A norm $|\cdot|$ of a Banach space $X$ is called locally uniformly rotund if $\lim |x_n - x| = 0$ whenever $x_n, x \in X$, and

$$\lim 2|x|^2 + 2|x_n|^2 - |x + x_n|^2 = 0.$$ 

It is shown that such an equivalent norm exists on every Banach space $X$ which possesses a projectional resolution $\{P_\alpha\}$ of the identity operator, for which all $(P_{\alpha+1} - P_\alpha)X$ admit such norms. This applies, for example, for the dual space of a space with Fréchet differentiable norm.

Projectional resolutions of the identity in nonseparable Banach spaces were first studied by Amir and Lindenstrauss [1]. It has proved to be a very powerful tool in the study of geometry of some nonseparable Banach spaces. For example, it was used to prove that every weakly compactly generated Banach space admits an equivalent locally uniformly rotund norm in [6], and that the dual space of a space with Fréchet smooth norm admits an equivalent strictly convex norm in [4]. In the projectional resolutions used in [6], the spaces $(P_{\alpha+1} - P_\alpha)X$ were all separable. Sometimes one

Received 31 October 1983. The author is deeply indebted to the Sonderforschungsbereich 72 der Universität Bonn for the support during the preparation of the paper. He also thanks the University of Dortmund for providing him with excellent working conditions during his stay there.
cannot ensure this separability requirement, but only the fact that
\((P_{\alpha+1} - P_{\alpha}) X\) all admit equivalent locally uniformly rotund norms. Then, a
slight variant of the construction from [6], given here, works to give the
result. This is, for example, the case mentioned in the abstract.

We will work in real Banach spaces. \(N\) will denote the set of all
positive integers.

**THEOREM 1.** Let \((X, \| \cdot \|)\) be a Banach space which possesses a family
\(\{P_\alpha\}, \alpha \in \Gamma,\) of bounded linear operators \(P_\alpha : X \to X,\) such that

(i) the map \(T\) defined on \(X\) by \(T(x) = \|P_\alpha x\|\) for \(x \in X,\)
maps \(X\) into \(c_0(\Gamma),\)

(ii) if \(x \in X,\) then \(x \in \text{sp}\{P_\alpha x\},\)

(iii) for each \(\alpha \in \Gamma,\) \(P_\alpha X\) admits an equivalent locally
uniformly rotund norm.

Then \(X\) admits an equivalent locally uniformly rotund norm.

An application of Theorem 1 is given below.

**COROLLARY 1.** Suppose that \(X\) is a Banach space which admits a real
valued continuously Fréchet differentiable function with bounded non-empty
support. Then \(X^*\) admits an equivalent locally uniformly rotund norm.

Proof. Using the method of [5], it was shown in [4] that under our
assumption on \(X,\) the identity operator on \(X^*\) admits a projectional
resolution \(\{P_\alpha\}\) for which each \(P_\alpha X^*\) is isometric to some \(X^*\) where
dens \(X_\alpha\) = dens \(X^*\) < dens \(X = dens X^*\). So, using a transfinite induction
argument on dens \(X,\) one can ensure the existence of operators needed in
Theorem 1, by the projections \(\{P_{\alpha+1} - P_{\alpha}\}\) on \(X^*\). This proves Corollary 1
from Theorem 1.

Proof of Theorem 1. A variant of that in [6]. Assume, without loss
of generality, that \(|P_\alpha| \leq 1, \alpha \in \Gamma.\) Let \(h_\alpha(x) = \|P_\alpha x\|_\alpha\) where \(\alpha \in \Gamma\)
and \(x \in X,\) and where \(|\cdot|_\alpha\) is an equivalent locally uniformly rotund
norm on \(P_\alpha X\) such that \(|\cdot| \leq |\cdot|_\alpha \leq 2|\cdot|.\) Furthermore, for \(k \in N,\) let
Locally uniformly rotund renorming

\[ r_j^k = \left\{ r_{j,1}^k, \ldots, r_{j,k}^k \right\}, \ j = 1, 2, \ldots \ \text{be a sequence of all (ordered)} \ \text{k-tuples of rational numbers. Let Q be a map which assigns to each finite subset } A \subset \Gamma \text{ an enumeration of } A \text{ to an (ordered) sequence } Q(A) = (\alpha_1, \ldots, \alpha_n), \ \alpha_i \text{ distinct.} \]

If \( A \subset \Gamma \), where \( \text{card } A = n \) and \( Q(A) = (\alpha_1, \ldots, \alpha_n) \) and if \( j, l \in \mathbb{N} \) define

\[
E(A, j, l)(x) = \left( \sum_{\alpha \in A} h_{\alpha}^2(x) + \frac{1}{l} \cdot \frac{1}{\sum_{i=1}^n |r_{j,i}^n| + 1} \left| x - \sum_{i=1}^n r_{j,i}^n \alpha_i x \right|^2 \right)^{\frac{1}{2}}
\]

for \( x \in X \). Furthermore, if \( j, l, n \in \mathbb{N} \) are fixed, let

\[
G(j, l, n)(x) = \sup \{ E(A, j, l)(x), A \subset \Gamma, \ \text{card } A = n \}
\]

for \( x \in X \).

Finally, let

\[
\| x \| = \left( |x|^2 + \sum_{j+l+n} G^2(j, l, n)(x) \right)^{\frac{1}{2}}
\]

for \( x \in X \). Then it is easy to see that \( \| \cdot \| \) is an equivalent norm on \( X \).

We shall show that it is locally uniformly rotund.

To this end suppose \( x_k, x \in X \), are such that \( |x| = 1 \) and

\[
\lim 2\| x \|^2 + 2\| x_k \|^2 - \| x + x_k \|^2 = 0.
\]

We need show that \( \lim |x_k - x| = 0 \). Thus, given \( \varepsilon > 0 \), we shall show that beginning with some index \( k_0 \), \( |x_k - x| < 4\varepsilon \). For that, first find a set \( A \subset \{ \alpha \in \Gamma, h_{\alpha}(x) \neq 0 \} \), \( Q(A) = (\alpha_1, \ldots, \alpha_n) \) such that \( \rho(z, sp\{P_{\alpha}x, \alpha \in A\}) < \varepsilon \).

Assume without loss of generality that

\[
\min_{\alpha \in A} h_{\alpha}^2(x) - \max_{\alpha \notin A} h_{\alpha}^2(x) = d > 0.
\]

Let
V. Zizler

\[ |x - \sum_{s=1}^{n} r_s P_{\alpha_s} x| < \varepsilon \]

where \( \{r_1, \ldots, r_n\} \) is a sequence of rationals.

Choose \( j \in \mathbb{N} \) so that \( r_{n_j} = (r_{n_1}, \ldots, r_{n_n}) = (r_1, \ldots, r_n) \).

Finally let \( \ell > 4/d \). Fix these \( n, j, \ell \). From (1) we have that

\[ a_k = 2G^2(j, \ell, n)(x) + 2G^2(j, \ell, n)(x_k) - G^2(j, \ell, n)(x+x_k) \to 0 \]  

Let \( A_k \subset \Gamma \), card \( A_k = n \), \( Q(A_k) = \left[ a_1, \ldots, a_n \right] \), be so that

\[ 0 \leq \sigma_k = G^2(j, \ell, n)(x+x_k) - E^2(A_k, j, \ell)(x+x_k) \to 0 \]  

Then

\[ a_k \geq 2G^2(A_k, j, \ell)(x) + 2G^2(A_k, j, \ell)(x_k) - E^2(A_k, j, \ell)(x+x_k) - \sigma_k = b_k - \sigma_k \]

for some nonnegative \( b_k \) and thus, since \( \lim a_k = \lim \sigma_k = 0 \), we have that \( \lim b_k = 0 \) as well. Thus

\[ b_k = 2 \sum_{\alpha \in A_k} h^2_\alpha(x) + \frac{2}{\ell} \left\{ 1/ \left( \sum_{i=1}^{n} |r_j, i|+1 \right) \right\}^2 \left| x - \sum_{i=1}^{n} r_j, i \alpha_i x \right|^2 \]

\[ + 2 \sum_{\alpha \in A_k} h^2_\alpha(x_k) + \frac{2}{\ell} \left\{ 1/ \left( \sum_{i=1}^{n} |r_j, i|+1 \right) \right\}^2 \left| x_k - \sum_{i=1}^{n} r_j, i \alpha_i x_k \right|^2 \]

\[ - \sum_{\alpha \in A_k} h^2_\alpha(x+x_k) \]

\[ - \frac{1}{\ell} \cdot 1/ \left( \sum_{i=1}^{n} |r_j, i|+1 \right) \left| x+x_k - \sum_{i=1}^{n} r_j, i \alpha_i x_k \right|^2 \to 0 \].

Thus by the convexity argument,

\[ 2 \sum_{\alpha \in A_k} h^2_\alpha(x) + 2 \sum_{\alpha \in A_k} h^2_\alpha(x_k) - \sum_{\alpha \in A_k} h^2_\alpha(x+x_k) \to 0 \]
and

\[(6) \quad \left| x - \sum_{i=1}^{n} r_{j,i}^n \frac{x}{\alpha_i^k} \right| - \left| x_k - \sum_{i=1}^{n} r_{j,i}^n \frac{P_k x_k}{\alpha_i^k} \right| \to 0.\]

We now show that, beginning with some index \(k_1\), \(A_k = A\) and so then \(\{\alpha_k, \ldots, \alpha_n^k\} = \{\alpha_1, \ldots, \alpha_n\}\). Indeed, if this were not the case, we would have for infinitely many \(k\), \(A_k \neq A\). But for these \(k\) we would have

\[
a_k \geq 2 \sum_{\alpha \in A} h^2(x_k) + \frac{2}{l} \left( \frac{1}{l} \left( \sum_{i=1}^{n} r_{j,i}^n \left| x_{i}^{n+1} \right| \right) \right)^2 \left| x - \sum_{i=1}^{n} r_{j,i}^n \frac{P_k x_k}{\alpha_i^k} \right| - \frac{2}{l} \sum_{\alpha \in A_k} h^2(x_{k+1})
\]

\[+ \frac{2}{l} \left( \frac{1}{l} \left( \sum_{i=1}^{n} r_{j,i}^n \left| x_{i}^{n+1} \right| \right) \right)^2 \left| x - \sum_{i=1}^{n} r_{j,i}^n \frac{P_k x_k}{\alpha_i^k} \right| - \frac{1}{l} \left( \frac{1}{l} \left( \sum_{i=1}^{n} r_{j,i}^n \left| x_{i}^{n+1} \right| \right) \right)^2 \left| x_k + x_k - \sum_{i=1}^{n} r_{j,i}^n \frac{P_k (x+x_k)}{\alpha_i^k} \right| - c_k \]

\[= 2 \left( \sum_{\alpha \in A} h^2(x_k) - \sum_{\alpha \in A_k} h^2(x_k) \right)
\]

\[+ \frac{2}{l} \left( \frac{1}{l} \left( \sum_{i=1}^{n} r_{j,i}^n \left| x_{i}^{n+1} \right| \right) \right)^2 \left| x - \sum_{i=1}^{n} r_{j,i}^n \frac{P_k x_k}{\alpha_i^k} \right| - \frac{2}{l} \sum_{\alpha \in A_k} h^2(x_{k+1})
\]

\[+ 2 \sum_{\alpha \in A_k} h^2(x_{k+1}) + \frac{2}{l} \left( \frac{1}{l} \left( \sum_{i=1}^{n} r_{j,i}^n \left| x_{i}^{n+1} \right| \right) \right)^2 \left| x_k - \sum_{i=1}^{n} r_{j,i}^n \frac{P_k x_k}{\alpha_i^k} \right| - \frac{1}{l} \left( \frac{1}{l} \left( \sum_{i=1}^{n} r_{j,i}^n \left| x_{i}^{n+1} \right| \right) \right)^2 \left| x_k + x_k - \sum_{i=1}^{n} r_{j,i}^n \frac{P_k (x+x_k)}{\alpha_i^k} \right| - c_k \]

\[\geq 2d - \frac{1}{l} - c_k \geq d - c_k,
\]

which is a contradiction with the fact that \(\lim a_k = \lim c_k = 0\), \(d > 0\).
Therefore, beginning with some index \( k_1 \), we must have that \( A_k = A \) and thus \( \{\alpha_1^k, \ldots, \alpha_n^k\} = (\alpha_1, \ldots, \alpha_n) \).

Now from (5) we have that, for each \( \alpha \in A \),

\[
2h_\alpha^2(x) + 2h_\alpha^2(x_k) - h_\alpha^2(x+x_k) \to 0
\]

and thus from the locally uniformly rotund of \( |\cdot|_\alpha \), we have that \( P_\alpha(x_k - x) \to 0 \). Therefore

\[
\left| \sum_{i=1}^n r_i^n P_\alpha z_i x - \sum_{i=1}^n r_i^n P_\alpha z_i x_k \right| \to 0
\]

and using this and (2) and (6) we have that beginning with some index \( k_0 \), we have that \( |x - x_k| < 4\varepsilon \). The proof is finished.

**Remark 1.** A version of Corollary 1 was published in [2]. However, we are afraid that the proof given in [2] is not correct.

**Remark 2.** The locally uniformly rotund norm constructed in Corollary 1 on \( X^* \) cannot generally be made to be a dual one (see [5]).

**References**


Institut für Angewandte Mathematik der Universität Bonn,
Wegelerstr. 6,
D-5300 Bonn,
West Germany

and

Department of Mathematics,
University of Alberta,
Edmonton T6G 2G1,
Alberta,
Canada.