A note on Fritz John sufficiency

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An elementary proof is given of a sufficient optimality condition recently proven by B.D. Craven. This proof avoids the use of a transposition theorem and this allows for a strengthening of Craven's result.

Recently Craven [2] has given a general sufficiency theorem for a Fritz John necessary condition [6] to imply optimality. This extended a sufficiency result for complex programmes given by Gulati [5] which was in turn stimulated by necessary conditions proved by Craven and Mond [3], [4].

It is the purpose of this note to correct an omission in the statements of the theorems in [2] and [5] and to provide a simpler proof of a more general result than in [2]. Our notation is as in [2]. Consider the non-linear programme

\[(P) \quad \min \{ \text{re } f(x) : -g(x) \in S, h(x) = 0, -k(x) \in N \}, \quad x \in U\]

where \(X, Y, Z, W\) are real or complex Banach spaces, \(U\) is open in \(X\), \(S \subset Y, T \subset Z, N \subset W\) are closed, convex cones, \(f : U \rightarrow \mathbb{R}\) (or \(\mathbb{C}\)), \(g : U \rightarrow Y\), \(h : U \rightarrow Z\) are Gâteaux differentiable, and \(k : X \rightarrow W\) is affine and continuous. The dual cone of a convex cone \(S\) is

\[S^* = \{ u \in Y' : \text{re } u(s) \geq 0 \text{ for all } s \in S \}, \]

where \(Y'\) is the topological dual of \(Y\). Let \(\mathbb{R}^+\) denote the non-negative real axis, \(\text{int } S\) denote the interior of \(S\).

The map \(g : U \rightarrow Y\) is (strictly) \(S\)-convex at \(a \in U\) if for each \(x \in U/\{a\}\),

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$g(x) - g(a) - g'(a)(x-a) \in S (\in \text{int } S)$.

(This latter supposes \( \text{int } S \neq \emptyset \).)

The map \( f : X \to \mathbb{R} \) is pseudoconvex at \( a \) if

\[ x \in U \text{ and } f(x) < f(a) \text{ implies } f'(a)(x-a) < 0 \,.

We now present our result.

**Theorem.** Suppose that \( a \in U \), \( \text{re } f \) is pseudoconvex, \( g \) is strictly \( S \)-convex, and \( h \) is strictly \( T \)-convex. Suppose there is a solution \( r, v, w, m \) to

\[
\begin{align*}
(i) \quad \text{re}(rf'(a)+vg'(a)+wh'(a)+mk'(a)) &= 0, \\
(ii) \quad \text{re } vg(a) &= 0, \quad \text{re } mk(a) = 0 ,
\end{align*}
\]

with \( r \in R^+, v \in S^*, w \in T^*, m \in N^* \), and such that not all of \( r, v, w \) are zero.

It follows that if \( a \) is feasible for (P) it is optimal for (P).

**Proof.** Suppose first that \( r = 0 \). If there is no \( x \neq a \), feasible for (P), we are done since \( a \) is assumed feasible. Suppose \( \bar{x} \neq a \) is feasible. Then

\[
\begin{align*}
(1) \quad g'(a)(\bar{x}-a) + g(a) &\in -\text{int } S \\
(2) \quad h'(a)(\bar{x}-a) + h(a) &\in -\text{int } T .
\end{align*}
\]

Then, since one of \( v, w \) is non-zero, we have \( v \in S^*, w \in T^* \)

\[
\text{re}(vg'(a)(\bar{x}-a)+vg(a)+wh'(a)(\bar{x}-a)+wh(a)) < 0 .
\]

Since \( \text{re } vg(a) = 0 \) by (ii) and \( wh(a) = 0 \) by the feasibility of \( a \), we have

\[
\text{re}(vg'(a)(\bar{x}-a)+wh'(a)(\bar{x}-a)) < 0 .
\]

Also \( \text{re } mk(a) = 0 \) by (ii), so

\[
\text{re}(mk'(a)(\bar{x}-a)) = \text{re}(mk(\bar{x})-mk(a)) \leq 0 ,
\]

since \( k \) is affine, \( \bar{x} \) is feasible, \( m \in N^* \), and \( k(\bar{x}) \in -N \). Adding (4) and (5) contradicts (P). Thus \( r \neq 0 \). We may assume that \( r = 1 \).

The optimality of \( a \in U \) now follows from the pseudoconvexity of \( \text{re } f \) and the convexity of \( G(x) = \text{re}(vg(x)+wh(x)+mk(x)) \) at \( a \), since
REMarks. (i) In both [2], [5], it is not assumed that \( \alpha \) is feasible. This is clearly necessary as is shown by the real programme

\[
\begin{align*}
\text{minimize } & \left\{ \frac{x^2}{2} : \frac{(x-1)^2}{2} \leq 0 \right\}
\end{align*}
\]

which satisfies the conditions of Theorem 1 of [2]. Now

\[
\text{re}\{rf'(\alpha)+vg'(\alpha)\} = 0 , \quad \text{re} \, vg(\alpha) = 0 , \quad r \in \mathbb{R}^+ , \quad v \in S^4 ,
\]

is solved by \( r = 1 , \quad v = 0 , \quad \alpha = 0 \), or \( r = 0 , \quad v = 1 , \quad \alpha = 1 \), and the former is not feasible; hence not optimal.

(ii) The proof presented here removes Craven's condition that either

\([k(\alpha)k'(\alpha)]\) is surjective or that \( k^T(N^*) \) is weak star closed by avoiding the use of a Transposition Theorem [2].

(iii) In the same manner as in Theorem 1 we can remove the extraneous condition on \( k \) in Theorems 2 and 3 of [2]. In the latter case this is just the observation that if one of \( r \) or \( v \) is nonzero we need only assume \( h \) is \( T \)-convex.

(iv) It seems to the author that Theorem 1 is more properly a Kuhn-Tucker Sufficiency Condition [1] than a Fritz John condition since it essentially gives a constraint qualification to force \( r \) to be nonzero. It would be interesting to see a "true" Fritz John condition that gave necessary and sufficient conditions for optimality in absence of any added convexity hypotheses.

References


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