LEFT IDEALS IN THE NEAR-RING OF AFFINE TRANSFORMATIONS

WOLFGANG MUTTER

In this paper we determine the left ideals in the near-ring \( \text{Aff}(V) \) of all affine transformations of a vector space \( V \). It is shown that there is a Galois correspondence between the filters of affine subspaces of \( V \) and those left ideals of \( \text{Aff}(V) \) which are not left invariant. In particular, the not left invariant finitely generated left ideals of \( \text{Aff}(V) \) are precisely the annihilators of the affine subspaces of \( V \). A similar correspondence exists between the filters of linear subspaces of \( V \) and the left invariant left ideals of \( \text{Aff}(V) \). If \( V \) is finite-dimensional, then all left ideals of \( \text{Aff}(V) \) are finitely generated.

1. INTRODUCTION

Let \( V \) be a vector space and let \( \text{Aff}(V) \) denote the collection of all affine transformations of \( V \). Under pointwise addition and under composition of mappings \( \text{Aff}(V) \) is a near-ring. In [2] Blackett showed that the set \( C \) of all constant transformations forms an ideal of \( \text{Aff}(V) \). If \( V \) is finite dimensional, then \( C \) is the only non-trivial ideal of \( \text{Aff}(V) \). Wolfson [5] determined all ideals of \( \text{Aff}(V) \) for an arbitrary vector space \( V \). He observed that \( C \) is contained in all non-trivial ideals of \( \text{Aff}(V) \) and that \( \text{Aff}(V)/C \) is isomorphic to the ring \( \text{Hom}(V, V) \) of all linear transformations of \( V \). Thus the ideals of \( \text{Aff}(V) \) are the sets \( T_v + C \) with \( T_v = \{ f \in \text{Hom}(V, V) \mid \text{Range } f < \aleph_v \} \), where \( \aleph_v \) is a cardinal number.

In this paper we investigate the structure of the left ideals of \( \text{Aff}(V) \). We use the results of Baer on the left ideals of the ring \( \text{Hom}(V, V) \) in [1, p.172 following], where he showed that the finitely generated left ideals of \( \text{Hom}(V, V) \) are precisely the annihilators of the linear subspaces of the vector space \( V \). In particular, Baer established a Galois correspondence between the left ideals of \( \text{Hom}(V, V) \) and the filters of linear subspaces of \( V \). Thus, by the second isomorphism theorem for near-rings (see for example Theorem 1.31 in [3]), the left invariant left ideals of \( \text{Aff}(V) \) are completely determined, since a left ideal of \( \text{Aff}(V) \) is left invariant if and only if it contains the ideal \( C \) of all constant transformations of \( V \).

The purpose of this paper is to show that there is a similar correspondence between the left ideals of \( \text{Aff}(V) \) which are not left invariant and the affine subspaces of \( V \), as
in the case of $\text{Hom}(V, V)$. If $V$ is finite dimensional, then all left ideals of $\text{Aff}(V)$ are finitely generated. In this case the left ideals of $\text{Aff}(V)$ which are not left invariant are precisely the annihilators of the affine subspaces of $V$. The left invariant left ideals of $\text{Aff}(V)$ are the sets $L + C$, where $L$ is the annihilator of a linear subspace of $V$.

## 2. Basic definitions and results

For details on near-rings and $N$-groups we refer the reader to [4]. According to [4] we consider right near-rings.

**Definition 2.1**: Let $(N, +, \cdot)$ be a near-ring. A subset $L$ of $N$ is called a left ideal of $N$ provided that

1. $(L, +)$ is a normal subgroup of $(N, +)$, and
2. $m(n + i) - mn \in L$ for all $i \in L$ and $m, n \in N$.

If $S$ is a subset of a near-ring $N$, let $(S)_L$ denote the left ideal generated by $S$. In particular, $(n_1, \ldots, n_k)_L$ denotes the left ideal generated by $n_1, \ldots, n_k \in N$. If a near-ring $N$ is regarded as a $N$-group in the usual way, the left ideals of $N$ are precisely the kernels of $N$-homomorphisms with domain $N$.

In general, a left ideal of a near-ring is not invariant under multiplication from the left. Therefore, we call a left ideal $L$ of a near-ring $N$ left invariant, if for all $n \in N$ and $i \in L$ the element $n \cdot i$ is in $L$. The left invariant left ideals of a near-ring can be characterised as follows:

**Lemma 2.2**: Let $N$ be a near-ring with constant part $N_c$ and let $L$ be a left ideal of $N$. Then $L$ is left invariant if and only if $N_c \subseteq L$.

**Proof**: If $L$ is left invariant and $n_c$ is in $N_c$, then $n_c = n_c \cdot i \in L$ for all $i \in L$. Conversely, if $N_c \subseteq L$, then for all $n \in N$ and $i \in L$ the element $n \cdot i = n \cdot i - n \cdot 0 + n \cdot 0$ is in $L$, since $n \cdot 0$ is in $N_c$.

If $V$ is a vector space and $S$ is a subset of $V$, then $\text{Ann}(S)$ denotes the annihilator $\{f \in \text{Aff}(V) \mid f(S) = 0\}$. If $p$ is an element of $V$, let $(p)$ denote the constant transformation of $V$ which carries all of $V$ onto $p$. Any affine transformation $f \in \text{Aff}(V)$ can be decomposed as $f = f - (f(0)) + (f(0))$ with $f - (f(0)) \in \text{Hom}(V, V)$ and $(f(0)) \in C$. $\text{Hom}(V, V)$ is a subnear-ring of $\text{Aff}(V)$ and

$$
\varphi : \text{Aff}(V) \to \text{Hom}(V, V) : f \mapsto f - (f(0))
$$

is a surjective near-ring homomorphism with $\ker \varphi = C$. By Lemma 2.2 and by the second isomorphism theorem for near-rings ([3, Theorem 1.31]) $\varphi$ induces a bijective correspondence between the left invariant left ideals of $\text{Aff}(V)$ and the left ideals of $\text{Hom}(V, V)$ by $L \to \varphi(L)$. 

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A left ideal $L$ of $\text{Aff}(V)$ which is not left invariant does not contain many constant transformations, for we have

**Lemma 2.3.** If $L$ is a not left invariant left ideal of $\text{Aff}(V)$, then $L \cap C = \{0\}$.

**Proof:** It is easy to show that $L \cap C$ is isomorphic to a submodule of the simple $\text{Hom}(V, V)$-module $V$. Hence, by Lemma 2.2, the assertion of the lemma is obvious. □

For an affine transformation $f$ let $Z(f)$ denote the zero-set of $f$, that is $Z(f) = \{p \in V \mid f(p) = 0\}$. If $Z(f)$ is not empty, then it is an affine subspace of $V$. Conversely, every affine subspace of a vector space is the zero-set of an affine transformation. More precisely:

**Lemma 2.4.** Let $A = p + U$ be an affine subspace of a vector space $V$, where $U$ is a linear subspace of $V$ and $p \in V$. Then there exists $f \in \text{Aff}(V)$ with $Z(f) = A$. In particular, if $W$ is a linear complement of $U$ in $V$, there exists $f \in \text{Aff}(V)$ with $Z(f) = A$ and $f(V) = W$.

**Proof:** By the Complementation Theorem in [1, p.12], there exists a linear subspace $W$ of $V$ with $V = U \oplus W$. If $\tau_{-p}$ denotes the translation by $-p$ and $\text{pr}_W$ is the projection map from $V$ onto $W$, then $f = \text{pr}_W \circ \tau_{-p}$ is an affine transformation of $V$ with the required properties. □

### 3. The Not Left Invariant Left Ideals

In this section we determine the left ideals of $\text{Aff}(V)$ which are not left invariant.

**Lemma 3.1.** Let $L$ be a left ideal of $\text{Aff}(V)$ and let $f_1, \ldots, f_n$ be in $L$ with $Z(f_1) \cap \cdots \cap Z(f_n) \neq \emptyset$. If $g$ is an affine transformation of $V$ with $Z(g) \supseteq Z(f_1) \cap \cdots \cap Z(f_n)$, then $g \in L$.

**Proof:** Since $Z(f_1) \cap \cdots \cap Z(f_n)$ is not empty, there exist an element $p \in V$ and a linear subspace $U$ of $V$ with $p + U = Z(f_1) \cap \cdots \cap Z(f_n)$. Let $\tau_p \in \text{Aff}(V)$ be given by $\tau_p(x) = x + p$. Then $\tau_p$ defines an $\text{Aff}(V)$-automorphism of $\text{Aff}(V)$ by $h \mapsto h \circ \tau_p$. Hence

$$U = Z(f_1 \circ \tau_p) \cap \cdots \cap Z(f_n \circ \tau_p)$$

and $U \subseteq Z(g \circ \tau_p)$. In particular, $f_1 \circ \tau_p, \ldots, f_n \circ \tau_p$ and $g \circ \tau_p$ are linear transformations of $V$. Since $\text{Hom}(V, V)$ is a left ideal of $\text{Aff}(V)$, the left ideal $\langle f_1 \circ \tau_p, \ldots, f_n \circ \tau_p \rangle_\ell$ generated by $f_1 \circ \tau_p, \ldots, f_n \circ \tau_p$ is obviously the smallest left ideal of the ring $\text{Hom}(V, V)$ which contains $f_1 \circ \tau_p, \ldots, f_n \circ \tau_p$. Hence $g \circ \tau_p \in \langle f_1 \circ \tau_p, \ldots, f_n \circ \tau_p \rangle_\ell$ by [1, p.173, Theorem A, and p.177, Theorem 1]. The second isomorphism theorem 1.30 for $N$-groups in [4] implies $g \in \langle f_1, \ldots, f_n \rangle_\ell \subseteq L$. □

In order to prove the next lemma, we need the following two propositions:
**Proposition 3.2.** Let $V$ be a vector space and let $A_1$ and $A_2$ be affine subspaces of $V$ with $A_1 \cap A_2 = \emptyset$. Then there exist maximal affine subspaces $M_1$ and $M_2$ of $V$ such that $A_1 \subseteq M_1$, $A_2 \subseteq M_2$ and $M_1 \cap M_2 = \emptyset$.

**Proof:** Let $p_1$, $p_2$ be in $V$ and let $U_1$, $U_2$ be linear subspaces of $V$ with $A_1 = p_1 + U_1$ and $A_2 = p_2 + U_2$. Since $A_1 \cap A_2 = \emptyset$, $p_1 - p_2$ is not in $U_1 + U_2$. By the Complementation Theorem in [1, p.12], there exists a linear subspace $U$ of $V$ such that $V$ can be decomposed as

$$V = \text{span}(p_1 - p_2) \oplus (U_1 + U_2) \oplus U.$$ 

Then $M_1 = p_1 + (U_1 + U_2 + U)$ and $M_2 = p_2 + (U_1 + U_2 + U)$ are maximal affine subspaces of $V$ with the required properties. \(\square\)

**Proposition 3.3.** If $L$ is a left ideal of $\text{Aff}(V)$ and $f \in L$ with $Z(f) = \emptyset$, then $L$ is left invariant.

**Proof:** $f(V)$ is an affine subspace of $V$. Thus by Lemma 2.4 there exists $g \in \text{Aff}(V)$ with $Z(g) = f(V)$. Furthermore the constant transformation

$$(-g(0)) = g \circ f - g \circ (0)$$

is in $L$. Moreover, $g(0)$ is not zero, since $0$ is not in $f(V)$. Hence the assertion of the lemma follows by Lemmas 2.2 and 2.3. \(\square\)

**Lemma 3.4.** Let $L$ be a left ideal of $\text{Aff}(V)$ and suppose there are $f, g \in L$ with $Z(f) \cap Z(g) = \emptyset$. Then $L$ is left invariant.

**Proof:** By Proposition 3.3 it suffices to show that there exists an affine transformation $h \in L$ with $Z(h) = \emptyset$. Therefore we may assume that $Z(f)$ and $Z(g)$ are not empty. By Proposition 3.2 there exist maximal subspaces $M_1$ and $M_2$ of $V$ such that $Z(f) \subseteq M_1$, $Z(g) \subseteq M_2$ and $M_1 \cap M_2 = \emptyset$. By Lemma 2.4 there exist nonzero elements $p_1$ and $p_2$ in $V$ and transformations $f_1, f_2 \in \text{Aff}(V)$ with $M_1 = Z(f_1)$, $M_2 = Z(f_2)$, $f_1(V) = \text{span}(p_1)$ and $f_2(V) = \text{span}(p_2)$. Lemma 3.1 implies $f_1, f_2 \in L$, since $Z(f_1) \supseteq Z(f)$ and $Z(f_2) \supseteq Z(g)$. Now we distinguish two cases:

Suppose dim $V > 1$. Then there exist nonzero elements $q_1, q_2 \in V$ with $\text{span}(q_1) \cap \text{span}(q_2) = \{0\}$. Let $h_1$ and $h_2$ be invertible linear transformations of $V$ with $h_1(p_1) = q_1$ and $h_2(p_2) = q_2$. Then $h_1 \circ f_1(V) = \text{span}(q_1)$ and $h_2 \circ f_2(V) = \text{span}(q_2)$. Furthermore the transformation $h = h_1 \circ f_1 - h_2 \circ f_2$ is in $L$. If $x \in V$, then

$$h(x) = 0 \iff h_1 \circ f_1(x) = h_2 \circ f_2(x) \iff h_1 \circ f_1(x) = h_2 \circ f_2(x) = 0 \iff f_1(x) = f_2(x) = 0.$$ 

Hence $Z(h) = \emptyset$, since $Z(f_1) \cap Z(f_2) = \emptyset$. This proves the assertion of the lemma for dim $V > 1$. \(\square\)
If \( \dim V = 1 \), then there exist distinct elements \( q_1 \) and \( q_2 \) in \( V \) with \( Z(f_1) = \{q_1\} \) and \( Z(f_2) = \{q_2\} \). An easy check shows that in this case \( f_1 \) and \( f_2 \) are injective. Hence there exist affine transformations \( h_1 \) and \( h_2 \) with \( h_1 \circ f_1 = h_2 \circ f_2 = \text{id} \). The constant transformation

\[
   h = (g_2(0) - g_1(0)) = (g_1 \circ f_1 - g_1 \circ (0)) - (g_2 \circ f_2 - g_2 \circ (0))
\]

is in \( L \) and is not zero, since \( h_1(0) = h_1(f_1(q_1)) = q_1 \) and \( h_2(0) = h_2(f_2(q_2)) = q_2 \). This completes the proof of the lemma.

Now we are in a position to establish a bijective correspondence between the left ideals of \( \text{Aff}(V) \), which are not left invariant, and the filters of affine subspaces of \( V \). First we need

**Definition 3.5:** A nonempty family \( \mathcal{F} \) of affine subspaces of a vector space \( V \) is called an \( \mathcal{A} \)-filter on \( V \) provided that

1. \( \emptyset \notin \mathcal{F} \),
2. if \( A_1, A_2 \in \mathcal{F} \), then \( A_1 \cap A_2 \in \mathcal{F} \), and
3. if \( A \in \mathcal{F} \) and \( A' \) is an affine subspace of \( V \) with \( A' \supseteq A \), then \( A' \in \mathcal{F} \).

For example, if \( A \) is an affine subspace of \( V \), the family \( \mathcal{F}_A \) of all affine subspaces of \( V \) which contain \( A \) is an \( \mathcal{A} \)-filter on \( V \). Obviously \( \mathcal{F}_A \) is the smallest \( \mathcal{A} \)-filter containing \( A \), hence we call \( \mathcal{F}_A \) the \( \mathcal{A} \)-filter generated by \( A \).

**Theorem 3.6.** Let \( V \) be a vector space.

1. If \( L \) is a left ideal of \( \text{Aff}(V) \) which is not left invariant, then

   \[
   Z[L] = \{Z(f) \mid f \in L\}
   \]

   is an \( \mathcal{A} \)-filter on \( V \).

2. If \( \mathcal{F} \) is an \( \mathcal{A} \)-filter on \( V \), then

   \[
   Z \leftarrow [\mathcal{F}] = \bigcup \{\text{Ann}(A) \mid A \in \mathcal{F}\}
   \]

   is a not left invariant left ideal of \( \text{Aff}(V) \).

Moreover, the mapping \( Z \) is one-one between the set of all not left invariant left ideals of \( \text{Aff}(V) \) and the \( \mathcal{A} \)-filters on \( V \).

**Proof:** 1. Let \( L \) be a left ideal of \( \text{Aff}(V) \) which is not left invariant. We have to show that \( Z[L] \) satisfies the properties 1 – 3 of Definition 3.5. Proposition 3.3 implies \( \emptyset \notin Z[L] \). Suppose now that \( A_1, A_2 \in Z[L] \). If \( A_1 \cap A_2 = \emptyset \), then by Lemma 3.4 \( L \) is left invariant, which contradicts the hypothesis. If \( A_1 \cap A_2 \neq \emptyset \), then according to Lemma 2.4 there exists \( f \in \text{Aff}(V) \) with \( Z(f) = A_1 \cap A_2 \). Lemma 3.1 implies \( f \in L \),...
hence $A_1 \cap A_2 \in Z[L]$. Finally, let $A \in Z[L]$ and let $A'$ be an affine subspace of $V$ with $A' \supseteq A$. By Lemma 2.4 there exists $f' \in \text{Aff}(V)$ with $A' = Z(f')$. Since $A \neq \emptyset$ by Lemma 3.4, Lemma 3.1 implies $f' \in L$. Therefore $A'$ is in $Z[L]$. Altogether, we have shown that $Z[L]$ is an $A$-filter on $V$.

2. The proof of the second assertion of the theorem is straightforward and therefore omitted.

3. In order to verify that the mapping $Z$ is one-one, we prove that $Z^-$ is the inverse mapping of $Z$. If $F$ is an $A$-filter on $V$ then clearly $Z[Z^-[F]] = F$. Furthermore it is obvious that any left ideal $L$ of $\text{Aff}(V)$ satisfies $L \subseteq Z^-[Z[L]]$. If, in addition, $L$ is not left invariant, we have seen that $Z[L]$ is an $A$-filter on $V$. Therefore, if $f$ is an affine transformation with $Z(f) \in Z[L]$, then $Z(f) \neq \emptyset$, and hence $f \in L$ by Lemma 3.1. This proves the converse inclusion $Z^-[Z[L]] \subseteq L$.

As a consequence of Theorem 3.6 we note that for an affine transformation $f$ with nonempty zero-set $Z(f)$ the left ideal $(f)_\ell$ generated by $f$ and the annihilator $\text{Ann}(Z(f))$ of $Z(f)$ coincide. Furthermore, we get the following

**Corollary 3.7.** The not invariant left invariant ideals $L$ of $\text{Aff}(V)$ are precisely the sets

$$Z^-[F] = \bigcup \{\text{Ann}(A) \mid A \in F\}$$

where $F$ is a filter of affine subspaces of $V$.

4. The Finitely Generated Left Ideals

Now we are in a position to determine the finitely generated left ideals of $\text{Aff}(V)$.

**Theorem 4.1.** Let $V$ be a vector space.

1. The finitely generated left invariant left ideals of $\text{Aff}(V)$ are precisely the sets $\text{Ann}(U) + C$, where $U$ is a linear subspace of $V$.

2. The finitely generated left ideals of $\text{Aff}(V)$, which are not left invariant, are precisely the annihilators $\text{Ann}(A)$, where $A$ is an affine subspace of $V$.

**Proof:** The first assertion of the theorem follows by Theorem A in [1, p.173], Theorem 1 in [1, p.177], the second isomorphism theorem for near-rings and Lemma 2.2. To show 2, suppose first that $L = (f_1, \ldots, f_n)_\ell$ is a finitely generated left ideal of $\text{Aff}(V)$ which is not left invariant. By Theorem 3.6 the family $Z[L]$ is an $A$-filter on $V$. Hence there exists $f \in L$ with $Z(f) = Z(f_1) \cap \cdots \cap Z(f_n)$. Moreover, $Z(f)$ is not empty. By the remarks following Theorem 3.6 the left ideal $(f)_\ell$ generated by $f$ agrees with the annihilator $\text{Ann}(Z(f))$. Therefore $\text{Ann}(Z(f)) \subseteq L$. Since $\text{Ann}(Z(f))$ is a left ideal of $\text{Aff}(V)$ containing $f_1, \ldots, f_n$, it follows that $\text{Ann}(Z(f)) = L$. 

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Conversely, if \( A \) is an affine subspace of \( V \), by Lemma 2.4 there exists \( f \in \text{Aff}(V) \) with \( A = Z(f) \). The remarks following Theorem 3.6 imply \( \text{Ann}(A) = \langle f \rangle \), hence \( \text{Ann}(A) \) is a finitely generated and obviously not left invariant left ideal of \( \text{Aff}(V) \). 

For the proof of the next theorem it will be convenient to have

**Lemma 4.2.** Let \( V \) be a vector space. Then the following statements are equivalent:

1. \( \dim V < \infty \).
2. Every \( A \)-filter on \( V \) is generated by an affine subspace of \( V \).

**Proof:** Let \( \dim V < \infty \) and let \( F \) be an \( A \)-filter on \( V \). Let \( A \in F \) such that \( \dim A \leq \dim A' \) for all \( A' \in F \). If \( A' \in F \), then \( A \cap A' \in F \) and so \( \dim A \leq \dim (A \cap A') \). This implies \( A \subseteq A' \). Hence \( F \) is contained in the \( A \)-filter \( F_A \) generated by \( A \). Since \( A \in F \), it follows that \( F = F_A \).

To show the converse, suppose that \( \dim V = \infty \). Then the family of all finite dimensional linear subspaces of \( V \) is an \( A \)-filter on \( V \) which is not generated by an affine subspace of \( V \).

**Theorem 4.3.** Let \( V \) be a vector space. Then the following statements are equivalent:

1. \( \dim V < \infty \).
2. All left ideals of \( \text{Aff}(V) \) are finitely generated.

**Proof:** Let \( V \) be a finite dimensional vector space and let \( L \) be a left ideal of \( \text{Aff}(V) \). If \( L \) is not left invariant, then according to Corollary 3.7 and Lemma 4.2 there exists an affine subspace \( A \) of \( V \) with \( L = \bigcup \{ \text{Ann}(A') \mid A' \in F_A \} = \text{Ann}(A) \). Therefore Theorem 4.1 implies that \( L \) is finitely generated.

If \( L \) is a left invariant left ideal of \( \text{Aff}(V) \), then \( L \) can be decomposed as \( L = L_0 + C \), where \( L_0 \) is a left ideal of \( \text{Hom}(V, V) \). In particular, \( L_0 \) is a left ideal of \( \text{Aff}(V) \) which is not left invariant. Hence \( L_0 \) is finitely generated. Furthermore, Lemma 2.3 implies that \( C \) is a finitely generated left ideal of \( \text{Aff}(V) \). Therefore \( L \) is finitely generated.

If conversely all left ideals of \( \text{Aff}(V) \) are finitely generated, then \( \dim V < \infty \) by Theorem 4.1, Lemma 4.2 and Corollary 3.7.

In particular, Theorem 4.1 and Theorem 4.3 show that for a finite dimensional vector space \( V \) there is a Galois correspondence between the left invariant left ideals of \( \text{Aff}(V) \) and the linear subspaces of \( V \) and a similar correspondence between the not left invariant left ideals of \( \text{Aff}(V) \) and the affine subspaces of \( V \).
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Mathematisches Institut
Universität Erlangen-Nürnberg
Bismarckstr. 1 ½
D-8520 Erlangen,
Federal Republic of Germany