A METRISATION THEOREM FOR PSEUDOCOMPACT SPACES

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In this paper we prove that a completely regular pseudocompact space with a quasi-regular-$G_δ$-diagonal is metrisable.

1. INTRODUCTION

Recently, we have considered the question of what topological properties imply metrisability in the presence of a weak diagonal property. For example, it is well-known that the existence of a quasi-$G_δ$-diagonal is sufficient for metrisability in countably compact spaces [7]. In [3] we proved that a manifold with a quasi-regular-$G_δ$-diagonal is metrisable. In this present paper, we give a diagonal condition on pseudocompact spaces to get metrisability.

A countable family $\{G_n\}_{n \in \mathbb{N}}$ of collections of open subsets of a space $X$ is called a quasi-$G_δ$-diagonal (quasi-$G_δ^*$-diagonal), if for each $x \in X$ we have $\bigcap_{n \in c(x)} st(x, G_n) = \{x\}$ where $c(x) = \{n : x \in G$ for some $G \in G_n\}$ and $st(x, G_n)$ is the union of all sets in $G_n$ which contain $x$.

A space $X$ has a quasi-regular-$G_δ$-diagonal [3] if and only if there is a countable sequence $\langle U_n : n \in \mathbb{N}\rangle$ of open subsets in $X^2$, such that for all $(x, y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin U_n$.

A space $X$ is called quasi-developable if there is a countable family $\{G_n : n \in \mathbb{N}\}$ of collections of open subsets of $X$ such that for all $x \in X$ the nonempty sets of the form $st(x, G_n)$ form a local base at $x$.

In this paper all spaces will be completely regular, unless we state otherwise.

2. THE MAIN RESULTS

Pseudocompact spaces were first defined and investigated by Hewitt in [4].

DEFINITION 2.1. A space $X$ is pseudocompact if every real-valued continuous function on $X$ is bounded.

The following characterisation of pseudocompactness may be found in [2].
Lemma 2.2. A space $X$ is pseudocompact if and only if for every decreasing sequence $\langle U_n : n \in \mathbb{N} \rangle$ of nonvoid open subsets of $X$, $\bigcap_{n \in \mathbb{N}} U_n \neq \emptyset$.

McArthur in [6] proved the following lemma.

Lemma 2.3. Let $X$ be a pseudocompact space. Suppose $\langle U_n : n \in \mathbb{N} \rangle$ is a decreasing sequence of open sets such that $\bigcap_{n \in \mathbb{N}} U_n = \{x\}$ for a point $x \in X$. Then the sets $U_n$ form a local neighbourhood base at $x$.

The proof of our main result relies on a metrisation theorem.

Theorem 2.4. [3] Let $X$ be a space with a sequence $\langle G_n : n \in \mathbb{N} \rangle$ of open families such that, for each $x \in X$, $\{st(x, G_n)\}_{n \in \mathbb{N}} - \emptyset$ (that is, the union of all sets $st(y, G_n)$ with $y \in st(x, G_n)$) is a local base at $x$. Then $X$ is metrisable.

Lemma 2.5. Let $X$ be a pseudocompact space with a quasi-regular-G$_\delta$-diagonal. Then $X$ is quasi-developable.

Proof: Let $\langle V_n : n \in \mathbb{N} \rangle$ be a quasi-regular-G$_\delta$-diagonal sequence for $X$. Without loss of generality we may assume that $V_1 = \{x\}$. Set $c_V(x) = \{n : st(x, V_n) \neq \emptyset\}$. Then $\bigcap_{n \in c_V(x)} V_n = \{x\}$. Let $F$ denote the set of non-empty finite subsets of $\mathbb{N}$. For each $F \in F$ set

$$G_F = \left\{ \bigcap_{i \in F} V_i : V_i \in V_i \right\}.$$

We show that $\{G_F : F \in F\}$ is a quasi-development of $X$. For each $n \in \mathbb{N}$, $x \in X$ put $F_n(x) = c_V(x) \cap \{1, 2, \ldots, n\}$. Then $F_n(x) \neq \emptyset$. Note that $st(x, G_{F_n(x)}) \subseteq st(x, V_m)$ for each $n \in \mathbb{N}$, each $x \in X$ and each $m \in F_n(x)$. Note also that

$$\bigcap_{n \in \mathbb{N}} st(x, G_{F_n(x)}) = \bigcap_{n \in \mathbb{N}} st(x, G_{F_n(x)}) = \{x\}. $$

By Lemma 2.3, $\{st(x, G_{F_n(x)}) : n \in \mathbb{N}\}$ forms a local neighbourhood base at $x$. Hence, $\{st(x, G_F) : F \in F\} - \emptyset$ forms a local neighborhood base at $x$. \hfill $\square$

Theorem 2.6. Let $X$ be a pseudocompact space with a quasi-regular-G$_\delta$-diagonal. Then $X$ is metrisable.

Proof: By Theorem 2.4, we only need to show that $X$ has a quasi-development $\langle G_n : n \in \mathbb{N} \rangle$ such that, for each $x \in X$, $\{st(x, G_n)\}_{n \in \mathbb{N}} - \emptyset$ is a local base at $x$.

Let $\langle U_n : n \in \mathbb{N} \rangle$ be as in the definition of quasi-regular-G$_\delta$-diagonal. So, the sets $U_n$ are open in $X^2$ and for all $(x, y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin U_n$. Put $H_n = \{H : H$ is open, $H \times H \subseteq U_n\}$. As in the proof of Lemma 2.5, let $F$ denote the set of non-empty finite subsets of $\mathbb{N}$, and for $F \in F$ put

$$G'_F = \left\{ \bigcap_{i \in F} H_i : H_i \in H_i \right\}.$$
We show that for each \( x \in X \), \( \{ \text{st}^2(x, G'_F) \}_{F \in \mathcal{F}} - \{0\} \) is a local base at \( x \). Take any \( x \in X \). For each \( n \in \mathbb{N} \) put \( F_n(x) = \{ s : \text{st}(x, \mathcal{H}_s) \neq \emptyset \} \cap \{1, 2, \ldots, n\} \). Without loss, \( \mathcal{H}_i = \{X\} \), so \( F_n(x) \neq \emptyset \). We prove that \( \bigcap_{n \in \mathbb{N}} \text{st}^2(x, G'_{F_n(x)}) = \{x\} \).

Suppose, for a contradiction, for all \( n \in \mathbb{N} \), \( y \in \text{st}^2(x, G'_{F_n(x)}) \) and \( x \neq y \). So by the definition of quasi-regular-\( G_\delta \)-diagonal, there is \( k \) such that \( (x, x) \in U_k \) but \( (x, y) \notin \overline{U}_k \).

By the same argument as in Lemma 2.5, we know that \( \{ G'_F : F \in \mathcal{F} \} \) is a quasi-development of \( X \). Therefore there exist \( I \) and \( J \in \mathcal{F} \) such that

\[
(x, y) \in \text{st}(x, G'_I) \times \text{st}(y, G'_J) \subseteq X^2 - \overline{U}_n.
\]

Choose \( m \geq \max\{I, k\} \), so that \( I \subseteq F_m(x) \). It follows that \( y \in \text{st}^2(x, G'_{F_m(x)}) \), so \( \text{st}^2(x, G'_{F_m(x)}) \cap \text{st}(y, G'_J) \neq \emptyset \). Then there exists \( G_1, G_2 \in \mathcal{G}_{F_m(x)} \) and \( G_3 \in \mathcal{G}_J \) such that \( y \in G_3 \), \( x \in G_1, G_1 \cap G_2 \neq \emptyset \) and \( G_2 \cap G_3 \neq \emptyset \). Let \( z_1 \in G_1 \cap G_2 \) and \( z_2 \in G_2 \cap G_3 \). Then \( (z_1, z_2) \in (G_1 \times G_3) \cap (G_2 \times G_2) \). Now, \( G_1 \in \mathcal{G}_{F_m(x)} \), \( G_3 \in \mathcal{G}_J \), so \( G_1 \times G_3 \subseteq \text{st}(x, G'_{F_m(x)}) \times \text{st}(y, G'_J) \). Also, \( G_2 \in \mathcal{G}_{F_m(x)} \) and \( k \in F_m(x) \), so \( G_2 \subseteq H \) for some \( H \in \mathcal{H}_k \). Therefore \( G_2 \times G_2 \subseteq H \times H \subseteq U_k \), so \( (z_1, z_2) \in U_k \).

In other words, \( (z_1, z_2) \in (G_2 \times G_2) \cap U_k \subseteq (\text{st}(x, G'_{F_m(x)}) \times \text{st}(y, G'_J)) \cap U_k \), and this is a contradiction. Therefore, \( \bigcap_{n \in \mathcal{G}'(x)} \text{st}^2(x, G'_{F_n(x)}) = \{x\} \). We conclude by Lemma 2.3 that for each \( x \in X \), \( \{ \text{st}^2(x, G'_F) \}_{F \in \mathcal{F}} - \{0\} \) is a local base at \( x \). Hence, \( X \) is metrisable. \( \square \)

**EXAMPLE 2.7.** The space \( E \cap [0,1] \) of [2, Problem 3] is submetrisable (that is, it is a space with a coarser metric topology) pseudocompact and Hausdorff. Since the space is not completely regular, it is not metrisable.

**EXAMPLE 2.8.** The Mrowka space \( \Psi \) (see [2, 1, 5]) is completely regular, pseudocompact and developable but does not have a quasi–regular-\( G_\delta \)-diagonal, and hence is not metrisable.

**REFERENCES**


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