setminus \{e\}$ by [2] Theorem 2.3. Following [4], we know the matrix ring $M_n(R)_e(\bar{g})$ has a left Morita duality for every natural number $n$ and every $\bar{g} \in G^n$.

Conversely, if the matrix ring $M_n(R)_e(\bar{g})$ has a left Morita duality for some natural number $n$ and every $\bar{g} \in G^n$, then, following [4], $R_e$ has a left Morita duality, and $R_{g_i h g_j^{-1}}$ and $\text{Hom}_{R_e}(R_{g_i h g_j^{-1}}, E)$ are linearly compact, $i, j = 1, 2, \ldots, n$, so $R_e$ has a left Morita duality and $R_g$ and $\text{Hom}_{R_g}(R_g, E)$ are linearly compact for every $g \in G \setminus \{e\}$. Following [2] Theorem 2.3, $R$ has a left Morita duality. 

**Theorem 2.2.** If $R$ is a strongly graded ring and $M_n(R)(\bar{g})$ has a left Morita duality for some natural number $n$ and some $\bar{g} \in G^n$, then $R$ has a left Morita duality.

**Proof:** If $R$ is a strongly graded ring, then $M_n(R)(\bar{g})$ is a strongly graded ring by [5] Theorem 1.5.6. $M_n(R)_e(\bar{g})$ has a left Morita duality, so $M_n(R)(\bar{g})$ has a left

If $U$ is a nonempty subset of $G$, let $R(U) = \sum_{z \in U} R_z$. Suppose $H$ is a subgroup of $G$, we define $R\{H\} \subseteq M_G(R)$ by

$$R\{H\} = \{\alpha \in M_G(R) \mid \alpha_{x,y} \in R_{xH_y^{-1}}\}.$$  

**Theorem 2.3.** If $R$ has a left Morita duality, then $R\{H\}$ has a left Morita duality for any subgroup $H$ of $G$. Conversely, if $R\{H\}$ has a left Morita duality for some subgroup $H$ of $G$, then $R$ has a left Morita duality.

**Proof:** $R$ has a left Morita duality, so $R\#G$ has a left Morita duality by [8] Theorem 3.9, so, $R\{H\}$ has a left Morita duality by [6] Lemma 1.2 and [2] Corollary 2.6 for every subgroup $H$ of $G$. Conversely, if $R\{H\}$ has a left Morita duality for some subgroup $H$ of $G$. Since $R\{H\}$ is a strongly graded ring by [6] Lemma 1.2, $R\#G$ has a left Morita duality by [2] Corollary 2.6. So $R$ has a left Morita duality by [8] Theorem 3.9. 

**References**
