PROLONGATIONS OF LINEAR CONNECTIONS TO THE FRAME BUNDLE

Luis A. Cordero and Manuel de Leon

In this paper we construct the prolongation of a linear connection \( \Gamma \) on a manifold \( M \) to the bundle space \( FM \) of its frame bundle, and show that such prolongated connection coincides with the so-called complete lift of \( \Gamma \) to \( FM \).

Introduction

The purpose of the present paper is to construct the prolongation of a linear connection on a manifold \( M \) to the bundle space \( FM \) of the frame bundle of \( M \). To do this, we use Morimoto's general theory of prolongations to tangential fibre bundles of \( p^r \)-jets of \( M \) [6] particularized when \( r = 1 \), as well as some result stated in [2].

In §1, we briefly recall some results which will be used in the remaining sections. In §2, the prolongation of a connection on a principal fibre bundle \( P \) to the principal bundle \( J^1_P \) of \( p^1 \)-jets of \( P \) is constructed. In §3, we apply the results in §2 for the case of linear connections and construct the prolongation \( \tilde{\Gamma} \) to \( FM \) of a linear connection \( \Gamma \) on \( M \), proving moreover that \( \tilde{\Gamma} \) coincides with the so-called complete lift \( \Gamma^C \) of \( \Gamma \) defined by Mok in [5]. Finally, in §4 we show that connections adapted to \( G \)-structures on \( M \) prolongate to

Received 18 July 1983.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/83 $A2.00 + 0.00.
connections adapted to the corresponding prolongations of these $G$-structures introduced in [2].

In this paper all manifolds and mappings are assumed to be differentiable of class $C^\infty$, entries of matrices are written as $a_{ij}^i$, $i$ being the row index and $j$ the column index, and summation over repeated index is always implied.

1. Preliminaries

Let $M$ be an $n$-dimensional manifold, $\mathbb{R}^P$ the Euclidean $p$-space and $J^1_pM$ the set of 1-jets at $0 \in \mathbb{R}^P$ of all differentiable mappings $\mathcal{G} : \mathbb{R}^P \to M$ defined on some open neighborhood of $0 \in \mathbb{R}^P$; if $j^1(\mathcal{G})$ denotes the 1-jet of $\mathcal{G}$ at $0$, the target map $\pi : J^1_pM \to M$ is defined by $\pi(j^1(\mathcal{G})) = \mathcal{G}(0)$ and is in fact a projection map from $J^1_pM$ onto $M$.

On $J^1_pM$ there exists a structure of $(n+pn)$-dimensional manifold, canonically induced from the manifold structure of $M$, which is given as follows: let $(U, x^\alpha)$ be a coordinate system in $M$, $U$ being the coordinate neighborhood and $\{x^\alpha\}$ the coordinate functions on $U$; then, on $J^1_pU = \pi^{-1}(U)$ we define a family of coordinate functions $\{x^\alpha^{\alpha_1}, x^\alpha_\alpha\}$ by setting

$$x^\alpha^{\alpha_1}(j^1(\mathcal{G})) = x^{\alpha_1}(\mathcal{G}(0)), \quad x^\alpha_\alpha(j^1(\mathcal{G})) = \frac{\partial(x^{\alpha_1}(\mathcal{G}))}{\partial x^\alpha_\alpha} \bigg|_0$$

$(1 \leq i \leq n, 1 \leq a \leq p)$ for any $j^1(\mathcal{G}) \in J^1_pU$, and where $\{x^1, \ldots, x^P\}$ are the canonical coordinate functions on $\mathbb{R}^P$. Then $\{J^1_pU, x^\alpha, x^\alpha_\alpha\}$ is a coordinate system in $J^1_pU$ which will be said to be induced by $(U, x^\alpha)$ in $M$.

Let $h : M \to N$ be a differentiable map; then $h^1 : J^1_pM \to J^1_pN$ will
denote the map canonically induced by \( h \) and given by
\[ h^1(j^1(\xi)) = j^1(h \circ \xi) \] for any \( j^1(\xi) \in J^1_P P \). If \((U, \tilde{x}^i), (U', \tilde{y}^j)\) are local coordinate systems in \( M \) and \( N \) respectively, and if we assume \( h : U \cup U' \) expressed by \( \tilde{y}^j = h^j(x^1, \ldots, x^n) \) then, with respect to the induced coordinate systems \( \{J^1_P U, x^i, x^\alpha\}, \{J^1_P U', \tilde{y}^j, \tilde{y}^\alpha\}\), \( h^1 \) is expressed by
\[ h^1 : \tilde{y}^j = h^j(x^1, \ldots, x^n), \quad y^\alpha = \frac{\partial h^j}{\partial x^k} \tilde{x}^k, \] where \( 1 \leq k \leq \dim M \), \( 1 \leq j \leq \dim N \) and \( 1 \leq \alpha \leq p \).

Let \( G \) be a Lie group; then \( J^1_P P G \) has also a Lie group structure, its product being defined as follows: for any \( j^1(\xi), j^1(g) \in J^1_P P G \), \( j^1(\xi) \cdot j^1(g) = j^1(\xi g) \), where \( \xi g : H^P \to G \) is defined by \( (\xi g)(t) = \xi(t)g(t), \quad t \in \text{dom} \xi \cap \text{dom} g \). The unit element \( e_p \) of \( J^1_P P G \) is then the 1-jet at \( 0 \in H^P \) of the constant map from \( H^P \) into the unit element \( e \) of \( G \).

Next, we shall recall some results to be used later.

1. Assume \( p = n = \dim M \). Then the bundle space \( FM \) of the principal fibre bundle of linear frames over \( M \) (briefly, the frame bundle of \( M \)) is an open (dense) submanifold of \( J^1_M \), and the induced structure on \( FM \) is the usual one with respect to which \( \pi_M : FM \to M \) is a \( \text{GL}(n) \)-principal bundle, \( \text{GL}(n) \) denoting the general linear group. If \( (U, \tilde{x}^i) \) is a local coordinate system in \( M \), the induced coordinate functions on \( FM = (\pi_M)^{-1}(U) \) will be written as \( \{\tilde{x}^i, \tilde{x}^\alpha\} \) if there is no confusion.

2. Assume \( p = 1 \). Then \( \pi : J^1_M \to M \) is nothing but the tangent bundle \( \pi_M : TM \to M \). In this case, if \( (U, \tilde{x}^i) \) is a local coordinate system in \( M \), the induced coordinate functions on \( TM = (\pi_M)^{-1}(U) \) will be
written as \( \{ x^i; x^j \} \). Note that the linear structure of this vector bundle is locally given as follows: let \( X, Y \) be tangent vectors at \( x = (x^1, \ldots, x^n) \in U \) with coordinates \( X = \{ x^i; x^j \} \), \( Y = \{ x^i; y^j \} \);
then \( X + Y = \{ x^i; x^j + y^j \} \). If \( \delta : M \to N \), we shall denote by
\[
T_{\delta} : TM \to TN
\]
the induced map.

(3) Let \( P(M, \pi, G) \) be a principal fibre bundle with bundle space \( P \), base space \( M \), projection \( \pi \) and structure group \( G \). Then
\[
\pi^{\perp}P \bigg|_{\pi^{\perp}M, \pi^{\perp}N, \pi^{\perp}G}
\]
is again a principal fibre bundle. In fact, if
\[
\phi_U : \pi^{\perp}(U) \to U \times G
\]
is the trivialization of \( P \) over \( U \subseteq M \), then, since
\[
(\pi^{\perp})^{-1}\left(j^1_PU\right) = j^1_P\pi^{\perp}(U)
\]
we define \( \tilde{\phi}_U : j^1_P\pi^{\perp}(U) \to j^1_PU \times j^1_PG \) by setting
\[
\tilde{\phi}_U\left(j^1(\delta)\right) = \left(j^1(\pi \circ \delta), j^1(\eta \circ \phi_U \circ \delta)\right)
\]
for any \( j^1(\delta) \in j^1_P\pi^{\perp}(U) \),
where \( \eta : U \times G \to G \) is the canonical projection.

(4) Let \( G = GL(n), \{ x^i_j \} \) be the canonical coordinates in \( GL(n) \),
\( \{ x^i_j, x^i_{j\alpha} \} \) the induced coordinates in \( j^1_nGL(n) \) and \( \{ y^A_B, 1 \leq A, B \leq n+n^2 \} \)
the canonical coordinates in \( GL(n+n^2) \); then, there exists a canonical embedding of Lie groups
\[
j_n : j^1_nGL(n) \to GL(n+n^2)
\]
given by
\[
j_n\left(\left[\begin{array}{c} x^i_j \\ x^i_{j1} \\ \vdots \\ x^i_{jn} \end{array}\right] \right) = \left[\begin{array}{c} x^i_j \\ 0 \\ \vdots \\ 0 \end{array}\right]
\]
that is, with respect to the coordinates above \( j_n \) is expressed by
Frame bundle connections

\[ y^i_j = x^i_j, \quad y^i_{\alpha} = 0, \]

\[ j_n : \]

\[ y^i_j = x^i_{j\alpha}, \quad y^i_{\alpha} = \delta^i_{\beta} x^\beta_j, \]

where \( i_{\alpha} = \alpha n + i, \ 1 \leq i, \ \alpha \leq n \). If we consider the Lie algebras of \( J_n^1 \text{Gl}(n) \) and \( \text{Gl}(n+n^2) \) identified with the tangent spaces at the respective unit elements \( e_n \) and \( e \), then the induced homomorphism

\[ j_n : T_{e_n} J_n^1 \text{Gl}(n) \to T_{e} \text{Gl}(n+n^2) \]

may be written as follows:

\[
j_n \left( \left( \delta^i_j, 0; A^i_j, B^i_{j\alpha} \right) \right) = \left( \begin{bmatrix} A^i_j & 0 & \ldots & 0 \\ B^i_j & A^i_j & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B^i_{jn} & 0 & \ldots & A^i_j \end{bmatrix} \right). \]

(5) Let \( F_n^M(M, \pi_M, \text{Gl}(n)) \) be the frame bundle of \( M \), \( J_n^1 F_n^M \) the induced \( J_n^1 \text{Gl}(n) \)-principal bundle and \( F_n^M J_n^1(M, \pi_M, \text{Gl}(n+n^2)) \) the frame bundle of the \( (n+n^2) \)-dimensional manifold \( J_n^1 M \). Then there exists a canonical injective homomorphism of principal bundles [2]

\[ j_M : J_n^1 F_n^M \to F_n^1 J_n^1 M \]

over the identity of \( J_n^1 M \), with associate Lie group homomorphism \( j_n \).

The homomorphism \( j_M \) is locally defined as follows: let \((U, x^i)\) be a local coordinate system in \( M \) and consider fibered coordinate functions \((x^i, x^i_\alpha, x^i_j, x^i_{j\alpha})\) on \( \pi_M^{-1} (J_n^1 U) \) and \((y^i, y^i_\alpha, y^i_j, y^i_{j\alpha})\) on \( F_n^1 U \); then, with respect to these coordinates, \( j_M \) is expressed by
Since the restriction \( F^1_M \) of \( F^1_M \) to the open submanifold \( \mathbb{F}_M \subset J^1_M \) is canonically isomorphic to the frame bundle \( FFM \) of \( FM \), then the homomorphism \( j_M \) above induces an injective homomorphism of principal bundles, noted again \( j_M : F^1_M \mathbb{F}_M \rightarrow FFM \), over the identity of \( FM \) and with associate Lie group homomorphism \( j_n \).

(6) Particularizing the general results of Morimoto ([6], Chapter IV), we can assert: let \( M \) be an \( n \)-dimensional manifold; then there exist canonical diffeomorphisms

\[
\alpha^p_{M}^{1} : TJ^1_{\mathbb{F}_M} \rightarrow J^1_{\mathbb{F}_M}, \quad \alpha^{1,p}_{M} : J^1_{\mathbb{F}_M} \rightarrow TJ^1_{\mathbb{F}_M},
\]

such that \( \alpha^p_{M}^{1} \) and \( \alpha^{1,p}_{M} \) are mutually inverse. Locally, \( \alpha^p_{M}^{1} \) is given as follows: let \( (U, x^i) \) be a local coordinate system in \( M \) and let \( (x^i, x^i_\alpha; x^i; x^i_\alpha), \quad \{y^i, \dot{y}^i, (y^i)_\alpha, (\dot{y}^i)_\alpha\} \) be the induced coordinate functions on \( TJ^1_{\mathbb{F}_M} \) and \( J^1_{\mathbb{F}_M} \) respectively. Then

\[
\alpha^p_{M}^{1} : y^i = x^i, \quad \dot{y}^i = \dot{x}^i, \quad (y^i)_\alpha = x^i_\alpha, \quad (\dot{y}^i)_\alpha = x^i_\alpha,
\]

with \( 1 \leq i \leq n, \ 1 \leq \alpha \leq p \). The local expression of \( \alpha^{1,p}_{M} \) is obvious. Moreover, if \( \phi : M \rightarrow N \) is a differentiable map, then the following diagram is commutative.
2. Prolongation of connections

Let \( P(M, \pi, G) \) be a principal fibre bundle and consider on \( P \) a connection whose connection form will be denoted by \( \omega \). Following Kobayashi [3], we shall consider this form \( \omega \) as a differentiable map \( \omega : TP \to TG \) which is a linear map of the tangent space \( T_uP \) with values in the tangent space \( T_{\tilde{\delta}}G \) for each point \( u \in P \), and satisfying:

\[
\omega(u \cdot \tilde{\delta}) = \delta^{-1} \cdot \tilde{\delta},
\]

\[
\omega(\tilde{u} \cdot \delta) = \delta^{-1} \cdot \omega(\tilde{u}) \cdot \delta,
\]

for every \( u \in P \), \( \delta \in G \), \( \tilde{u} \in T_uP \) and \( \tilde{\delta} \in T_{\tilde{\delta}}G \), and where by definition \( \tilde{u} \cdot \delta = TR_\delta(\tilde{u}) \), \( u \cdot \tilde{\delta} = TL_\delta(\tilde{u}) \), \( R_\delta : P \to P \) and \( L_\delta : G \to G \) being the canonical maps.

Let \( \omega : TP \to TG \) be a connection form on \( P(M, \pi, G) \) and define a differentiable map \( \omega_1 : T^1P \to T^1G \) by setting

\[
(2.1) \quad \omega_1 = \alpha^{1P}_G \circ \omega^1 \circ \alpha^{1P}_{1P}.
\]

Then, from Morimoto's general results [6], we know that

\[
\text{Im} \omega_1 \subset T_{\tilde{\delta}}G \setminus \tilde{J}_G^1P,
\]

\[
\omega_1(\tilde{u} \cdot \tilde{\delta}) = \delta^{-1} \cdot \tilde{\delta},
\]

\[
\omega_1(\tilde{\tilde{u}} \cdot \delta) = \delta^{-1} \cdot \omega_1(\tilde{u}) \cdot \tilde{\delta},
\]
for every \( \tilde{\xi} \in J^1_p G \), \( \tilde{\mu} \in J^1_p P \), \( \tilde{\xi} \in T_{\tilde{\xi}} J^1_p G \) and \( \tilde{\mu} \in T_{\tilde{\mu}} J^1_p P \). Hence, to prove that \( \omega \) is actually a connection form on the principal bundle \( J^1_p \left( J^1_p M, \pi, J^1_p G \right) \) it suffices to prove that \( \omega : T_{\bar{u}} J^1_p P \to T_{\bar{u}} J^1_p G \) is a linear map for any \( \bar{u} \in J^1_p P \).

To do this we proceed as follows.

Let \( (U, x^i), (U', y^a) \) be local coordinate systems in \( P \) and \( G \), respectively, with \( u = \pi(\bar{u}) \in U \), \( e \in U' \) and \( 1 \leq i \leq \dim P \), \( 1 \leq a \leq \dim G \). Then, with respect to the induced coordinate systems \( (TU, x^i, \dot{x}^i), (TU', y^a, \dot{y}^a) \) in \( TP \) and \( TG \) respectively, \( \omega \) is expressed by

\[
\omega : y^a = \omega^a(x^i; \dot{x}^i) = \dot{y}^a(e) , \quad \dot{y}^a = \omega^a(x^i; \dot{x}^i),
\]

and, therefore, for any \( i \) and \( a \),

\[
(2.2) \quad \frac{\partial \omega^a}{\partial x^i} = \frac{\partial \omega^a}{\partial \dot{x}^i} = 0 .
\]

On the other hand, if \( \bar{u}, \bar{u}' \in T_{\bar{u}} P \) are given by \( \bar{u} = \left( x^i; \dot{x}^i \right) \), \( \bar{u}' = \left( x^i; \dot{x}^{i'} \right) \) then the linearity of \( \omega : T_{\bar{u}} P \to T_{\bar{e}} G \) implies

\[
(2.3) \quad \omega^a(x^i; \dot{x}^i + \dot{x}^{i'}) = \omega^a(x^i; \dot{x}^i) + \omega^a(x^i; \dot{x}^{i'}),
\]

and therefore

\[
(2.4) \quad \frac{\partial \omega^a}{\partial x^i} (x^i; \dot{x}^i + \dot{x}^{i'}) = \frac{\partial \omega^a}{\partial x^i} (x^i; \dot{x}^i) + \frac{\partial \omega^a}{\partial \dot{x}^i} (x^i; \dot{x}^{i'}),
\]

\[
\frac{\partial \omega^a}{\partial x^i} (x^i; \dot{x}^i + \dot{x}^{i'}) = \frac{\partial \omega^a}{\partial x^i} (x^i; \dot{x}^i) + \frac{\partial \omega^a}{\partial \dot{x}^i} (x^i; \dot{x}^{i'}).
\]

Now, let \( \left( x^i, \dot{x}_a^i, \dot{x}_a, \dot{x}_a^{i'} \right), \left( y^a, y_a^a, y_a, y_a^a \right) \) be the induced coordinate functions on \( J^1_p U \) and \( J^1_p U' \) respectively. Then, taking into account the local expressions of \( \alpha^1_p, \alpha^1_G \) and \( \omega^1 \) as well as (2.2), a
Frame bundle connections

direct computation leads to the following local expression of $\omega_1$:

$$y^\alpha = y^\alpha(\epsilon), \quad y^\alpha_\alpha = 0,$$

$$\omega_1 : y^\alpha = \omega^\alpha(x^i; \dot{x}^i),$$

$$y^\alpha_\alpha = \frac{\partial \omega^\alpha}{\partial x^k} (x^i; \dot{x}^i) \cdot x^k + \frac{\partial \omega^\alpha}{\partial \dot{x}^k} (x^i; \dot{x}^i) \cdot \dot{x}^k.$$

Therefore, if $\tilde{u} \in \mathcal{J}^1_P$ has coordinates $\tilde{u} = \{x^i, \dot{x}^i\}$ and

$$\tilde{u}, \tilde{u}' \in T_{\tilde{u}} \mathcal{J}^1_P$$

are given by $\tilde{u} = \{x^i, x^i; \dot{x}^i, \dot{x}^i\}$, $\tilde{u}' = \{x^i, x^i; \dot{x}^i, \dot{x}^i\}$

then $\tilde{u} + \tilde{u}' = \{x^i, x^i; \dot{x}^i + \dot{x}^i, \dot{x}^i + \dot{x}^i\}$ and a straightforward computation, using (2.3) and (2.4), leads to

$$y^\alpha_\alpha(\omega_1(\tilde{u} + \tilde{u}')) = y^\alpha_\alpha(\omega_1(\tilde{u})) + y^\alpha_\alpha(\omega_1(\tilde{u}')),$$

$$y^\alpha_\alpha(\omega_1(\tilde{u} + \tilde{u}')) = y^\alpha_\alpha(\omega_1(\tilde{u})) + y^\alpha_\alpha(\omega_1(\tilde{u}')).$$

Thus we have proved the following theorem

**THEOREM 2.1.** Let $\omega : TP \rightarrow TG$ be a connection form on a principal fibre bundle $P(M, \pi, G)$. Then $\omega_1 : T\mathcal{J}^1_P \rightarrow T\mathcal{J}^1_G$ given by (2.1) is a connection form on the principal fibre bundle $\mathcal{J}^1_P \{\mathcal{J}^1_M, \pi^1, \mathcal{J}^1_G\}$. We shall call $\omega_1$ the prolongation of the connection $\omega$ to $\mathcal{J}^1_P$.

We remark that, for $p = 1$, $\omega_1$ coincides with the connection tangential to $\omega$ due to Kobayashi ([3], p. 152), also obtained by Morimoto in [7].

3. Prolongation of linear connections to the frame bundle

In this section we apply the result in the previous section to the linear connections on a manifold. From now on the indices

$h, i, j, k, \ldots, \alpha, \beta, \gamma, \ldots$ have range in $\{1, 2, \ldots, n\},$

$A, B, C, \ldots$ in $\{1, 2, \ldots, n+n^2\}$ and $\dot{i}_\alpha$ stands for $\alpha n + i$. 


Let \( F_M(M, \pi_M, \text{Gl}(n)) \) and \( FFM(F_M, \pi_{FM}, \text{Gl}(n+n^2)) \) be the frame bundles of \( M \) and \( F_M \) respectively.

**Theorem 3.1.** Let \( \Gamma \) be a linear connection on a manifold \( M \). Then there exists canonically a linear connection \( \tilde{\Gamma} \) on the frame bundle \( F_M \) of \( M \), which will be called the prolongation of \( \Gamma \) to \( F_M \).

**Proof.** Let \( \omega \) be the connection form on \( F_M \) defining the connection \( \Gamma \). The prolongation \( \omega_1 \) of \( \omega \) is a connection form on \( J^1 F_M \), \( n^i = \dim M \). Then, using the bundle homomorphism \( j_M : J^1 F_M \mapsto FFM \) described in §1, (5), we canonically obtain a connection \( \tilde{\Gamma} \) on the principal fibre bundle \( FFM \).

Next, we shall compute the local components \( \tilde{\Gamma}_{BC} \) of the prolongation \( \tilde{\Gamma} \) of \( \Gamma \) to \( F_M \).

Let \( \omega : TFM \rightarrow T\text{Gl}(n) \) be the connection form of \( \Gamma \), \( (U, x^i) \) a local coordinate system in \( M \), \( \{x^i, x^j_i\} \) the induced coordinate functions on \( F_M \), \( \{y^i, y^j_i\} \) the canonical coordinates in \( \text{Gl}(n) \) and \( \{\xi^i, \xi_j^i, \xi_i^j, \xi_j^i\} \), \( \{y^i, y^j_i\} \) the induced coordinate functions on \( TFM \) and \( T\text{Gl}(n) \), respectively. Then \( \omega \) is locally expressed by

\[
\begin{align*}
\omega &= \omega_j^i \left( x^h, x^h_k, \xi^h_k \right) = \delta_j^i, \\
\omega &: \omega_j^i \left( y^h, y^h_k, \xi^h_k \right),
\end{align*}
\]

and thus, if \( \{e_j^i\} \) denotes the canonical basis of \( \text{gl}(n) \equiv T_e \text{Gl}(n) \), we can set

\[
\omega \left( x^h, x^h_k, \xi^h_k \right) = \omega_j^i \left( x^h, x^h_k, \xi^h_k \right) e_j^i \in T_e \text{Gl}(n).
\]

Let \( \sigma : U \rightarrow F_M \) be the natural cross section of \( F_M \) over \( U \), that is \( \sigma(x) = \{x^i, \xi_j^i\} \) for any \( x = (x^1, \ldots, x^n) \in U \), and set \( \omega_U = \sigma^* \omega \).
Then \( \omega_{\mathcal{U}} \) defines the local components \( \Gamma_{jkl}^i \) of \( \Gamma \) on \( \mathcal{U} \) by the equation
\[
\omega_{\mathcal{U}} = \left[ \Gamma_{jkl}^i \, dx^j \right] e^k_i
\]
and, using Proposition 7.3 in [4], one easily finds
\[
\omega_{j}^{i}(x, x^{h}; x^{h}; x^{h}; x^{h}) = \gamma_{j}^{i} + \gamma_{j}^{i} \gamma_{k}^{h} x^{j} x^{h} + \gamma_{j}^{i} \gamma_{k}^{h} \gamma_{j}^{h}
\]
where \( \left\{ \gamma_{j}^{i} \right\} = \left( x_{k}^{i} \right)^{-1} \). Consequently, at the point \( q = \{ x^{h}; x^{h}; x^{h}; x^{h} \} \) we have
\[
\begin{align*}
\frac{\partial \omega_{j}^{i}}{\partial x^{k}}(q) &= \gamma_{j}^{i} \left( \frac{\partial \Gamma_{h}^{l} \gamma_{m}^{h}}{\partial x^{j}} \right) x_{m}^{l} x^{h}, \\
\frac{\partial \omega_{j}^{i}}{\partial x^{k}}(q) &= -\gamma_{j}^{i} \gamma_{h}^{l} x_{m}^{l} x_{j}^{m} + \gamma_{j}^{i} \gamma_{l}^{m} \delta_{k}^{j} x_{m}^{h} - \gamma_{j}^{i} \gamma_{h}^{l} \gamma_{j}^{m}, \\
\frac{\partial \omega_{j}^{i}}{\partial x^{k}}(q) &= \gamma_{j}^{i} \gamma_{l}^{h} \gamma_{h}^{l} x^{h}, \\
\frac{\partial \omega_{j}^{i}}{\partial x^{k}}(q) &= \gamma_{j}^{i} \delta_{k}^{j}.
\end{align*}
\]

Now, let \( \tilde{\omega} \) denote the connection form of the extension of \( \omega_{\perp} \) to \( F_{\mathcal{U}}^{1} M \) via the homomorphism \( \mathcal{J}_{M} : F_{\mathcal{U}}^{1} M \rightarrow F_{\mathcal{U}}^{1} M \); then \( \mathcal{J}_{M} \tilde{\omega} = \mathcal{J}_{n} \circ \omega_{\perp} \). If \( \tilde{\sigma}^{1} : J^{1}_{n} U \rightarrow F_{\mathcal{U}}^{1} M \) denotes the cross-section of \( J^{1}_{n} M \) induced by
\[
\begin{align*}
\sigma : U \rightarrow F_{\mathcal{U}}^{1} M,
\end{align*}
\]
then the composition \( \tilde{\sigma} = \mathcal{J}_{M} \circ \sigma^{1} \) is easily proved to be the natural cross-section of \( F_{\mathcal{U}}^{1} M \) over \( J^{1}_{n} U \).

Let \( \left\{ \Gamma_{BC}^{A} \right\} \) still denote the local components of the linear connection on \( J^{1}_{n} M \) which is defined by \( \tilde{\omega} \), with respect to the induced coordinate system \( \left\{ J^{1}_{n} U, x^{j}, x_{\alpha}^{j} \right\} \). Then, if \( \left\{ E_{B}^{A} \right\} \) denotes the canonical basis of \( gl(n+n^{2}) = T_{\epsilon} gl(n+n^{2}) \), we have...
\[ \tilde{\omega} \left( \frac{\partial}{\partial x^j} \tilde{u} \right) = \tilde{T}_A E^B, \quad \tilde{\omega} \left( \frac{\partial}{\partial x^j} \tilde{u} \right) = \tilde{T}_A E^B, \]

\[ \tilde{u} = \tilde{\partial}(u) \text{ for any point } u \in J^n U. \text{ On the other hand, setting } \]

\[ u_\perp = \sigma^1(u), \]

\[ (T_j M) \left( \frac{\partial}{\partial x^j} u_\perp \right) = \left( \frac{\partial}{\partial x^j} \tilde{u} \right), \quad (T_j M) \left( \frac{\partial}{\partial x^j} u_\perp \right) = \left( \frac{\partial}{\partial x^j} \tilde{u} \right), \]

and hence

\[ \tilde{\omega} \left( \frac{\partial}{\partial x^j} \tilde{u} \right) = j_n \left[ \omega_1 \left( \frac{\partial}{\partial x^j} u_1 \right) \right], \]

\[ \tilde{\omega} \left( \frac{\partial}{\partial x^j} \tilde{u} \right) = j_n \left[ \omega_1 \left( \frac{\partial}{\partial x^j} u_1 \right) \right]. \]

Then, if \( u = \left( x^j, x^\alpha \right) \), we have

\[ \omega_1 \left( \frac{\partial}{\partial x^j} u_1 \right) = \omega_1 \left( x^i, x^i, 0, 0, \delta^i_j, 0, 0, 0 \right) \]

\[ \left[ I, 0; \omega^h \left( x^i, x^i; 0, 0, \delta^i_j \right) \right] \]

\[ \left[ I, 0; \omega^h \left( x^i, x^i; 0, 0, \delta^i_j \right) \right] \]

\[ \omega_1 \left( \frac{\partial}{\partial x^j} u_1 \right) = \omega_1 \left( x^i, x^i, 0, 0, 0, 0, \delta^i_j, 0 \right) \]

\[ \left[ I, 0; \omega^h \left( x^i, x^i; 0, 0, 0 \right) \right] \]

\[ \left[ I, 0; \omega^h \left( x^i, x^i; 0, 0, 0 \right) \right] \]

where \( I \) and \( 0 \) denote the unit matrix and the zero matrix, respectively.

Therefore
and, restricting to $\mathbb{F}M$, that is to the coordinate neighborhood $\mathbb{F}U$, we obtain the local components of the prolongation $\tilde{\Gamma}$ of $\Gamma$ to $\mathbb{F}M$:

\[
\begin{align*}
\tilde{h}_{ji} &= h_{ji}^1, & \tilde{h}_{ji}^0 &= 0, & \tilde{h}_{j\gamma}^\alpha &= 0, & \tilde{h}_{j\gamma j}^\alpha &= 0, & \tilde{h}_{j\gamma j}^\alpha &= 0, \\
\tilde{h}^\alpha_{ji} &= x^k_{\alpha} \tilde{h}^k_{ji}, & \tilde{h}^\alpha_{j\gamma} &= \delta^\alpha_h^{\beta j}, & \tilde{h}^\alpha_{j\gamma j} &= \delta^\alpha_h^{\beta j j}.
\end{align*}
\]

Now, comparing with Mok's result in ([5], p. 81), we deduce

**THEOREM 3.2.** Let $\Gamma$ be a linear connection on $M$. Then the prolongation $\tilde{\Gamma}$ of $\Gamma$ to the frame bundle $\mathbb{F}M$ of $M$ coincides with the complete lift $\Gamma^G$ of $\Gamma$ to $\mathbb{F}M$ defined by Mok [5].

### 4. Prolongation of connections adapted to $G$-structures

We begin this section proving a lemma.

**LEMMA 4.1.** Let $P(M, \pi, G)$ be a reduced bundle of the principal fibre bundle $P'(M, \pi, G')$, and let $\omega'$ be a connection form on $P'$ reducible to the connection form $\omega$ on $P$. Then $J^1_P\left(J^1_M, \pi, J^1_P G\right)$ is a reduced bundle of $J^1_{P'}\left(J^1_M, \pi, J^1_{P'} G\right)$, and the prolongation $\omega'_1$ of $\omega'$ to $J^1_{P'}$ is reducible to the prolongation $\omega_1$ of $\omega$ to $J^1_P$.

**Proof.** Let $\phi : P \to P'$ be the injective homomorphism of principal bundles which yields the reduction of $G'$ to $G$, and denote also by $\phi : G \to G'$ the corresponding Lie group homomorphism. Then a straightforward computation shows that the induced bundle homomorphism $\phi^1 : J^1_P \to J^1_{P'}$ yields a reduction of $J^1_P G'$ to $J^1_P G$ whose associate Lie group homomorphism is the induced one, $\phi^1 : J^1_P G \to J^1_{P'} G'$. On the other
hand, that $\omega'$ is reducible to $\omega$ means that the following diagram commutes:

\[
\begin{array}{ccc}
TP & \xrightarrow{\omega} & TG \\
\downarrow T\tilde{\delta} & & \downarrow T\tilde{\delta} \\
TP' & \xrightarrow{\omega'} & TG'.
\end{array}
\]

Therefore, from (6) in §1 we obtain a new commutative diagram

\[
\begin{array}{cccc}
TJ^1_p & \xrightarrow{\alpha^{p,1}_P} & J^1_p^TP & \xrightarrow{\omega^1} & J^1_p^TG & \xrightarrow{\alpha^{1,p}_G} & TJ^1_G \\
\downarrow T\tilde{\delta}^1 & & \downarrow (T\tilde{\delta})^1 & & \downarrow (T\tilde{\delta})^1 & & \downarrow T\tilde{\delta}^1 \\
TJ^1_p' & \xrightarrow{\alpha^{p,1}_{P'}} & J^1_p^TP' & \xrightarrow{(\omega')^1} & J^1_p^TG' & \xrightarrow{\alpha^{1,p}_{G'}} & TJ^1_G'.
\end{array}
\]

which implies that $\omega'_1$ is reducible to $\omega_1$. #

Let $G$ be a Lie subgroup of $GL(n)$ and denote

\[
\tilde{G} = j_n\left(J^1_n^G\right) \subset GL(n+2^n). \]

Assume that $P(M, \pi, G)$ is a reduced bundle of the frame bundle $FM(M, \pi_M, GL(n))$ of $M$, $n = \dim M$, that is $P$ is a $G$-structure on $M$. In [2] we have defined the prolongation of the $G$-structure $P$ on $M$ to a $\tilde{G}$-structure $\tilde{P}$ on $FM$ as follows: we consider the injective bundle homomorphism $i^1 : J^1_n^P \rightarrow J^1_n^{FM}$ induced by

\[
i : P \rightarrow FM \quad \text{and define } \tilde{P} = (j_M \circ i^1)(J^1_n^P)|_{FM}.
\]

As usual, we say that a linear connection $\Gamma$ on $M$ is adapted to the $G$-structure $P(M, \pi, G)$ on $M$ if $\Gamma$ is reducible to a connection on $P$. Then, taking into account Lemma 4.1 and the results in the previous section, we easily deduce

**Theorem 4.2.** Let $\Gamma$ be a linear connection on $M$ adapted to a $G$-structure $P(M, \pi, G)$ on $M$. Then the prolongation $\tilde{\Gamma}$ of $\Gamma$ to $FM$ is adapted to the $\tilde{G}$-structure $\tilde{P}(FM, \pi, \tilde{G})$ on $FM$, prolongation of $P$ to $FM$.

We remark that Theorem 4.2 improves some particular results in [1] and
[5] where only the prolongations (or complete lift) of \( G \)-structures on \( M \) defined by tensor fields of types \((0, s)\) and \((1, s)\) have been considered.

References


Departamento de Geometria y Topologia, Facultad de Matematicas, Universidad de Santiago de Compostela, Spain.