TESTING ON NULL SEQUENCES IS ENOUGH FOR BOCHNER INTEGRABILITY

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Dedicated to Paul R. Halmos on the occasion of his eightieth birthday.

Let \( E \) be a normed space, a Fréchet space or a complete \((DF)\)-space satisfying the dual density condition. Let \( \Omega \) be a Radon measure space. We prove that a function \( f : \Omega \to E \) is Bochner \( p \)-integrable if (and only if) \( f \) is \( p \)-integrable with respect to the topology of uniform convergence on the norm-null sequences from \( E' \).

1. INTRODUCTION

Our first question was: Can one deduce that a function with values in a Banach space is Bochner integrable from the fact that it is integrable for a coarser topology? Of course the answer is negative for the weak topology (see [3, II.3.3 on p.53] for a concrete example or use the Dvoretzky-Rogers theorem in general). In this paper, we want to show that the answer is "Yes" for the topology of uniform convergence on the null sequences of the dual.

Let \( \Omega \) be the measure space, \( E \) be the Banach space and \( \tau \) be the coarser topology. If \( f : \Omega \to E \) is the \( \tau \)-integrable function candidate to be Bochner integrable, two problems are involved here: to prove that absolute integrability with respect to the \( \tau \)-seminorms implies that \( t \to \|f(t)\| \) is in \( L^1 \) and to show that \( f \) is norm-measurable if it is \( \tau \)-measurable. The aim of this paper is to show that there is a (somehow natural) class of spaces for which these two problems have a solution and that this class includes normed spaces, Fréchet spaces, strict \((LF)\)-spaces and complete \((DF)\)-spaces satisfying the dual density condition of Bierstedt and Bonet.

Received 8th November, 1995
This research has been supported by La Dirección General de Investigación Científica y Técnica, project PB94-1460, and by La Consejería de Educación y Ciencia de la Junta de Andalucía

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2. TERMINOLOGY AND NOTATION

In what follows \((\Omega, \Sigma, \mu)\) stands for a \(\sigma\)-finite Radon measure space, where \(\Omega\) is a locally compact \(\sigma\)-compact topological space. Let \((E, \tau)\) be a locally convex space with a topology defined by a family of continuous seminorms \(Q(E, \tau)\). We consider measurability of functions in the sense of Lusin: we say that a function \(f : \Omega \to E\) is \(\tau\)-measurable if there is a sequence \((K_n)\) (that we may take either disjoint or increasing) of compact sets such that the restriction \(f|_{K_n}\) is continuous for every \(n \in \mathbb{N}\), and 
\[
\mu\left(\Omega \setminus \bigcup_n K_n\right) = 0.
\]
When \((E, \tau)\) is metrisable, the notion of a \(\tau\)-measurable function in the sense of Lusin coincides with the usual definition of a strongly measurable function as the \(\mu\)-almost everywhere limit of a sequence of simple functions. If \(\tau_1\) and \(\tau_2\) are two topologies defined on \(E\), the identity \((E, \tau_1) \to (E, \tau_2)\) is said to be universally measurable if (among several equivalent conditions) every \(\tau_1\)-measurable function is also \(\tau_2\)-measurable (for arbitrary \(\Omega\) and \(\mu\)).

A function \(f : \Omega \to E\) is said to be integrable with respect to \(\tau\), or simply \(\tau\)-integrable, if it is \(\tau\)-measurable and the scalar functions \(q(f) : t \in \Omega \to q(f(t)) \in \mathbb{R}\) are in \(L^1(\mu)\) for every \(q \in Q(E, \tau)\). When \((E, \tau)\) is a Banach space, \(\tau\)-integrability equals Bochner integrability. \(L^1(E, \tau)\) will denote the space of all \((\mu\)-almost everywhere equal) \(\tau\)-integrable functions endowed with the locally convex topology defined by the family of seminorms \(f \mapsto \|q(f)\|_1\) as \(q \in Q(E, \tau)\). For \(1 < p \leq \infty\), the space \(L^p(E, \tau)\) is defined in the analogous way.

We say that a locally convex space \((E, \tau)\) has property \((B)\) of Pietsch if for each bounded subset \(M\) of the space \(\ell^1(E, \tau)\) of all absolutely summable sequences in \((E, \tau)\), there exists a disc \(B \subset E\) such that for all \((x_n) \in M\) the following hold: \(x_n \in E_B\) for each \(n\) and \(\sum_n p_B(x_n) \leq 1\), where \(E_B\) is the linear span of \(B\) and \(p_B\) is its natural norm, the gauge of \(B\). In other words, each bounded subset of \(\ell^1(E, \tau)\) is a bounded subset of some \(\ell^1(E_B, p_B)\). Metrisable and \((df)\)-spaces have property \((B)\), for instance. A locally convex space \((E, \tau)\) is said to have property \((BM)\) if it has property \((B)\) and the topology \(\tau\) is metrisable when restricted to bounded sets. Metrisable or, more generally, strict \((LF)\)-spaces have property \((BM)\). For a quasi-complete locally convex space \((E, \tau)\) with property \((BM)\) the identity \((E, \sigma(E, E')) \to (E, \tau)\) is universally measurable [4, 4.13].

We introduced in [4, 3.5] the notion of fundamental \(L^p\)-boundedness as an extension of property \((B)\). Let \(1 \leq p \leq \infty\). A locally convex space \((E, \tau)\) is said to be fundamentally \(L^p\)-bounded, with respect to \((\Omega, \Sigma, \mu)\), if each bounded subset \(M\) of \(L^p(E, \tau)\) is contained in a bounded set of the form
\[
[U_p, B] := \{f \in L^p(E, \tau) : f(t) \in E_B \text{ almost everywhere and } p_B(f) \in U_p\},
\]
where \(B\) is a disc in \(E\) and \(U_p\) stands for the unit ball of \(L^p(\mu)\). This definition...
applied to the particular case of the counting measure on the power set of \( \mathbb{N} \), tells us that fundamental \( \ell^1 \)-boundedness is just property (B) (this is the terminology of [7], by the way).

The dual density condition was introduced by Bierstedt and Bonet in connection with their solution to the problem of when a Köthe echelon space is distinguished. They proved [1, Theorem 5] that a \((DF)\)-space \( E \) satisfies the dual density condition if and only if every bounded subset of \( E \) is metrisable, or if and only if \( \ell^\infty(E,\tau) \) is quasi-barrelled. \((DF)\)-spaces satisfying the dual density condition are quasi-barrelled (but not the opposite!). In particular, for \((DF)\)-spaces property \((BM)\) equals the dual density condition.


3. Results

**Main Theorem.** Let \((E,\tau)\) be a locally convex space. Let \( \tau_0 \) be another locally convex topology on \( E \) coarser that \( \tau \) and such that

1. the identity \((E,\tau_0) \to (E,\tau)\) is universally measurable,
2. every \( \tau_0 \)-bounded subset of \( E \) is also \( \tau \)-bounded,
3. the space \((E,\tau_0)\) is fundamentally \( L^p \)-bounded for some \( p \in [1,\infty) \).

Then a function \( f : \Omega \to E \) is \( p \)-integrable with respect to \( \tau \) if (and only if) \( f \) is \( p \)-integrable with respect to \( \tau_0 \), that is

\[
L^p(E,\tau) = L^p(E,\tau_0)
\]

holds as an equality of vector spaces.

**Proof:** Let \( f : \Omega \to E \) be \( \tau_0 \)-integrable. Since the identity \((E,\tau_0) \to (E,\tau)\) is universally measurable, it follows that the function \( f \) is \( \tau \)-measurable. It remains to prove that for every \( \tau \)-continuous seminorm \( q \) the scalar function \( t \to q(f(t)) \) is in \( L^p(\mu) \). The space \((E,\tau_0)\) is fundamentally \( L^p \)-bounded, therefore there exists a disc \( B \) in \((E,\tau_0)\) such that the scalar function \( t \to p_B(f(t)) \) is in \( L^p(\mu) \). Since \( B \) is also \( \tau \)-bounded, it is contained in some multiple of the unit ball of \( q \), hence \( t \to q(f(t)) \) is in \( L^p(\mu) \), as desired.

We shall give several applications of this theorem.

**Corollary 1.** Let \( E \) be a normed space and let \( \tau_0 \) be the topology of uniform convergence on the sequences that converge to zero in \( E' \). Let \( p \in [1,\infty) \). Then a function \( f : \Omega \to E \) is Bochner \( p \)-integrable if (and only if) \( f \in L^p(E,\tau_0) \).
PROOF: We only have to check that conditions (1) and (3) above hold in this case. A consequence of the Grothendieck-Phillips theorem [9, Part II, I.1. Theorem 3 on p.162] or [10, p.50], is that for a Banach space $E$ the identity $(E, \sigma(E, E')) \rightarrow (E, ||||)$ is universally measurable. It is easy to see, by passing to its completion, that the same holds when $E$ is a non-complete normed space. This proves (1). To see (3) note that $(E, \tau_0)$ is a $(df)$-space [5, 12.4–5], that is, it has a fundamental sequence of bounded sets (the integer multiples of the unit ball) and every norm-null sequence in $E'$ is equicontinuous. Now [4, 3.10] states that $(df)$-spaces are fundamentally $L^p$-bounded for every $p \in [1, \infty)$. 

REMARKS. Given $p \in [1, \infty)$, Corollary 1 tells us, in other words, that if $E$ is a normed space and $f : \Omega \rightarrow E$ is strongly measurable then $f$ is Bochner $p$-integrable provided that for every null sequence $(x_n')$ from $E'$, the scalar function

$$t \in \Omega \rightarrow \sup \{|(f(t), x_n')| : n \in \mathbb{N}\}$$

is $p$-integrable.

The $(df)$-space $(E, \tau_0)$ is not complete if $E$ is not reflexive [5, 12.5.1 and 2]. If $A$ is a measurable set, the integral $\int_A f \, d\mu$ of a function $f \in L^1(E, \tau_0)$ is obtained as the limit of a net of Riemann sums so that they belong, a priori, to the completion of $(E, \tau_0)$ and this completion coincides (as a vector space) with the bidual $E''$ [5, 12.5.1]. However, it follows from Corollary 1 that these integrals are, indeed, elements of $E$.

Corollary 1 also holds for every locally convex topology between $\tau_0$ and the norm topology. For all of these topologies $E$ is again a $(df)$-space.

Let us consider now the situation on the dual $E'$ of a Banach space $E$. The difficulty is to lift integrability from the topology $\tau_0'$ of uniform convergence on the norm-null sequences on $E$ to the norm topology in $E'$. The main problem will be that the behaviour of the measurability is not so good; a measurable function with respect the weak*-topology $\sigma(E', E)$ may not be measurable with respect the norm topology on $E'$ as the cases [9, Exercise 1 and 2 on p.168] $E = \ell^1$ or $E = C[0, 1]$ show.

**Corollary 2.** Let $E$ be a Banach space with dual $E'$ and $p \in [1, \infty)$. Then the following hold.

(a) A strongly measurable function $f : \Omega \rightarrow E'$ is Bochner $p$-integrable if and only if for every null sequence $(x_n)$ in $E$ the scalar function

$$t \in \Omega \rightarrow \sup \{|(x_n, f(t))| : n \in \mathbb{N}\}$$

is in $L^p(\mu)$.

(b) If $E'$ is separable, then a function $f : \Omega \rightarrow E'$ is Bochner $p$-integrable if and only if it is $p$-integrable with respect to the topology $\tau_0'$ of uniform convergence on the norm-null sequences in $E$. 

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https://doi.org/10.1017/S0004972700017767
PROOF: Part (a) can be proved as in the Main Theorem using the fact that $(E', \tau_0')$ is also a $(df)$-space and so is fundamentally $L^p$-bounded. Part (b) follows from a theorem due to Meyer and Schwartz [9, Part I, II.3 Corollary 2 of Theorem 10 on pp.122-124] [10, p.51] stating that if $E'$ is separable then the identity $(E', \sigma(E', E)) \rightarrow (E', ||\cdot||)$ is universally measurable.

By the Banach-Dieudonné theorem, $\tau_0'$ equals the topology of uniform convergence on the compact subsets of $E$ [5, 9.4.3]. Moreover, $(E', \tau_0')$ is not only a $(df)$-space; it is also a complete, Schwartz $(gDF)$-space [5, 9.4.1-3, 11.1.4 and 12.5.2 and 6].

**Corollary 3.** Let $p \in [1, \infty)$ and $(E, \tau)$ be a complete $(DF)$-space with the dual density condition. Let $\tau_0$ be the topology of uniform convergence on the sequences from $E'$ that converge to zero in the strong topology $\beta(E', E)$. Then a function $f : \Omega \rightarrow E$ is $p$-integrable with respect to $\tau$ if (and only if) $f$ is $p$-integrable with respect to $\tau_0$.

**Proof:** Since every $(DF)$-space with the dual density condition has property $(BM)$, condition (1) is a particular case of [4, 4.13]. On the other hand, it is clear that if $(E, \tau)$ is a $(DF)$-space then $(E, \tau_0)$ is a $(df)$-space and the proof finishes as in the proof of Corollary 1.

Corollary 3 can be also obtained as a consequence of Corollary 4 below — the corresponding result for quasi-complete spaces having property $(BM)$ — but the proof of the latter requires more work.

**Lemma.** Let $(E, \tau)$ be a quasi-complete locally convex space with property $(BM)$ and let $\tau_0$ be the topology of uniform convergence on the sequences that converge to zero in $(E', \beta(E', E))$. Then $(E, \tau_0)$ is fundamentally $L^p$-bounded for each $p \in [1, \infty)$.

**Proof:** We start by proving that $(E, \tau_0)$ has property $(B)$. Since $(E, \tau)$ has property $(B)$, it will be enough to prove that if $M$ is a bounded subset of $\ell^1\{E, \tau_0\}$ then $M$ is also bounded in $\ell^1\{E, \tau\}$. Let $q$ be any $\tau$-continuous seminorm and assume that

$$
\sup \left\{ \sum_{n=1}^{\infty} q(x_n) : (x_n) \in M \right\} = \infty.
$$

Then there exists a sequence $\{(x_n^{(k)}) : k = 1, 2, \ldots\} \subset M$ and an increasing sequence of indices $(n_k)$ such that

$$
\sum_{n=1}^{n_k+1} q(x_n^{(k)}) > 2^{2k}.
$$

Let $V \subset E'$ be the polar of the unit ball associated to $q$. Then we can find a sequence
\[(v_n) \subset V\text{ such that }
\sum_{n=1+n_k}^{n_{k+1}} \langle x^{(k)}_{n}, v_n \rangle > 2^{2k} \quad \text{for every } k = 1, 2, \ldots .\]

Since \(V\) is \(\beta(E', E)\)-bounded it follows that the sequence
\[
\frac{v_1}{2^1}, \ldots, \frac{v_{n_1}}{2^1}, \frac{v_{1+n_1}}{2^1}, \ldots, \frac{v_{n_2}}{2^2}, \frac{v_{1+n_2}}{2^2}, \ldots, \frac{v_{n_2}}{2^2}, \ldots
\]
converges to zero in the strong topology \(\beta(E', E)\) and satisfies
\[
\sum_{n=1+n_k}^{n_{k+1}} \langle x^{(k)}_{n}, v_n \rangle > 2^k \quad \text{for every } k = 1, 2, \ldots ,
\]
contradicting the fact that \(M\) is bounded in \(\ell^1\{E, \tau_0\}\).

We now prove that \((E, \tau_0)\) is fundamentally \(L^1(\mu)\)-bounded. By [4, 4.13] the identity \((E, \sigma(E, E')) \to (E, \tau)\) is universally measurable so that the identity \((E, \tau_0) \to (E, \tau)\) will also be universally measurable. (This proves, by the way, that condition (1) of the Main Theorem is satisfied.) Therefore, given \(f \in L^1(E, \tau_0)\) there is a disjoint sequence \((K_n)\) of compact sets such that the restriction \(f|_{K_n}\) is \(\tau\)-continuous for every \(n \in \mathbb{N}\), and \(\mu(\Omega \setminus \cup_n K_n) = 0\). In particular (see [2]), \(f\) will be integrable on every measurable set \(A\) contained in some \(K_n\); that is, there is an element \(\int_A f \, d\mu \in E\) such that
\[
\langle \int_A f \, d\mu, \nu \rangle = \int_A (f(t), \nu) \, d\mu \quad \text{for every } \nu \in E'.
\]

Let \(F \subset L^1(E, \tau_0)\) be a bounded set. For each \(q \in \mathcal{Q}(E, \tau_0)\) take
\[
\rho_q := \sup \left\{ \int_{\Omega} q(f) \, d\mu : f \in F \right\} < \infty.
\]

Let \(F_0\) be the set of all \(E\)-valued sequences of the form
\[
\left( \int_{A_1} f \, d\mu, \int_{A_2} f \, d\mu, \ldots \right)
\]
where \(f \in F\) and \(A_1, A_2, \ldots\) is a sequence of pairwise disjoint, measurable sets with positive and finite measure such that each one of them is contained in some compact set where \(f\) is \(\tau\)-continuous. As pointed out above, for each \(f \in F\) there is at least one such sequence \((A_n)\). For each seminorm \(q \in \mathcal{Q}(E, \tau)\) and each of these sequences, we have
\[
\sum_{n=1}^{\infty} q(\int_{A_n} f \, d\mu) \leq \sum_{n=1}^{\infty} \int_{A_n} q(f) \, d\mu \leq \int_{\Omega} q(f) \, d\mu \leq \rho_q.
\]
This tells us that $F_0$ is a bounded subset of $\ell^1\{E, \tau_0\}$. We have proved above that $(E, \tau_0)$ has property $(B)$, hence there is a closed disc $B \subset E$ such that for every sequence $(z_n) \in F_0$ we have $z_n \in E_B$ for each $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} p_B(z_n) \leq 1$. We shall prove that for every $f \in F$ we have (i) $f(t) \in E_B$ almost everywhere and (ii) the function $t \mapsto p_B(f(t))$ is in the unit ball of $L^1(\mu)$.

(i) If there is $f \in F$ such that $\mu \{ t \in \Omega : f(t) \notin E_B \} > 0$, then there is a compact set $K \subset \Omega$ with positive measure such that $f : K \to E$ is $\tau$-continuous and $f(t) \notin E_B$ for all $t \in K$. By [4, 3.7] for every $n \in \mathbb{N}$ there exists a simple function $z_n : K \to B^\circ \subset E'$ such that $\text{Re} \left( f(t), z_n(t) \right) > n$ for all $t \in K$. If we write $z_n = \sum_{i=1}^{k} v_i \chi_{A_i}$, where $\{A_1, A_2, \ldots, A_k\}$ is a measurable partition of $K$ and $\{v_1, v_2, \ldots, v_k\} \subset B^\circ$, then the sequence

$$\left( \int_{A_1} f d\mu, \int_{A_2} f d\mu, \ldots, \int_{A_k} f d\mu, 0, 0, \ldots \right)$$

is in $F_0$. However, we also have

$$\sum_{i=1}^{k} p_B \left( \int_{A_i} f d\mu \right) \geq \sum_{i=1}^{k} \left| \text{Re} \left( \int_{A_i} f d\mu, v_i \right) \right| = \sum_{i=1}^{k} \left| \int_{A_i} \text{Re} \left( f, v_i \right) d\mu \right| \geq n \sum_{i=1}^{k} \mu(A_i) = n\mu(K),$$

contradicting the boundedness of $F_0$ in $\ell^1\{E_B, p_B\}$.

(ii) Assume that the set of functions $\{p_B(f) : f \in F\}$ is not contained in the unit ball of $L^1(\mu)$. This can happen because this set is not contained in $L^1(\mu)$ at all, or simply because $\|p_B(f)\|_1 > 1$ for some $f \in F$. In either case, we can find a function $f \in F$ and a compact set $K \subset \Omega$, such that the functions $f : K \to E$ and $p_B(f) : K \to \mathbb{R}$ are $\tau$-continuous, and $\|p_B(f) \cdot \chi_K\|_1 > 1 + \delta$, for some positive $\delta$. It is well-known that for $\varphi \in L^1(\mu)$ one has

$$\|\varphi\|_1 = \sup \left\{ \left| \int_{\Omega} \varphi \cdot \theta \, d\mu \right| : \theta \text{ a simple function with } \|\theta\|_\infty \leq 1 \right\},$$

so we can find a simple function $\theta$ in the unit ball of $L^\infty(\mu)$ such that $\int_{K} p_B(f) \cdot \theta \, d\mu > 1 + \delta$; note that we may assume that $\theta$ is non-negative. Again by [4, 3.7], given $\varepsilon > 0$ there is a simple function $z : K \to B^\circ \subset E'$ such that

$$p_B(f(t)) < \text{Re} \left( f(t), z(t) \right) + \varepsilon \quad \text{for all } t \in K.$$

Write $\theta$ and $z$ as

$$\theta = \sum_{i=1}^{k} \alpha_i \chi_{A_i} \quad z = \sum_{i=1}^{k} v_i \chi_{A_i}.$$
where the sets \((A_i)\) are pairwise disjoint and have positive finite measure, and each \(\alpha_i\) is in \([0,1]\). Take the sequence \((x_i) \subset E\) defined by \(x_i := \int_{A_i} f \, d\mu\), for \(i = 1,2,\ldots,k\) and \(x_i = 0\) afterwards. Then \((x_i)\) is in \(F_0\) because each \(A_i\) is contained in \(K\), where \(f\) is \(\tau\)-continuous. Now, since each \(\alpha_i\) is in \([0,1]\) and \(F_0\) is contained in the unit ball of \(\ell^1\{E_B,p_B\}\), we have that

\[
1 \geq \sum_{i=1}^{k} \alpha_i p_B(x_i) = \sum_{i=1}^{k} \alpha_i p_B\left(\int_{A_i} f \, d\mu\right) \geq \sum_{i=1}^{k} \alpha_i \text{Re} \left(\int_{A_i} f \, d\mu, v_i\right)
\]

\[
= \sum_{i=1}^{k} \alpha_i \int_{A_i} \text{Re} \left(f, v_i\right) \, d\mu \geq \sum_{i=1}^{k} \alpha_i \int_{A_i} \left(p_B(f) - \varepsilon\right) \, d\mu
\]

\[
= \int_{K} \theta_{p_B}(f) \, d\mu - \varepsilon \|\theta \chi_K\|_1 > 1 + \delta - \varepsilon \mu(K),
\]

where the last inequality holds because \(\int_{K} p_B(f) \cdot \theta \, d\mu > 1 + \delta\), on the one hand, and \(\|\theta \chi_K\|_1 \leq \|\theta\|_\infty \|\chi_K\|_1 \leq \mu(K)\), on the other. Since \(\varepsilon\) was arbitrary, we obtain a contradiction.

To finish the proof of the Lemma apply \([4,3.6]\); this result tells us that fundamental \(L^1(\mu)\)-boundedness implies fundamental \(L^p\)-boundedness for every \(p \in [1,\infty]\).

**Corollary 4.** Let \(p \in [1,\infty)\) and let \((E,\tau)\) be a Fréchet space or, more generally, a strict \((LF)\)-space. Let \(\tau_0\) be the topology of uniform convergence on the sequences from \(E'\) that converge to zero in the strong topology \(\beta(E',E)\). Then a function \(f : \Omega \to E\) is \(p\)-integrable with respect to \(\tau\) if (and only if) \(f\) is \(p\)-integrable with respect to \(\tau_0\).

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