HANKEL MEASURES ON HARDY SPACE

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We characterise the complex measures $\mu$ on the open unit disk $D$ such that
$\left| \int_D f^2 \, d\mu \right| \leq C\|f\|^2_{H^2}$ for all $f$ in the Hardy space $H^2$. The characterisation involves Carleson measures, the duality between $H^1$ and $BMOA$, and Hankel operators.

Let $D$ be the unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbb{C}$ and denote by $dm$ the two-dimensional Lebesgue measure on $D$. The boundary $\{z : |z| = 1\}$ of $D$ will be written as $\partial D$.

For $p \in [1, \infty)$, define $H^p$ to be the Hardy space of all holomorphic functions $f$ on $D$ for which
$$\|f\|_{H^p} = \sup_{r \in (0,1)} \left[ \frac{1}{2\pi} \int_{\partial D} |f(r\zeta)|^p \, |d\zeta| \right]^{1/p} < \infty.$$ If $f \in H^p$, the radial limit $\lim_{r \to 1} f(r\zeta)$, written $f(\zeta)$, exists for almost every $\zeta \in \partial D$; we may thus identify $f$ with its boundary value.

The Fefferman-Stein duality theorem [4] tells us that $[H^1]^* \cong BMOA$ under the pairing:
$$\langle f, g \rangle = \frac{1}{2\pi} \int_{\partial D} f(\zeta) \overline{g(\zeta)} \, |d\zeta|,$$
where $BMOA$ is the space of all $f \in H^1$ such that
$$\|f\|_{BMOA} = |f(0)| + \sup_{w \in D} \|f \circ \phi_w - f(w)\|_{H^1} < \infty,$$ and $\phi_w : D \to D$ is the Möbius transformation $z \mapsto (w - z)/(1 - \overline{w}z)$, which interchanges $w$ and $0$ (see [1]).

When the author visited Lund University, Sweden, J. Peetre posed the following problem:

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PEETRE’S PROBLEM. Let \( \mu \) be a complex-valued measure on \( D \). What geometric properties must \( \mu \) have in order that

\[
\left| \int_D f^2 \, d\mu \right| \leq C \|f\|_{H^2}^2 \quad \forall f \in H^2 \quad ?
\]

Here and throughout the article, the letter \( C \) stands for a (variable) positive constant.

Before providing an answer to the problem, we would like to make two observations. First, if \( I \) is a subarc of \( \partial D \) with arclength \( |I| \), then \( S(I) \) denotes the Carleson box

\[
\left\{ \tau \zeta \in D : 1 - \frac{|I|}{2\pi} \leq r < 1, \ \zeta \in I \right\}.
\]

A complex measure \( \mu \) on \( D \) satisfying

\[
\sup_{I \in \partial D} \frac{|\mu|(S(I))}{|I|} < \infty
\]

is called a Carleson measure (see [7]). It is known that \( \mu \) is a Carleson measure (see [5, p.63], [10, p.170]) if and only if

\[
\int_D |f|^2 \, d|\mu| \leq C \|f\|_{H^2}^2 \quad \forall f \in H^2.
\]

A complex measure \( \mu \) on \( D \) that satisfies condition (1) will be called a Hankel measure, because Peetre’s problem originates from the study of Hankel matrices (see also [6, 9]). It is clear that a Carleson measure must be a Hankel measure, but not conversely.

Second, any Hankel measure is Möbius invariant in the following sense. Let \( \phi : D \to D \) be the Möbius mapping \( z \mapsto (az + b)/(bz + a) \), where \( a, b \in \mathbb{C} \) and \(|a|^2 - |b|^2 = 1\). Then if \( f \in H^2 \), so too is the function \( \tilde{f} : z \mapsto [f \circ \phi(z)]/(\bar{b}z + \bar{a}) \), and the mapping \( f \mapsto \tilde{f} \) is an isometry on \( H^2 \). So if \( d\mu(z) \) changes to \( (d\mu \circ \phi(z))((\bar{b}z + \bar{a})^2) \), then the best constant in Peetre’s problem does not change. This feature corresponds to the conformal invariance of Carleson measures. See for example [5, p.239].

Before we state our solution to Peetre’s problem, we need two more definitions. Given a measure \( \mu \) on \( D \), we define the function \( P_\mu \) and the Hankel operator \( K_\mu \) (see [8]) by the formulas

\[
P_\mu(z) = \int_D \frac{1}{1 - zw} \, d\mu(w);
\]

\[
(K_\mu f)(z) = \int_D \frac{f(w)}{1 - wz} \, d\mu(w).
\]
THEOREM. Let $\mu$ be a complex measure on $D$. Then the following are equivalent:

(a) $\mu$ is a Hankel measure.

(b) $\left| \int_D f \, d\mu \right| \leq C \|f\|_H^1 \, \forall f \in H^1$.

(c) $P_\mu$ is in $BMOA$.

(d) $\sup_{t \in \partial D} \frac{1}{|t|} \left| \int_{S(t)} \int_D \frac{\overline{w} \, d\mu(w)}{(1 - \overline{w}z)^2} \right|^2 \left(1 - |z|^2\right) \, dm(z) < \infty$.

(e) $\left| \int_D f_1 f_2 \, d\mu \right| \leq C \|f_1\|_{H^2} ||f_2||_{H^2} \, \forall f_1, f_2 \in H^2$.

(f) $K_\mu$ is a bounded operator on $H^2$.

(g) There exists $F \in L^\infty(\partial D)$ such that

$$\int_D f \, d\mu = \frac{1}{2\pi i} \int_{\partial D} f(\zeta) F(\zeta) \, d\zeta \, \forall f \in H^1.$$

(h) $\sup_A |\int_D A \, d\mu| < \infty$, where the supremum ranges over all holomorphic atoms.

PROOF: Step 1. We show that (a), (b), (c) and (d) are equivalent.

Trivially, (b) implies (a). On the other hand, suppose (a) holds. To prove (b), take $f \in H^1$ with $f \neq 0$. By [3, Theorem 2.8, p. 24], there exist an inner factor $g$ and an outer factor $h$ such that $f = gh$ and $\|h\|_{H^1} \leq ||f||_{H^1}$. Set $f_1 = (g - 1)h/2$ and $f_2 = (g + 1)h/2$. Then $f = f_1 + f_2$, and $||f_k||_{H^1} \leq ||f||_{H^1}$, $k = 1, 2$. Since both $f_1$ and $f_2$ are not equal to 0 anywhere on $D$, there are $g_1, g_2 \in H^2$ such that $f_1 = g_1^2$ and $f_2 = g_2^2$. Consequently,

$$\left| \int_D f \, d\mu \right| \leq \left| \int_D g_1^2 \, d\mu \right| + \left| \int_D g_2^2 \, d\mu \right| \leq C(||g_1||_{H^2}^2 + ||g_2||_{H^2}^2) \leq C||f||_{H^1},$$

that is, (b) holds.

By definition of $P_\mu$,

$$\int_D f(z) \, d\mu(z) = \int_D \left[ \frac{1}{2\pi} \int_{\partial D} \frac{f(\zeta)}{1 - z\overline{\zeta}} \, |d\zeta| \right] \, d\mu(z)$$

$$= \frac{1}{2\pi} \int_{\partial D} f(\zeta) \left[ \int_D \frac{1}{1 - z\overline{\zeta}} \, d\mu(z) \right] \, |d\zeta|$$

$$= \frac{1}{2\pi} \int_{\partial D} f(\zeta) P_\mu(\zeta) \, |d\zeta|$$

$$= \langle f, P_\mu \rangle.$$
Thus, (c) holds if and only if (b) does, by the isomorphism between \([H^1]^*\) and \(BMOA\).

An \(H^2\)-function \(f\) belongs to \(BMOA\) if and only if \((1 - |z|^2)|f'(z)|^2 \, dt\) is a Carleson measure (see [10, p.178]). Accordingly, (c) is equivalent to (d).

**STEP 2.** We verify that (b), (e) and (f) are equivalent.

An application of Schwarz's inequality to \(f_1 f_2\) (where \(f_1, f_2 \in H^2\)) shows that (b) implies (e). On the other hand, let (e) hold. By [5, p.87, Exercise 1], we see that every \(f \in H^1\) can be factored as \(f = f_1 f_2\) with \(f_1, f_2 \in H^2\) and \(\|f_1\|_H^2 = \|f_2\|_H^2 = \|f\|_{H^1}\), so that (b) follows from the estimate

\[
\left| \int_D f \, d\mu \right| \leq C\|f_1\|_{H^2}\|f_2\|_{H^2} = C\|f\|_{H^1}.
\]

When \(f, g \in H^2\), the definition of \(K_\mu\) implies that

\[
(K_\mu f, g) = \frac{1}{2\pi i} \int_{\partial D} (K_\mu f)(\zeta) g(\overline{\zeta}) \, d\zeta
\]

\[
= \frac{1}{2\pi} \int_{\partial D} \left[ \int_D \frac{f(w)}{1 - \zeta w} \, dt\mu(w) \right] g(\overline{\zeta}) \, d\zeta
\]

\[
= \int_D f(w) \left[ \frac{1}{2\pi} \int_{\partial D} \frac{g(\zeta)}{1 - \zeta w} \, d\mu(\zeta) \right] d\mu(w)
\]

\[
= \int_D f(w) g(\overline{w}) \, d\mu(w).
\]

This formula shows that (e) and (f) are equivalent.

**STEP 3.** We prove the equivalence of (b), (g) and (h).

It is obvious that (g) implies (b), thanks to the inequality

\[
\left| \int_D f \, d\mu \right| = \left| \frac{1}{2\pi i} \int_{\partial D} f(\zeta) F(\zeta) \, d\zeta \right| \leq \|F\|_{\infty}\|f\|_{H^1} \quad \forall f \in H^1.
\]

Conversely, if (b) is valid, then the linear functional \(T\), defined by \(T(f) = \int_D f \, d\mu\), belongs to \([H^1]^*\). The Hahn–Banach theorem allows us to extend \(T\) to \(L^1(\partial D)\), and hence produce \(G \in L^\infty(\partial D)\) such that

\[
T(f) = \frac{1}{2\pi} \int_{\partial D} f(\zeta) G(\zeta) \, d\zeta \quad \forall f \in L^1(\partial D).
\]

Take \(F\) such that \(F(z) = \overline{z} G(z)\); then \(F \in L^\infty(\partial D)\) and

\[
\int_D f \, d\mu = T(f) = \frac{1}{2\pi i} \int_{\partial D} f(\zeta) F(\zeta) \, d\zeta \quad \forall f \in H^1,
\]
and \((g)\) holds.

In [2], atoms are defined to be functions \(a : \partial D \to \mathbb{C}\) which are either identically 1 or are supported on an open subarc \(I\) of \(\partial D\), and satisfy

\[
\|a\|_\infty \leq \frac{1}{|I|} \quad \text{and} \quad \int_{\partial D} a(\zeta) \, |d\zeta| = 0.
\]

Holomorphic atoms \(A\) are the holomorphic projection of atoms \(a\), that is,

\[
A(z) = \frac{1}{2\pi} \int_{\partial D} \frac{a(\zeta)}{1 - \overline{z} \zeta} \, d\zeta.
\]

By [2, Theorem V], every \(f \in H^1\) can be represented as

\[
\sum_{j=1}^{\infty} \lambda_j A_j,
\]

where \(A_j\) are holomorphic atoms and \(\lambda_j \in \mathbb{C}\) with \(\sum_{j=1}^{\infty} |\lambda_j| \leq C\|f\|_{H^1}\). This decomposition, together with [2, Proposition VI], demonstrates that \((b)\) implies \((h)\) and vice versa. The proof of the theorem is complete. □

**REMARK.** Since \(f_w(z) = (1 - |w|^2)/(1 - \overline{w} z)^2\) lies in \(H^1\) and \(\sup_{w \in D} \|f_w\|_{H^1} < \infty\), a necessary condition for \(\mu\) to be a Hankel measure is that

\[
\sup_{w \in D} \left| \int_D \frac{1 - |w|^2}{(1 - \overline{w} z)^2} \, d\mu(z) \right| < \infty.
\]

We do not know whether this condition is sufficient, too. However, it is worth mentioning (see [5, p.139]) that \(\mu\) is a Carleson measure if and only if

\[
\sup_{w \in D} \int_D \frac{1 - |w|^2}{|1 - \overline{w} z|^2} \, d|\mu|(z) < \infty.
\]

As in [6] or [8], it is not hard to check that \(K_\mu\) is a classical Hankel operator on \(H^2\), and hence \(K_\mu : H^2 \to H^2\) is a compact or \(S_p\) (the Schatten–von Neumann \(p\)-class, for \(p \in (1, \infty)\)) operator if and only if \(P_\mu\) belongs to \(VMOA\), the space of all \(f \in H^1\) with

\[
\lim_{w \to \partial D} \left\| f \circ \phi_w - f(w) \right\|_{H^1} = 0,
\]

or to \(B_p\), the \(p\)-Besov space of all holomorphic functions \(f\) on \(D\) such that

\[
\int_D |f'(z)|^p (1 - |z|^2)^{p-2} \, dm(z) < \infty,
\]

respectively. □
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