POINTWISE CHAIN RECURRENT MAPS OF THE SPACE $Y$

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Let $Y = \{ z \in C : z^3 \in [0,1] \}$ (equipped with subspace topology of the complex space $C$) and let $f : Y \rightarrow Y$ be a continuous map. We show that if $f$ is pointwise chain recurrent (that is, every point of $Y$ is chain recurrent under $f$), then either $f^{12}$ is the identity map or $f^{12}$ is turbulent. This result is a generalisation to $Y$ of a result of Block and Coven for pointwise chain recurrent maps of the interval.

1. INTRODUCTION

In this paper we characterise the dynamics of maps of the space $Y = \{ z \in C : z^3 \in [0,1] \}$ equipped with the subspace topology for with every point is chain recurrent. We prove the following.

**MAIN THEOREM.** Let $f$ be a continuous map of $Y$ to itself. If $f$ is pointwise chain recurrent, then either $f^{12}$ is the identity map or $f^{12}$ is turbulent.

Block and Coven (see [4]) proved that a pointwise chain recurrent map $h$ of the interval must satisfy that either $h^2$ is the identity map or $h^2$ is turbulent. So our theorem extends this result to maps of the space $Y$.

Firstly some notation and definitions are established. Let $(X, d)$ be a compact metric space and $g : X \rightarrow X$ be a continuous map. If $g^n(x) = x \neq g^k(x), k = 1, 2, \ldots, n - 1$, for some $x \in X$ and some positive integer $n$, then the point $x$ is called a periodic point of period $n$, where $g^0 = id, \ g^i = g \circ (g^{i-1})(i \geq 1)$. In particular, if $g(x) = x$, then $x$ is called a fixed point of $g$. Denoted by $P(g)$ and $F(g)$ the set of periodic points and fixed points set of $g$ respectively. For $x, y \in X$ and $\varepsilon > 0$, an $\varepsilon$-chain from $x$ to $y$ is a finite sequence $x = x_0, x_1, \ldots, x_{n-1}, x_n = y$ with $d(g(x_i), x_{i+1}) < \varepsilon$ for $0 \leq i \leq n - 1$. We say $x$ is chain recurrent under $g$, if for each $\varepsilon > 0$, there is an $\varepsilon$-chain from $x$ to $x$. The map $g$ is said to be pointwise chain recurrent, if every point of $X$ is chain recurrent under $g$.

The following facts about chain recurrent are standard observations:

(a) If $g$ is pointwise chain recurrent, then $g$ maps $X$ onto $X$.

(b) $g$ is pointwise chain recurrent if and only if $g^n$ is pointwise chain recurrent for every $n > 0$.

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(c) [5, Theorem A] If $X$ is connected and $g : X \to X$ is pointwise chain recurrent, then there is no nonempty open set $U \neq X$ such that $g(U) \subseteq U$.

Being chain recurrent is an important dynamical property of a system and has been studied intensively in recent years. For more details see [1, 3, 5, 6, 9].

The space $Y$ is obviously a tree (see [7]) in which there are exactly three ends, denoted by $e_1$, $e_2$, and $e_3$, and exactly one vertex, denoted by $o$. For $a, b \in Y$, we shall use $[a, b]$, called a closed subinterval of $Y$, to denote the smallest closed connected subset containing $a$ and $b$. We define $(a, b) = [a, b] \setminus \{a, b\}$ and we can similarly define $(a, b]$ and $[a, b)$. For a subset $A$ of $Y$, we use $\text{int}(A)$, $\overline{A}$ and $\partial A$ to denote the interior, the closure and the boundary of $A$, respectively.

A map $g : Y \to Y$ is called turbulent if there are closed subintervals $J$ and $K$ with disjoint interiors such that $g(J) \cap g(K) \supseteq J \cup K$. Clearly, if $f$ is turbulent then $f^n$ is turbulent for any $n \geq 2$.

From the above definition of turbulence and the proof of [8, Theorem 1], the following result is clear.

**Theorem 1.1.** Let $f$ be a continuous map of space $Y$. If $f$ is turbulent, then $f$ has more than one fixed point.

Let $e \in \{e_1, e_2, e_3\}$. A partial order $<_e$ on $Y$ defined as follows, which will be useful in dealing with continuous maps of the space $Y$. For $x, y \in Y$, $x <_e y$ if $x \in [y, e]$ and $x \neq y$.

Throughout this paper, $f$ denotes a pointwise chain recurrent map of $Y$ into itself. This paper is organised as follows. In Section 2 and Section 3, the pointwise chain recurrent maps of $Y$ with more than one fixed point are characterised, where the fixed points set is disconnected in Section 2 and connected in Section 3. In Section 4, the pointwise chain recurrent maps of $Y$ with exactly one fixed point are discussed.

**Examples.** Clearly, $Y = I \cup \{xe^{(2/3)\pi i} \mid x \in I\} \cup \{xe^{(4/3)\pi i} \mid x \in I\}$, where $I = [0, 1]$.

1. $f : Y \to Y$, $f(x) = xe^{(2/3)\pi i}$, $f(xe^{(2/3)\pi i}) = x$ and $f(xe^{(4/3)\pi i}) = xe^{(4/3)\pi i}$ for any $x \in [0, 1]$. Then $f$ is pointwise chain recurrent such that $f^2 = id_Y$, but $f \neq id_Y$.

2. $f : Y \to Y$ is a rotation of period 3. Then $f$ is pointwise chain recurrent such that $f$ has exactly one fixed point.

2. **Pointwise chain recurrent maps of $Y$ with disconnected fixed points set**

In this section, we assume that $f$ has a disconnected fixed points set. Then there exist two fixed points $a, b$ of $f$ with $(a, b) \cap F(f) = \emptyset$.

**Theorem 2.1** If the closure of some component of $Y \setminus \{o\}$ contains $(a, b)$, then $f^2$ is turbulent.
**Theorem 2.2.** If a, b lie in two distinct components of $Y \setminus \{o\}$, $f^2$ is turbulent.

**Proof:** Without loss of generality, assume that $b \in (o, e_1)$, $a \in (o, e_2)$. 

**Case 1.** $f(x) <_{e_1} x$ for all $x \in (a, b)$. Then $b \neq e_1$, for otherwise $U = [e_1, a')$ satisfies $f(U) \subseteq U$ for any $a' \in (a, b)$. Let $c$ be the largest point in $(a, e_1]$ relative to $<_{e_1}$ such that $f(c) = a$. (If no such $c$ exists, then there exists $b' \in (a, b)$ such that $f(x) <_{e_1} b'$ for all $x \in (a, e_1]$.) But then $U = (b', e_1]$ satisfies $f(U) \subseteq U$. Let $d \in (a, c)$ be the point with $f(d) = c$. (Again if no such $d$ exists, then there exists $d' \in (b, c)$ such that $d' <_{e_1} f(x)$ for all $x \in (a, c]$.) But then $U = (d', d')$ satisfies $f(U) \subseteq U$ for some $d' \in (a, b)$. Then $J = [a, d]$ and $K = [d, c]$ show that $f$ is turbulent, and hence $f^2$ is turbulent.

**Case 2.** $x <_{e_1} f(x)$ for all $x \in (a, b)$. There exists $c \in Y \setminus [a, e_1]$ such that $f(c) = b$, for otherwise, $U = Y \setminus [b', e_1)$ for some $b <_{e_1} b' <_{e_1} a$ satisfies $f(U) \subseteq U$. The following three subcases are considered.

**Subcase 2.1.** There exists $c_i \in [e_i, a]$ such that $f(c_i) = b, i = 2, 3$, and there exists $d_2 \in [c_2, b]$ such that $f(d_2) = c_2$. (or there exists $c_i \in [e_i, a]$ such that $f(c_i) = b, i = 2, 3$, and there exists $d_3 \in [c_3, b]$ such that $f(d_3) = c_3$, the proof of this case is similar and omitted.) Taking $J = [c_2, d_2]$ and $K = [d_2, b]$, one gets that $f(J) \cap f(K) \supseteq J \cup K$ and then $f$ is turbulent. Thus $f^2$ is turbulent.

**Subcase 2.2.** $b <_{e_1} f(x)$ for all $x \in [e_2, a]$ and there exists $c \in [e_2, o)$ such that $f(c) = b$. (or $b <_{e_1} f(x)$ for all $x \in [e_2, a]$ and there exists $c \in [e_3, o)$ such that $f(c) = b$, the proof of this case is similar and omitted.) Assume that such point $c$ is the largest one in $[e_2, o)$ relative to $<_{e_2}$. Then there exists $d \in [e_3, b] \cup [c, o)$ such that $f(d) = c$. (If no such $d$ exists, then $U = [e_3, b'] \cup (o, c')$ for some $b' \in (a, b)$ and some $c' \in (o, c)$ satisfies $f(U) \subseteq U$.) If $d \in (c, b)$, then, taking $J = [c, d]$ and $K = [d, b]$, one gets that $f(J) \cap f(K) \supseteq J \cup K$ and thus $f^2$ is turbulent. Now, assume $[c, b] \cap f^{-1}(c) = \phi$ and such $d \in [e_3, o)$ is the largest one in $[e_3, o)$ relative to $<_{e_3}$. Then there exists $t \in [c, b] \cup [o, d]$ such that $f(t) = d$. (If no such $t$ exists, then $U = (c', b') \cap (d', d)$ for some $c' \in (o, c)$, some $b' \in (a, b)$ and some $d' \in (o, d)$ satisfies $f(U) \subseteq U$.) If $t \in (c, b)$, then, taking $J = [c, t]$, $K = [t, b]$, one gets $f^2(J) \cap f^2(K) \supseteq J \cap K$ and thus $f^2$ is turbulent. If $t \in (o, d)$, then taking $J = [d, t]$, $K = [t, b]$, one gets $f^2(J) \cap f^2(K) \supseteq [c, b] \cup [d, o] \supseteq J \cup K$ and thus $f^2$ is turbulent.

**Subcase 2.3.** $b <_{e_1} f(x)$ for all $x \in [o, a]$ and there exists $c_i \in [e_i, o)$ such that $f(c_i) = b$ and $f([c_i, b] \cup \{c_i\}) = \phi, i = 2, 3$. Assume that such $c_i$ is the largest one in $[e_i, o)$ relative to $<_{e_i}, i = 2, 3$. Then there exists $d_2 \in [c_2, o)$ such that $f(d_2) = c_2$ or $d_3 \in [c_3, o)$ such that $f(d_3) = c_2$. (If none of such $d_2, d_3$ exists, then $U = (c_2', b') \cup (c_3', o)$ for some $c_2' \in (c_2, o)$, some $c_3' \in (c_3, o)$, and some $b' \in (a, b)$ satisfies $f(U) \subseteq U$.) Furthermore, assume that such $d_i$ is the largest one in $[c_i, o)$ relative to $<_{e_i}, i \in \{2, 3\}$. Now a similar argument as that in Subcase 2.2 yields that $f^2$ is turbulent. The proof is complete. 

**Theorem 2.2.** If a, b lie in two distinct components of $Y \setminus \{o\}$, $f^2$ is turbulent.

**Proof:** Without loss of generality, assume that $b \in (o, e_1), a \in (o, e_2)$.
CASE 1. $x < e_1 f(x)$ for all $x \in (a, b)$ (or $x < e_2 f(x)$ for all $x \in (a, b)$), the proof of this case is similar and omitted.) A similar proof as that of case 2 in Theorem 2.1 implies that $f^2$ is turbulent.

CASE 2. $f(x) < e_1 x$ for all $x \in (a, b)$ and $f(x) < e_2 x$ for all $x \in (a, b)$. Then $[a', b'] \cap F(f) \neq \phi$ for any $a' \in (a, b)$ and any $b' \in (a, b)$ (according to the proof of [8, Theorem 1], in fact, we have $o \in F(f)$). There is a contradiction. Therefore case 2 is impossible and proof is complete.

3. POINTWISE CHAIN RECURRENT MAPS OF $Y$ WITH CONNECTED FIXED POINTS SET

In this section, we assume that $f$ has connected fixed points set. Then $F(f)$ is a connected closed subset of $Y$. If $F(f)$ is degenerated, then $f$ has exactly one fixed point. This case will be discussed in section 4. Now assume that $F(f)$ is nondegenerated.

**THEOREM 3.1** If $F(f)$ is contained in the closure of a component of $Y \setminus \{o\}$, then $f^2 = id_Y$ but $f \neq id_Y$ or $f^2$ is turbulent.

**Proof:** Without loss of generality, assume that $F(f) = [p, q] \subset [o, e_1]$ and $p < e_1 q$.

We first claim that $q = o$. Suppose not. Then $f(x) < e_1 x$ for all $x \in [o, q]$. Note that $p, q$ are fixed points of $f$. There exists $q' \in (o, q)$ such that $f([q', p']) \subset (q', p')$ for some $p' \in (p, e_1)$ (if $p \neq e_1$) or $f([q', e_1]) \subset (q', e_1]$ (if $p = e_1$). There is a contradiction. By the claim, the following two cases will be considered.

CASE 1. $p \neq e_1$. Clearly, we have $x < e_1 f(x)$ for all $x \in (p, e_1]$; $x < e_2 f(x)$ for all $x \in (o, e_2]$ and $x < e_3 f(x)$ for all $x \in (o, e_3)$. Since $f$ is onto, there exists $x_0 \in [e_2, e_3] \setminus \{o\}$ such that $f(x_0) = e_1$. Without loss of generality, we assume that $x_0 \in [e_2, o)$. Then, by the continuity of $f$, there exists $r \in (o, x_0)$ such that $f(r) = p$. Furthermore, we may assume that such $r$ is the largest one in $[e_2, o)$ relative to $< e_2$.

**SUBCASE 1.1** $p < e_1 f(x)$ for all $x \in (o, e_3]$. Then there exists $s \in (o, r) \cup (o, e_3]$ such that $f(s) = r$. (If no such $s$ exists, then $U = (r', e_3) \cup (o, p')$ for some $r' \in (o, r)$ and some $p' \in (p, e_1)$ satisfies $f(U) \subset U$.) Furthermore, we have $s \in (o, e_3]$ (for otherwise $(o, r) \cap F(f) \neq \phi$) and assume that such $s$ is the largest one in $(o, e_3]$ relative to $< e_3$. There exists $t \in (o, r) \cup (o, s)$ such that $f(t) = s$. (If no such $t$ exists, then $U = (r', s') \cup (o, p')$ for some $r' \in (o, r)$, some $s' \in (o, s)$ and some $p' \in (p, e_1)$ satisfies $f(U) \subset U$.) Furthermore, we have $t \in (o, r)$ (for otherwise, $(o, s) \cap F(f) \neq \phi$). Taking $J = [o, t], K = [t, r)$, one gets $f^2(J) \cap f^2(K) \supset J \cup K$ and thus $f^2$ is turbulent.

**SUBCASE 1.2** There exists $r_1 \in (o, e_3]$ such that $f(r_1) = p$. Without loss of generality, assume that such $r_1$ is the largest one in $[e_3, o)$ relative to $< e_3$. Then there exists $s \in (o, r_1)$ such that $f(s) = r$ or $s_1 \in (o, r_1)$ such that $f(s_1) = r_1$. (If none of such $s, s_1$ exists, then $U = (r', r'_1) \cup (o, p')$ for some $r' \in (o, r), r'_1 \in (o, r_1)$ and some $p' \in (p, e_1)$ satisfies $f(U) \subset U$.) Without loss of generality, we assume that there exists $s \in (o, r_1)$ such...
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If \( f(s) = r \). (If there exists \( s_1 \in (o, r) \) such that \( f(s_1) = r_1 \), the proof of this case is similar and omitted.) A similar argument as that in subcase 1.1 yields that \( f^2 \) is turbulent.

**Case 2.** \( p = e_1 \). Clearly, we have \( x <_{e_2} f(x) \) for all \( x \in (o, e_2) \) and \( x <_{e_3} f(x) \) for all \( x \in (o, e_3) \).

If there exists \( a \in [e_2, e_3] \setminus \{o\} \) such that \( f(a) \in (o, e_1) \), then we can get \( b \in (o, a) \) \( \cup (o, e_3) \) (without loss of generality, assume that \( a \in (o, e_2) \). For \( a \in (o, e_3) \), a similar argument will be done.) such that \( f(b) = a \). (If no such \( b \) exists, then there exists \( a' \in (o, a) \) such that \( a' <_{e_2} f(x) \) for all \( x \in (o, a) \cup (o, e_3) \). But then \( U = [e_1, e_3] \cup (o, a') \) satisfies \( f(U) \subseteq U \).) In fact, we have \( b \in (o, e_3) \). (For otherwise, \( F(f) \cap (o, a) \neq \emptyset \).) Without loss of generality, assume that such \( b \) is the largest one in \( (o, e_3) \) relative to \( <_{e_3} \) such that \( f(b) = a \). Furthermore, let \( c \) be any point in \((a, b)\) such that \( f(c) = b \). (Again if no such \( c \) exists, then there exists \( b' \in (o, b) \) such that \( b' <_{e_3} f(x) \) for all \( x \in [a, b] \cup (o, e_1) \). But then \( U = (a', b') \cup (o, e_1) \) satisfies \( f(U) \subseteq U \) for some \( a' \in (o, a) \).) In fact, we have \( c \in (o, a) \) (for otherwise, \( F(f) \cap (o, e_3) \neq \emptyset \)). Taking \( J = [o, c], K = [a, c], \) one gets \( f^2(J) \cap f^2(K) \supseteq J \cup K \) and thus \( f^2 \) is turbulent.

If \( f^{-1}((o, e_1)] \cap [e_2, e_3] \neq \emptyset \), then \( f|_{[e_2, e_3]} : [e_2, e_3] \to [e_2, e_3] \) is pointwise chain recurrent and has exactly one fixed point. It follows from [4, Theorem] that \( f^2|_{[e_2, e_3]} = id|_{[e_2, e_3]} \) or \( f^2|_{[e_2, e_3]} \) is turbulent. If \( f^2|_{[e_2, e_3]} = id|_{[e_2, e_3]} \) then \( f^2 = id_Y \) but \( f \neq id_Y \); if \( f^2|_{[e_2, e_3]} \) is turbulent, then \( f^2 \) is certainly turbulent.

The proof is complete.

**Theorem 3.2.** There does not exist \( f \) such that \( o \in \text{int} F(f) \) except the identity map \( id_Y \).

**Proof:** Assume that such \( f \) exists and \( f \) is not the identity. Let \( F(f) \cap [o, e_i] = [o, p_i], i \in \{1, 2, 3\} \). Note that each \( p_i \) is the smallest fixed point in \([o, e_i]\) relative to \( <_{e_i} \). Then there exists \( p_i' \in (p_i, e_i) \) (if \( p_i \neq e_i \)) such that \( x <_{e_i} f(x) <_{e_i} p_j \) (\( i \in \{1, 2, 3\} \), and \( j \neq i \)) for all \( x \in (p_i, p_i') \). Thus, taking

\[
U = U_1 \cup U_2 \cup U_3,
\]

where each \( U_i = [o, p'_i] \) if \( p_i \neq e_i \); \( [o, e_i] \) if \( p_i = e_i \), one gets that \( f(U) \subseteq U \). There is a contradiction. The proof is complete.

4. Pointwise chain recurrent of \( Y \) with exact one fixed point

In this section, we assume that \( f \) has exactly one fixed point, written by \( p \).

**Lemma 4.1.**

1. If \( p = o \), then \( f^2 \) has exactly one fixed point too, but then \( f^3 \) has more than one fixed point.

2. If \( p \neq o \), then \( f^2 \) has more than one fixed point.
PROOF: (1) Assume that \( f^2 \) has a fixed point \( p' \) different from \( o \). Without loss of generality, we assume that \( p' \in (o,e_1) \), then \( f(p') \in (o,e_2] \cup (o,e_2] \) (for otherwise, there exists at least one fixed point of \( f \) in \( (o,e_1) \)). Without loss generality, we assume that \( f(p') \in (o,e_2] \). Since \( f \) is onto, there exist \( a_1 \in (o,e_2] \cup (o,e_3] \) such that \( f(a_1) = e_1 \), \( a_2 \in (o,e_1] \cup (o,e_3] \) such that \( f(a_2) = e_2 \) and \( a_3 \in (o,e_1] \cup (o,e_2] \) such that \( f(a_3) = e_3 \). If \( a_1 \in (o,e_2] \), then we claim that \( a_2 \in (o,e_3] \) and \( a_3 \in (o,e_1] \) (If \( a_1 \in (o,e_3] \), we must have \( a_2 \in (o,e_2] \) and \( a_3 \in (o,e_2] \). A similar argument will be done.) In fact, if \( a_2 \in (o,e_1] \), then \( a_3 \in (o,e_2] \) or \( a_3 \in (o,e_1] \). Without loss of generality, we assume that \( a_3 \in (o,e_2] \) (If \( a_3 \in (o,e_1] \), the proof of this case is similar and omitted.) Furthermore, we assume that \( a_1 < e_2 \ a_3 \) (If \( a_3 < e_2 \ a_1 \), the proof of this case is similar and omitted.), then by the continuity of \( f \), \( f(a_3) \in [o, e_1] \), which contradicts \( f(a_3) = e_3 \). Thus, we have \( p', a_1 \in (o,e_1] \). By the continuity of \( f \), if \( p' < e_1 \ a_3 \), then \( f(a_3) \in [o, f(p')] \), which contradicts \( f(a_3) = e_2 \); If \( a_3 < e_1 \ p' \), then \( f(p') \in [o, e_3] \), which contradict \( f(p') \in (o,e_2] \).

From the above discussion, we see that either there exist \( a_1 \in (o,e_2] \), \( a_2 \in (o,e_3] \), \( a_3 \in (o,e_1] \), or \( a_1 \in (o,e_3] \), \( a_2 \in (o,e_1] \), \( a_3 \in (o,e_2] \) such that \( f(a_1) = e_1 \), \( f(a_2) = e_2 \), \( f(a_3) = e_3 \). Since the proofs of the above two cases are similar. We only prove the former. Clearly, \([o, a_1] \subseteq f^3([o, a_1])\), hence there exists \( a \in [o, a_1] \) such that \( f^3(a) = a_1 \). Then \( f^3 \) has a fixed point in \([a, e_2] \).

(2) In fact, if \( p \neq o \), then we must have \( p \) is in one component of \( Y \setminus \{o\} \) and \( p \not\in \{e_1, e_2, e_3\} \)(For otherwise, there exist more than one fixed point of \( f \)). The proof of this case is similar to that of [4, Lemma 3] and omitted. \( \square \)

THEOREM 4.1.

(1) If \( p = o \), then \( f^2 \) can not be turbulent. But \( f^6 \) is turbulent or identity map.

(2) If \( p \neq o \), then \( f^4 \) is turbulent or identity map.

PROOF: By the previous results, the theorem is clear. Now to prove the main theorem, by Theorems 2.1, 2.2, 3.1, 3.2 and Lemma 4.1, either \( f^{12} \) is the identity map or \( f^{12} \) is turbulent. \( \square \)

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