Sums of Deficiencies of Algebroid Functions

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Let \( f(z) \) be an \( n \)-valued algebroid function of finite lower order. In the present paper, we give a spread relation of \( f(z) \) and some applications of the spread relation.

1. Introduction

Let \( f(z) \) be an \( n \)-valued algebroid function of finite lower order, defined by an irreducible equation

\[
A_0 f^n + A_1 f^{n-1} + \cdots + A_{n-1} f + A_n = 0
\]

where \( A_0, A_1, \ldots, A_n \) are entire functions without common zeros.

Fix a sequence \( (r_j) \) of Pólya peaks of order \( \mu \) of \( f(z) \) (or \( T(r, f) \)). Let \( f_j(z) \) be the \( j \)th determination of \( f(z) \) and \( \Lambda(r) \) a positive function with

\[
\Lambda(r) = o(T(r, f)), \quad r \to \infty.
\]

Define the sets of arguments \( E'_\Lambda(r, a) \subset (-\pi, \pi] \) by

\[
E'_\Lambda(r, a) = \{ \theta : \min_{1 \leq j \leq n} |f_j(re^{i\theta}) - a| < e^{\Lambda(r)}, a \neq \infty \},
\]

\[
E'_\Lambda(r, \infty) = \{ \theta : \max_{1 \leq j \leq n} |f_j(re^{i\theta})| > e^{\Lambda(r)} \},
\]

and let

\[
\sigma'_\Lambda(a) = \liminf_{j \to \infty} \text{meas } E'_\Lambda(r_j, a)
\]

\[
\sigma'(a) = \inf_{\Lambda} \sigma'_\Lambda(a)
\]

where the infimum is taken over all functions \( \Lambda(r) \) satisfying (1.2). Niino ([5]) proved the following spread relation

\[
\sigma'(a) \geq \min \left\{ 2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a, f)}{2}} \right\}.
\]

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Now we assume that
\[ \|A(z)\| = \left( |A_0|^2 + |A_1|^2 + \cdots + |A_n|^2 \right)^{1/2}, \]
\[ \|a\| = \left\{ \begin{array}{ll} \left( |a|^{2n} + |a|^{2n-2} + \cdots + |a|^2 + 1 \right)^{1/2}, & a \neq \infty \\ 1, & a = \infty, \end{array} \right. \]
\[ F(z, a) = \left\{ \begin{array}{ll} A_0 a^n + A_1 a^{n-1} + \cdots + A_{n-1} a + A_n, & a \neq \infty \\ A_0, & a = \infty, \end{array} \right. \]
\[ m(r, a, A) = \frac{1}{2\pi} \int_0^{2\pi} \log \left( \frac{\|A\| \cdot \|a\|}{|F(z, a)|} \right) d\theta, \quad z = re^{i\theta}, \]
\[ \mu(r, A) = \frac{1}{2\pi} \int_0^{2\pi} \log \max_{0 \leq j \leq n} |A_j(re^{i\theta})| d\theta. \]

Set
\[ T(r, a, A) = m(r, a, A) + N(r, 0, F(z, a)); \]
by Jensen’s formula, we have
\[ T(r, a, A) = \frac{1}{2\pi} \int_0^{2\pi} \log (\|A\| \cdot \|a\|) d\theta + O(1). \]

Since
\[ \max_{0 \leq j \leq n} |A_j(z)| \leq \|A(z)\| \leq (n + 1)^{1/2} \max_{0 \leq j \leq n} |A_j(z)|, \]
we have
\[ |T(r, a, A) - n\mu(r, A)| = O(1). \]

By using Valiron’s result ([8]), we get
\[ |T(r, a, A) - nT(r, f)| = O(1), \]
so that
\[ \delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, 0, F(z, a))}{T(r, a, A)}. \]

With these notations, we define the sets of arguments \( E_A(r, a) \subset (-\pi, \pi) \) by
\[ E_A(r, a) = \{ \theta : \frac{\|A\| \cdot \|a\|}{|F(z, a)|} > e^A(r), \quad z = re^{i\theta} \} \]
and let
\[ \sigma_A(a) = \liminf_{j \to \infty} \text{meas} E_A(r_j, a) \]
\[ \sigma(a) = \inf_A \sigma_A(a) \]
where the infimum is taken over all functions satisfying (1.2).

In the present paper, we prove a spread relation analogous to (1.3) with the spread \( \sigma'(a) \) replaced by \( \sigma(a) \) and give some applications of the spread relation.
2. SPREAD RELATIONS

In the following statements the notations of the introduction are taken for granted. For a complex number \( a \), we set

\[
m^*(z, a) = \sup_E \frac{1}{2\pi} \int_E \log \frac{||A|| \cdot ||a||}{|F(\xi, a)|} \, d\omega, \quad \xi = re^{i\omega},
\]

\[
(z = re^{i\theta}, \ 0 < r < \infty, \ 0 < \theta \leq \pi)
\]

where the supremum is taken over all measurable sets \( E \subset (-\pi, \pi) \) of Lebesgue measure \( 2\theta \), and

\[
T^*(z) = T^*(z, a) = m^*(z, a) + N(r, 0, F(z, a)).
\]

The function \( T^*(z) \) is defined on the set

\[
H_1 = \{ z: \ \text{Im} \ z \geq 0, \ z \neq 0 \}.
\]

It follows from the definition of this function that for arbitrary \( r \) such that \( 0 < r < \infty \) and a complex number \( a \) we have

\[
(2.1) \quad \sup T^* (re^{i\theta}) = T(r, a, A),
\]

\[
(2.2) \quad T^*(r) = N(r, 0, F(z, a)).
\]

**Lemma 2.1.** \( T^*(z) \) is subharmonic in the half plane \( \text{Im} \ z > 0 \) and is continuous in \( H_1 \).

**Proof:** By a result of Goldberg ([3]), we know that \( \log ||A|| \) is subharmonic so that \( \log (||A|| : ||a||) \) is subharmonic. Since \( F(z, a) \) is an entire function, we have \( \log |F(z, a)| \) is a subharmonic function. By the Theorem A' in [2], Lemma 2.1 follows. \( \square \)

**Theorem 2.1.** Let \( f(z) \) be an \( n \)-valued algebroid function of lower order \( \mu (0 < \mu < \infty) \), defined by the equation (1.1); then

\[
(2.3) \quad \sigma(a) \geq \min \left\{ 2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a, f)}{2}} \right\},
\]

where \( a \) is a deficient value of \( f(z) \).

**Proof:** We consider the following two cases.

(1) \( 4 \arcsin \sqrt{(\delta(a, f)/2)/\mu} < 2\pi \).

To deduce inequality (2.3) we should use Lemma 2.1 and the proof of (1.4) in [1]; let us outline the method of the proof of inequality (2.3) (for details see the proof of relation (1.4) in [1, p.429–434].
We set

\[
\gamma = \frac{2}{\pi \mu} \arcsin \sqrt{\frac{\delta(a, f)}{2}}.
\]

The following inequality is fulfilled for the function

\[
v(z) = \begin{cases} 
0, & z = 0 \\
T^*(z\gamma), & z = re^{i\theta}, \ 0 < r < \infty, \ 0 \leq \theta \leq \pi
\end{cases}
\]

which is subharmonic in the half plane \(\text{Im} \ z > 0\) (see Lemma 2.1):

\[
v(re^{i\theta}) \leq \int_{-R}^{R} v(t)A(t, r, \theta, R)dt + \int_{0}^{\pi} v(Re^{i\varphi})B(\varphi, r, \theta, R)d\varphi
\]

where \(A\) and \(B\) are kernels (see [1, p.430]).

We use the estimates

\[
B(\varphi, r, \theta, R) < \frac{32}{\pi r^2}, \ (0 < \theta < \pi, \ 0 < r < R/2)
\]

and

\[
A(t, r, \theta, R) \leq P(t, r, \pi - \theta), \ A(-t, r, \theta, R) \leq P(t, r, \theta),
\]

where

\[
P(t, r, \theta) = \frac{1}{\pi t^2 + 2rt \cos \theta + r^2}.
\]

Taking into account properties (2.1) and (2.2) of the function \(T^*(z)\), we get

\[
v(re^{i\theta}) \leq \int_{0}^{R} N(t\gamma, 0, F(z, a))P(t, r, \pi - \theta)dt
\]

\[
+ \int_{0}^{R} T(t\gamma, a, A)P(t, r, \theta)dt + 32(r/R)T(R\gamma, a, A)
\]

\[(0 < \theta < \pi, \ 0 < r < R/2).
\]

Let \((r_j)\) be a sequence of Polya peaks of order \(\mu\) of \(T(r, a, A)\) (or \(T(r, f)\)) and \((r_j')\) be the sequence occurring in the definition of Polya peaks (see [1, p.418]) such that \(r_j'/r_j \to \infty \ (j \to \infty)\).

Let us set

\[
s_j = (r_j)^{1/\gamma} \text{ and } s'_j = (r_j')^{1/\gamma}.
\]

The following relations are valid:

\[
\int_{0}^{s_j'} N(t\gamma, 0, F)P(t, s_j, \pi - \theta)dt \leq (1 - \delta(a, f))T(r_j, a, A)
\]

\[
\times \left\{ \frac{\sin (\pi - \theta)\gamma \mu}{\sin \pi \gamma \mu} + o(1) \right\},
\]

\[
\int_{0}^{s_j'} T(t\gamma, a, A)P(t, s_j, \theta)dt \leq T(r_j, a, A) \left\{ \frac{\sin \theta \mu \gamma}{\sin \pi \mu \gamma} + o(1) \right\},
\]

\[(j \to \infty, \ 0 < \theta < \pi),
\]
where \( o(1) \) does not depend on \( \theta \),

\[
(2.8) \quad \frac{s_j}{s_j'} T \left( (s_j')^7, a, A \right) = o(T(r_j, a, A)), \quad j \to \infty.
\]

Setting \( r = s_j \) and \( R = s_j' \), in (2.5) and taking the relations (2.6), (2.7), (2.8) into account, we get \( (j \to \infty, 0 < \theta < \pi) \)

\[
(2.9) \quad v(s_je^{i\theta}) \leq T(r_j, a, A) \left\{ \frac{\sin \theta \gamma \mu + (1 - \delta(a, f)) \sin (\pi - \theta) \gamma \mu}{\sin \pi \gamma \mu} + o(1) \right\}.
\]

From the definition of \( \gamma \) we have

\[
1 - \delta(a, f) = \cos \pi \gamma \mu.
\]

We write the inequality (2.9) in the form

\[
v(s_je^{i\theta}) \leq T(r_j, a, A) \{\cos (\pi - \theta) \gamma \mu + \alpha_j\},
\]

\( (j = 1, 2, \ldots, 0 < \theta < \pi) \)

where \( \alpha_j \to 0 \) as \( j \to \infty \). Further, following [1, p. 433–434], we arrive at the relation (2.3).

\[
(2) \quad 4/\mu \arcsin \sqrt{(\delta(a, f)/2)} \geq 2\pi.
\]

In this case, we choose a number \( d \) such that

\[
0 < d < \delta(a, f)
\]

and

\[
\frac{4}{\mu} \arcsin \sqrt{\frac{d}{2}} < 2\pi.
\]

Set

\[
\gamma = \frac{2}{\pi \mu} \arcsin \sqrt{\frac{d}{2}};
\]

by similar reasoning, we arrive at

\[
\sigma(a) \geq \frac{4}{\mu} \arcsin \sqrt{\frac{d}{2}}.
\]

Letting \( d \uparrow d_0 = 2 \sin^2 (\mu \pi/2) \), we obtain the desired result

\[
\sigma(a) = 2\pi.
\]

Theorem 2.1 is proved.
THEOREM 2.2. Let \( f(z) \) be an \( n \)-valued algebroid function of lower order \( \mu (0 < \mu < \infty) \), defined by the equation (1.1), and \( q \geq 2\mu \) be an integer. If

\[
\delta(a, f) \geq 1 - \cos \frac{\mu \pi}{q},
\]

then

\[
(2.11)\quad \sigma(a) \geq \frac{2\pi}{q}.
\]

PROOF: The proof of this theorem is similar to the proof of case (1) in Theorem 2.1. Let us only observe that we must choose \( \gamma = 1/q \) and apply inequality (2.10) to relation (2.9). Relation (4.16) from [1, p.433] reduces to the desired inequality (2.11). □

3. APPLICATIONS

LEMMA 3.1. Let \( f(z) \) be an \( n \)-valued algebroid function of lower order \( \mu (0 < \mu < \infty) \) and let \( a_i (i = 0, 1, \ldots, n) \) be any \( n + 1 \) distinct complex numbers. Choose \( \Lambda(r) = (T(r, f))^1/2 \) and define the sets \( E_\Lambda(r, a_j) \) in \((-\pi, \pi]\) by

\[
(3.1)\quad E_\Lambda(r, a_j) = \left\{ \theta : \frac{\|A(z)\| \cdot \|a_j\|}{|F(z, a_j)|} > e^{\Lambda(\tau)}, z = e^{i\theta}r \right\} (j = 0, 1, \ldots, n),
\]

Then there exists a positive number \( r_0 > 0 \) such that \( r \geq r_0 \)

\[
\bigcap_{j=0}^{n} E_\Lambda(r, a_j) = \emptyset.
\]

PROOF: We assume that \( a_j \neq \infty (j = 0, 1, \ldots, n) \) without loss of generality. Suppose that

\[
E(r) = \bigcap_{j=0}^{n} E_\Lambda(r, a_j) \neq \emptyset.
\]

We choose \( \theta_0 \in E(r) \) and consider the following system of \( n + 1 \) equations.

\[
F(re^{i\theta_0}, a_j) = \sum_{k=0}^{n} A_k(re^{i\theta_0})a_j^{n-k} \quad (j = 0, 1, \ldots, n).
\]

Since the determinant of the coefficients

\[
\det (a_j^n, a_j^{n-1}, \ldots, a_j, 1) \neq 0,
\]

available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972700028367
we can solve this system for the unknowns $A_j(r e^{i\theta_0})$ $(0 \leq j \leq n)$ and obtain (for some constants $b_{jk}$):

$$A_k(r e^{i\theta_0}) = \sum_{j=0}^{n} b_{jk} F(r e^{i\theta_0}, a_j), \quad (k = 0, 1, \ldots, n)$$

so that

$$|A_q(r e^{i\theta_0})| = \max_{0 \leq k \leq n} |A_k(r e^{i\theta_0})|$$

$$\leq \max_{0 \leq k \leq n} \sum_{j=0}^{n} |b_{jk}| \cdot |F(r e^{i\theta_0}, a_j)|$$

$$\leq C |F(r e^{i\theta_0}, a_s)|, \quad (0 \leq s \leq n)$$

where $C$ is a constant and

$$|F(r e^{i\theta_0}, a_s)| = \max_{0 \leq j \leq n} |F(r e^{i\theta_0}, a_j)|.$$

This means that for fixed $r$

$$\frac{\|A(r e^{i\theta_0})\| \cdot \|a_s\|}{|F(r e^{i\theta_0}, a_s)|} \leq \frac{(n + 1)^{1/2} |A_q(r e^{i\theta_0})| \cdot \|a_s\|}{|F(r e^{i\theta_0}, a_s)|}$$

$$\leq (n + 1)^{1/2} C \|a_s\| = \text{constant},$$

which for sufficiently large $r$ contradicts the assumption that $\theta_0$ belongs to $E_{\Lambda}(r, a_s)$. Lemma 3.1 is thus proved.

**Lemma 3.2.** Let $f(z)$ be an $n$-valued algebroid function of lower order $\mu$ $(0 < \mu < \infty)$, defined by the equation (1.1) and

$$\Lambda(r) = (T(r, f))^{1/2}.$$

Then, on summing all the deficient values $a$ of $f(z)$, we have

$$\sum_a \sigma(a) \leq \sum_a \sigma_{\Lambda}(a) \leq 2n\pi.$$

**Proof:** Let $a_j$ $(j = 1, 2, \ldots, N)$ be any $N$ deficient values of $f(z)$. The sets $E_{\Lambda}(r, a_j)$ $(1 \leq j \leq N)$ are defined by (3.1). Since for each $\theta_0 \in (-\pi, \pi]$, $\theta_0$ can belong to at most $n$ of the sets $E_{\Lambda}(r, a_j)$ $(1 \leq j \leq N)$ for sufficiently large $r$,

$$\sum_{k=1}^{N} \sigma(a_k) \leq \sum_{k=1}^{N} \sigma_{\Lambda}(a_k) = \sum_{k=1}^{N} \lim_{j \to \infty} \text{meas} E_{\Lambda}(r_j, a_k) \leq 2n\pi.$$
Since $N$ can be arbitrarily large, Lemma 3.2 is thus proved.

**Theorem 3.1.** Let $f(z)$ be an $n$-valued algebroid function of lower order $\mu (0 < \mu < \infty)$, defined by the equation (1.1) and $q \geq 2\mu$ be an integer. If $f(z)$ has more than $nq$ deficient values, then there are at most $nq - 1$ deficient values $a_k (k = 1, 2, \ldots, nq - 1)$ such that

$$\delta_k = \delta(a_k, f) \geq 1 - \cos \frac{\mu\pi}{q}, \quad (k = 1, 2, \ldots, nq - 1).$$

**Proof:** Assume that the assertion is false; we choose $nq + 1$ distinct deficient values $a^*_k (k = 1, 2, \ldots, nq + 1)$ of $f(z)$ such that

$$\delta_1 \geq \delta_2 \geq \cdots \delta_{nq} \geq 1 - \cos \frac{\mu\pi}{q},$$

$$\delta_{nq+1} > 0, \quad (q \geq 2\mu, \delta_k = \delta(a_k, f), 1 \leq k \leq nq).$$

Choosing the integer $s \geq nq$ large enough, (3.3) implies

$$\delta_{nq+1} \geq 1 - \cos \frac{\mu\pi}{s}.$$

Now let $(r_j)$ be a sequence of Pólya peaks of $T(r, f)$ and let

$$\Lambda(r) = \left(T(r, f)\right)^{1/2};$$

by Theorem 2.2, (3.2) and (3.4) imply

$$\sigma(a_k) \geq \frac{2\pi}{q}, \quad \sigma(a_{nq+1}) \geq \frac{2\pi}{s}, \quad k = 1, 2, \ldots, nq.$$

So

$$\sum_{k=1}^{nq+1} \sigma_h(a_k) \geq \sum_{k=1}^{nq+1} \sigma(a_k) \geq 2\pi + \frac{2\pi}{s}.$$ 

This contradicts Lemma 3.2, so Theorem 3.1 is proved.

**Theorem 3.2.** Let $f(z)$ be an $n$-valued algebroid function of lower order $\mu (0 \leq \mu \leq \infty)$, defined by the equation (1.1). Then on summing over all the deficient values $a$ of $f(z)$, we have

$$\sum_a \sqrt{\delta(a, f)} \leq n\left(\sqrt{2\mu\pi} + 2\mu + 1\right).$$

**Proof:** We consider the following two cases.

(1) If $\mu = 0$, by a result of Gu ([4]), $f(z)$ has at most $n$ deficient values, so that Theorem 3.2 holds.
(2) If $0 < \mu < \infty$, we assume that
\begin{equation}
\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots
\end{equation}
are all the deficient values of $f(z)$ and assume that (3.5) has been ordered so that
\[\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n \geq \cdots,\]
where $\delta_k = \delta(a_k, f)$, $k = 1, 2, \cdots, n, \cdots$.

Let $q = \lfloor 2\mu \rfloor + 1$ and $m$ be an integer. If $m \leq nq$, it is trivial that
\[\sum_{i=1}^{m} \sqrt{\delta(\alpha_i, f)} \leq nq \leq n(2\mu + 1).\]

If $m > nq$, by Theorem 3.1 we have
\[\delta_{nq} < 1 - \cos \frac{\mu \pi}{q}.\]

Hence, with each $\delta_{nq+i} > 0$ ($1 \leq i \leq m - nq$), we may associate a positive integer $q_i$ such that
\begin{equation}
1 - \cos \frac{\mu \pi}{q_i + 1} \leq \delta_{nq+i} < 1 - \cos \frac{\mu \pi}{q_i}, \ i \geq 1.
\end{equation}

By Lemma 3.2 and Theorem 2.2, we get
\[\sum_{i=1}^{m-nq} \frac{2\pi}{q_i + 1} \leq \sum_{i=nq+1}^{m} \sigma(\alpha_i) \leq 2n\pi,
\]
so that
\begin{equation}
\sum_{i=1}^{m-nq} \frac{1}{q_i} \leq \sum_{i=1}^{m-nq} \frac{2}{q_i + 1} \leq 2n.
\end{equation}

From the second inequality in (3.6), we deduce
\[\delta_i^{1/2} < \sqrt{2} \sin \frac{\mu \pi}{2q_i} < \frac{\pi \mu}{q_i \sqrt{2}},\]
and hence
\[\sum_{i=nq+1}^{m} \sqrt{\delta(\alpha_i, f)} \leq \sum_{i=1}^{m-nq} \frac{\mu \pi}{\sqrt{2}q_i} \leq \sqrt{2}n\mu \pi.
\]

Therefore
\begin{equation}
\sum_{i=1}^{m} \sqrt{\delta(\alpha_i, f)} \leq \sqrt{2}n\mu \pi + nq \leq n \left(\sqrt{2} \mu \pi + 2\mu + 1\right).
\end{equation}

Since $m$ can be arbitrarily large, Theorem 3.2 follows from (3.8). □
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