On a conjecture of Mahler

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Let $R$ be the field of real numbers. For $a$ in $R$, let $||a||$ be the distance of $a$ from the nearest integer. The following conjecture of Kurt Mahler [Bull. Austral. Math. Soc. 14 (1976), 463-465] is proved.

Let $m, n$ be two positive integers $n \geq 2m$. Let $S$ be a finite or infinite set of positive integers with the following properties:

(Q1) $S$ contains the integers $m, m+1, \ldots, n-m$;

(Q2) every element of $S$ satisfies

$$||a/n|| \geq m/n .$$

Then

$$\sup_{a \in R} \inf_{s \in S} ||sa|| = m/n .$$

1. Introduction

Let $R$ be the field of real numbers. For $a \in R$, let $||a||$ be the distance of $a$ from the nearest integer. Mahler [3] has proved the following:

**Theorem.** Let $S$ be a finite or infinite set of positive integers with the following two properties:

(i) $S$ contains the integers $1, 2, \ldots, n-1$;

(ii) $S$ does not contain any multiple of $n$.

Received 23 September 1976. Communicated by Kurt Mahler. The authors are grateful to Professor R.P. Bambah for some useful comments.
Then
\[ \sup_{\alpha \in \mathcal{R}} \inf_{s \in S} \|s\alpha\| = \frac{1}{n}. \]
Mahler also conjectured the following generalization.

**CONJECTURE.** Let \( m, n \) be two positive integers such that \( 2m \leq n \). Let \( S \) be a finite or infinite set of positive integers with the following two properties:

(Q₁) \( S \) contains the integers \( m, m+1, \ldots, n-m \);

(Q₂) every element of \( S \) satisfies the inequality
\[ \|s/n\| \geq \frac{m}{n}. \]
Then
\[ \sup_{\alpha \in \mathcal{R}} \inf_{s \in S} \|s\alpha\| = \frac{m}{n}. \]

Our object in this note is to prove the above conjecture.

2.

We shall use the following:

**LEMMA.** Let \( m \) be a positive integer. Let \( K \) be a convex body in the \( n \)-dimensional euclidean space \( \mathbb{R}^n \) with centre \( 0 \) and volume 
\[ V(K) > 2^n m. \] Then there are \( m \) non-zero points \( X_1, \ldots, X_m \) of the integral lattice such that

(i) \( X_i \in K, \ 1 \leq i \leq m \),

(ii) \( X_i - X_j \in K, \ 1 \leq i, j \leq m \),

(iii) \( 0 < X_1 < X_2 < \ldots < X_m \), where \( \prec \) is the lexicographic ordering in \( \mathbb{R}^n \).

**REMARK.** This result is essentially due to van der Corput [11], but he did not bring out the fact that the points \( X_i \) can be chosen to satisfy (ii) and (iii) also. Here we indicate the necessary modifications to ensure (ii) and (iii).
Proof. van der Corput's result rests on the fact that \( V(\frac{1}{2}K) > m \) implies the existence of a point \( Z \in \mathbb{R}^n \) and \((m+1)\) distinct points \( Y_0, Y_1, \ldots, Y_m \) such that \( Z \in \frac{1}{2}K + Y_i \), \( 0 \leq i \leq m \) (see, for example, Lekkerkerker [2], p. 44). We can suppose that the points \( Y_i \) are arranged in the lexicographic order

\[
Y_0 < Y_1 < \ldots < Y_m .
\]

Let

\[
X_i = Y_i - Y_0 , \quad 1 \leq i \leq m .
\]

Then \( 0 < X_1 < X_2 < \ldots < X_m \). Also since \( \frac{1}{2}K \) is symmetric convex with centre 0, we have

\[
X_i = 2 \frac{(Z-Y_0)-(Z-Y_i)}{2} \in K
\]

and

\[
X_i - X_j = Y_i - Y_j = 2 \frac{(Z-Y_j)-(Z-Y_i)}{2} \in K .
\]

This proves the lemma.

3.

Proof of Conjecture. For \( \alpha = 1/n \) the condition \((Q_2)\) implies that

\[
\inf_{s \in S} \|s\alpha\| \geq m/n .
\]

Therefore

\[
\sup_{\alpha \in \mathbb{R}} \inf_{s \in S} \|s\alpha\| \geq m/n .
\]

It remains to prove that for every \( \alpha \in \mathbb{R} \),

\[
\inf_{s \in S} \|s\alpha\| \leq m/n .
\]

Let \( T = \{m, m+1, \ldots, n-m\} \).

Since \( S \supset T \) it suffices to prove that there is a \( t \in T \) such that

\[
\|t\alpha\| \leq m/n .
\]
If \( n = 2m \), then \( T = \{m\} \) and \( \|ma\| \leq \frac{1}{2} = m/n \) for every real number \( a \).

Let \( n > 2m \). Let \( 0 < \varepsilon < (n/2m) - 1 \), so that \( \frac{m(1+\varepsilon)}{n} < \frac{1}{2} \). Since \( T \) is a finite set it is enough to prove that for every such \( \varepsilon \), there is a \( t \in T \) such that

\[
(1) \quad \|ta\| \leq \frac{m(1+\varepsilon)}{n}.
\]

Consider the parallelogram \( \Pi \) with centre 0 defined by

\[
(2) \quad |\alpha x - y| < \frac{m(1+\varepsilon)}{n},
\]

\[
(3) \quad |x| < n.
\]

If \((x, y)\) is an integral point in \( \Pi \), then clearly

\[
\|xa\| < \frac{m(1+\varepsilon)}{n}.
\]

The area of \( \Pi \) is equal to \( 4m(1+\varepsilon) \geq 4m \). By the lemma it follows that \( \Pi \) contains \( m \) non-zero integral points

\[
X_i = (x_i, y_i), \quad i = 1, 2, \ldots, m, \quad 0 \leq x_1 \leq x_2 \leq \ldots \leq x_m,
\]

and

\[
X_i - X_j \in \Pi \quad \text{for} \quad 1 \leq i, j \leq m.
\]

We observe that

(i) \( x_i > 0 \) for each \( i \), because otherwise (2) implies that

\[
|y_i| < \frac{m(1+\varepsilon)}{n} < \frac{1}{2} \quad \text{and hence} \quad y_i = 0;
\]

(ii) \( x_i \neq x_j \) when \( i \neq j \), because \( x_i = x_j \) implies, by (2),

\[
|y_i - y_j| < \frac{2m(1+\varepsilon)}{n} \leq \frac{1}{2} \quad \text{and hence} \quad y_i = y_j; \quad \text{that is,} \quad X_i = X_j \quad \text{so that} \quad i = j.
\]

Hence

\[
1 \leq x_1 < x_2 < \ldots < x_m \leq n - 1.
\]
If $x_i \notin T$ for some $i$, then $\|x_i \alpha\| < \frac{m(1+\epsilon)}{n}$ and the result follows. Suppose that $x_i \in T$ for $i = 1, 2, \ldots, m$. Let

$$1 \leq x_1 < x_2 < \ldots < x_\gamma < m < n-m < x_{\gamma+1} < \ldots < x_m \leq n-1.$$  

Clearly $1 \leq \gamma \leq m-1$, and $1 \leq i \leq \gamma$. If the integers $x_i + n - m$, $1 \leq i \leq \gamma$ are distinct from the integers $x_{\gamma+j}$, $1 \leq j \leq m-\gamma$, then the interval $n-m+1 \leq x \leq n-1$ contains at least $\gamma + (m-\gamma) = m$ integers, which is impossible because the length of the interval is $m-2$. Therefore there exist $i$ and $j$, $1 \leq i \leq \gamma$, $1 \leq j \leq m-\gamma$, such that $x_i + n - m = x_{\gamma+j}$. Then $x_{\gamma+j} - x_i = (n-m, y)$ is an integral point in $\Pi$. Therefore

$$\|(n-m)\alpha\| < \frac{m(1+\epsilon)}{n}$$

and $n-m \in T$. Thus the conjecture is proved.

References


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