CENTRALISERS ON RINGS AND ALGEBRAS

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In this paper we investigate identities related to centralisers in rings and algebras. We prove, for example, the following result. Let $A$ be a semisimple $H^*$-algebra and let $T : A \rightarrow A$ be an additive mapping satisfying the relation $T(x^{m+n+1}) = x^mT(x)x^n$ for all $x \in A$ and some integers $m \geq 1, n \geq 1$. In this case $T$ is a left and a right centraliser.

Throughout, $R$ will represent an associative ring with centre $Z(R)$. Given an integer $n \geq 2$, a ring $R$ is said to be $n$-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. Recall that a ring $R$ is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. An additive mapping $T : R \rightarrow R$ is called a left centraliser in case $T(xy) = T(x)y$ holds for all $x, y \in R$. The concept appears naturally in $C^*$-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \rightarrow R_R$ is a homomorphism of a ring module $R$ into itself. For a semiprime ring $R$ all such homomorphisms are of the form $T(x) = qx$ where $q$ is an element of the Martindale right ring to quotients $Q_r$ (see Chapter 2 by Beidar and Martindale). In case $R$ has the identity element $T : R \rightarrow R$ is a left centraliser if and only if $T$ is of the form $T(x) = ax$ for some $a \in R$. An additive mapping $T : R \rightarrow R$ is called a left Jordan centraliser in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of right centralizer and right Jordan centralizer should be self-explanatory. In case $T : R \rightarrow R$ is a left and right centraliser, where $R$ is a semiprime ring with extended centroid $C$, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see [2, Theorem 2.3.2]).

Zalar [12] has proved that any left (right) Jordan centraliser on a 2-torsion free semiprime ring is a left (right) centraliser. Molnár [7] has proved that in case we have an additive mapping $T : A \rightarrow A$, where $A$ is a semisimple $H^*$-algebra, satisfying the relation $T(x^3) = T(x)x^2$ (respectively $T(x^3) = x^2T(x)$) for all $x \in A$, then $T$ is a left (right) centraliser. Let us recall that a semisimple $H^*$-algebra is a semisimple Banach $^*$-algebra whose norm is a Hilbert space norm such that $(x, yz^*) = (xz, y) = (z, x^*y)$.

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is fulfilled for all $x, y, z \in A$ (see [1]). The result of Benkovič and Eremita [3] states that in case we have a prime ring $R$ and an additive mapping $T : R \to R$ satisfying the relation $T(x^n) = T(x)x^{n-1}$ for all $x \in R$, where $n \geq 2$ is a fixed integer, then $T$ is a left centraliser in case $\text{char}(R) = 0$ or $\text{char}(R) \geq n$. Some results concerning centralisers on semiprime rings can be found in [3, 6] and [8, 9, 10, 11]. Let $X$ be a real or complex Banach space and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on $X$ and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subset L(X)$ is said to be standard in case $F(X) \subset A(X)$. Let us point out that any standard algebra is prime, which is a consequence of Hahn-Banach theorem. We denote by $X^*$ the dual space of a Banach space $X$ and by $I$ the identity operator on $X$.

It is our aim in this paper to prove the following result.

**Theorem 1.** Let $A$ be a semisimple $H^*$-algebra and let $T : A \to A$ be an additive mapping satisfying the relation

$$T(x^{m+n+1}) = x^m T(x)x^n$$

for all $x \in A$ and some integers $m \geq 1$, $n \geq 1$. In this case $T$ is a left and a right centraliser.

For the proof of the theorem above we need the result below which is of independent interest.

**Theorem 2.** Let $X$ be a Banach space over a real or complex field $F$ and let $A(X) \subset L(X)$ be a standard operator algebra. Suppose there exists an additive mapping $T : A(X) \to L(X)$ satisfying the relation

$$T(A^{m+n+1}) = A^m T(A)A^n$$

for all $A \in A(X)$ and some integers $m \geq 1$, $n \geq 1$. In this case $T$ is of the form $T(A) = \lambda A$ for some $\lambda \in F$.

In the proof of Theorem 2 we shall use some ideas similar to those used in [7] and the following purely algebraic results proved by Brešar [4] and Zalar [12].

**Theorem A.** ([4, Theorem 2]). Let $R$ be a 2-torsion free prime ring. Suppose there exists an additive mapping $F : R \to R$ satisfying the relation $[F(x), x, x] = 0$ for all $x \in R$. In this case $[F(x), x] = 0$ holds for all $x \in R$.

**Theorem B.** ([12, Proposition 1.4]). Let $T$ be a 2-torsion free semiprime ring and let $T : R \to R$ be a left (right) Jordan centraliser. In this case $T$ is a left (right) centraliser.

**Proof of Theorem 2:** We have the relation

$$T(A^{m+n+1}) = A^m T(A)A^n.$$
Let us first consider the restriction of $T$ on $F(X)$. Let $A$ be from $F(X)$ and let $P \in F(X)$ be a projection with $AP = PA = A$. From the above relation one obtains $T(P) = PT(P)P$, which gives

\[(2) \quad T(P)P = PT(P) = PT(P)P.\]

Putting $A + P$ for $A$ in the relation (1), we obtain

\[(3) \quad \sum_{i=0}^{m+n+1} \binom{m+n+1}{i} T(A^{m+n+1-i}P^i) = \left( \sum_{i=0}^{m} \binom{m}{i} A^{m-i}P^i \right)(T(A) + B) \left( \sum_{i=0}^{n} \binom{n}{i} A^{n-i}P^i \right),\]

where $B$ stands for $T(P)$. Using (1) and rearranging the equation (3) in the sense of collecting together terms involving an equal number of factors of $P$ we obtain:

\[(4) \quad \sum_{i=1}^{m+n} f_i(A, P) = 0,\]

where $f_i(A, P)$ stands for the expression of terms involving $i$ factors of $P$. Replacing $A$ by $A + 2P$, $A + 3P$, \ldots, $A + (m + n)P$ in turn in the equation (1), and expressing the resulting system of $m + n$ homogeneous equations in the variables $f_i(A, P)$, $i = 1, 2, \ldots, m+n$, we see that the coefficient matrix of the system is a van der Monde matrix

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
2 & 2^2 & \cdots & 2^{m+n} \\
\vdots & \vdots & \ddots & \vdots \\
(m+n) & (m+n)^2 & \cdots & (m+n)^{m+n}
\end{bmatrix}.
\]

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution.

In particular

\[
f_{m+n-1}(A, P) = \left( \frac{m+n+1}{m+n-1} \right) T(A^2) - \left( \frac{m}{m-2} \right) \binom{n}{m} A^2 B - \left( \frac{m}{n-2} \right) B A^2 - \left( \frac{m}{m-1} \right) \binom{n}{m} P T(A) P - \left( \frac{m}{n-1} \right) \binom{n}{m} P T(A) A - \left( \frac{m}{m-1} \right) \binom{n}{n-1} A B A = 0,
\]
and

\[ f_{m+n}(A, P) = \binom{m+n+1}{m+n} T(A) - \binom{m}{m-1} \binom{n}{n} AB - \binom{m}{m} \binom{n}{n-1} BA - \binom{m}{m} \binom{n}{n} PT(A)P \]

\[ = 0. \]

The above equations reduce to

(5) \((m+n+1)(m+n)T(A^2) = m(m-1)A^2B + n(n-1)BA^2 + 2mnABA + 2mAT(A)P + 2nPT(A)A,\)

and

(6) \((m+n+1)T(A) = mAB + nBA + PT(A)P.\)

Right multiplications of the relation (6) by \(P\) gives

(7) \((m+n+1)T(A)P = mAB + nBA + PT(A)P.\)

Similarly one obtains

(8) \((m+n+1)PT(A) = mAB + nBA + PT(A)P.\)

Combining (7) with (8) gives

\[ T(A)P = PT(A), \]

which reduces the relations (5) to

(9) \((m+n+1)(m+n)T(A^2) = m(m-1)A^2B + n(n-1)BA^2 + 2mnABA + 2mAT(A)P + 2nPT(A)A,\)

and the relation (7) to

(10) \((m+n)T(A)P = mAB + nBA.\)

Combining (10) with (6) gives

(11) \(T(A) = T(A)P.\)

From the above relation one can conclude that \(T\) maps \(F(X)\) into itself. Further from

(11), (10) reduces to

(12) \((m+n)T(A) = mAB + nBA.\)
From this we can conclude that $T$ is linear on $F(X)$. Further apply (12) we obtain

$$2mnABA = n(mAB)A + mA(nBA)$$

$$= n((m + n)T(A) - nBA)A + mA((m + n)T(A) - mAB)$$

$$= (m + n)(nT(A)A + mAT(A)) - n^2BA^2 - m^2A^2B.$$  

We have therefore

$$2mnABA = (m + n)(nT(A)A + mAT(A)) - n^2BA^2 - m^2A^2B.$$  

Applying (12) and the relation above to (9) we obtain

$$2mnABA = (m + n)(nT(A)A + mAT(A)) - n^2BA^2 - m^2A^2B.$$  

Applying (12) and the relation above to (9) we obtain

$$(m + n)T(A^2) = nT(A)A + mAT(A),$$

and multiplying by $(m + n)$ we obtain

$$(m + n)^2T(A^2) = n(m + n)T(A)A + mA(m + n)T(A).$$

Applying the above relation on both sides of (12) we obtain

$$(m + n)(mA^2B + nBA^2) = n(mAB + nBA)A + mA(mAB + nBA),$$

which reduces to

$$[A, B], A = 0.$$  


$$[T(A), A] = [B, A]A.$$  

Then applying (14) we obtain $[[T(A), A], A] = [[B, A]A, A] = [[B, A], A]A = 0$. Thus we have

$$[[T(A), A], A] = 0,$$

for any $A \in F(X)$. We have therefore an additive mapping $T$ which maps $F(X)$ into itself satisfying the relation above for any $A \in F(X)$. Since $F(X)$ is prime all the assumptions of Theorem A are fulfilled which means that

$$[T(A), A] = 0,$$

holds for any $A \in F(X)$. Applying this in (13), one obtains that $T(A^2) = T(A)A$ and $T(A^2) = AT(A)$ holds for all $A \in F(X)$. In other words, $T$ is a left and a right
Jordan centraliser on $F(X)$. By Theorem B it follows that $T$ is a left and also a right centraliser of $F(X)$.

We intend to prove that there exists $C \in L(X)$, such that

$$\tag{15} T(A) = CA, \quad \text{for all } A \in F(X).$$

For any fixed $x \in X$ and $f \in X^*$ we denote by $x \otimes f$ an operator from $F(X)$ defined by $(x \otimes f)y = f(y)x$, for all $y \in X$. For any $A \in L(X)$ we have $A(x \otimes f) = ((Ax) \otimes f)$. Let us choose $f$ and $y$ such that $f(y) = 1$ and define $Cx = T(x \otimes f)y$. Obviously, $C$ is linear. Using the fact that $T$ is a left centraliser on $F(X)$ we obtain

$$(CA)x = C(Ax) = T((Ax) \otimes f)y = T(A(x \otimes f))y = T(A)(x \otimes f)y = T(A)x, \ x \in X.$$ 

We have therefore $T(A) = CA$, for any $A \in F(X)$. Since $T$ a right centraliser on $F(X)$ we obtain $C(AP) = T(AP) = AT(P) = ACP$, where $A \in F(X)$ and $P$ is an arbitrary one-dimensional projection. We have therfore $[A, C]P = 0$. Since $P$ is arbitrary one-dimensional projection it follows that $[A, C] = 0$, for any $A \in F(X)$. Using the closed graph theorem one can easily prove that $C$ is continuous. Since $C$ commutes with all operators from $F(X)$ one can conclude that $Cx = \lambda x$ holds for any $x \in X$ and some $\lambda \in F$, which together with the relation (15) gives that $T$ is of the form

$$\tag{16} T(A) = \lambda A$$

any $A \in F(X)$ and some $\lambda \in F$.

It remains to prove that the above relation holds for any $A \in A(X)$ as well. Let us introduce $T_1 : A(X) \to L(X)$ by $T_1(A) = \lambda A$ and consider $T_0 = T - T_1$. The mapping $T_0$ is, obviously additive and satisfies the relation (1). Besides, $T_0$ vanishes on $F(X)$.

It is our aim to prove that $T_0$ vanishes on $A(X)$ as well. Let $A \in A(X)$, let $P$ be a one-dimensional projection and let $S = A + PAP - (AP + PA)$. Note that $S$ can be written in the form $S = (I - P)A(I - P)$, where $I$ denotes the identity operator on $X$. Since, obviously, $S - A \in F(X)$, we have $T_0(S) = T_0(A)$. Besides, $SP = PS = 0$. We have therefore the relation

$$\tag{17} T_0(A^{m+n+1}) = A^m T_0(A) A^n,$$

for all $A \in A(X)$. Applying the above relation we obtain

$$S^m T_0(S) S^n = T_0(S^{m+n+1}) = T_0((S + P)^{m+n+1})$$

$$= (S + P)^m T_0(S + (S + P)^m) T_0(S^n + P)$$

$$= S^m T_0(S) S^n + PT_0(S) S^n + S^n T_0(S) P + PT_0(S) P.$$
We have therefore

(18) \[ PT_0(S)S^n + S^mT_0(A)P + PT_0(A)P = 0. \]

Multiplying the above relation from both sides by \( P \) we obtain

(19) \[ PT_0(A)P = 0, \]

which reduces (18) to

(20) \[ PT_0(A)S^n + S^mT_0(A)P = 0. \]

Right multiplication by \( P \) then gives

(21) \[ S^mT_0(A)P = 0. \]

We intend to prove that

(22) \[ S^{m-1}T_0(A)P = 0. \]

Putting \( A + B \) for \( A \), where \( B \in F(X) \), in (21) and using the fact that \( T_0 \) vanishes on \( F(X) \), we obtain

\[
(S_1S^{m-1} + SS_1S^{m-2} + \cdots + S^{m-1}S_1)T_0(A)P = 0,
\]

where \( S_1 \) stands for \((I - P)B(I - P)\) (see [5]). The substitution \( T_0(A)PB \) for \( B \) in the above relation gives because of (19)

\[
(T_0(A)PBS^{m-1} + ST_0(A)PBS^{m-2} + \cdots + S^{m-1}T(A)PB)T_0(A)P = 0.
\]

Multiplying from the left side by \( S^{m-1} \) and applying (21) we obtain

\[
(S^{m-1}T_0(A)P)B(S^{m-1}T_0(A)P) = 0,
\]

for all \( B \in F(X) \). Then it follows \( S^{m-1}T_0(A)P = 0 \) by the primeness of \( F(X) \), which proves (22).

Now, (21) implies (22), one can conclude by induction that \( ST_0(A)P = 0 \), which gives

\[
AT_0(A)P - PAT_0(A)P = 0,
\]

because of (19). Then putting \( A + B \) for \( A \), where \( B \in F(X) \), we obtain

\[
0 = (A + B)T_0(A)P - P(A + B)T_0(A)P = BT_0(A)P - PBT_0(A)P.
\]

We have therefore proved that

\[
BT_0(A)P - PBT_0(A)P = 0
\]
holds for all $A \in A(X)$ and all $B \in F(X)$. The substitution $T_0(A)PB$ for $B$ in the above relation gives, because of (19), $(T_0(A)P)B(T_0(A)P) = 0$, for all $B \in F(X)$. Thus it follows $T_0(A)P = 0$ by the primeness of $F(X)$. Since $P$ is an arbitrary one-dimensional projection, one can conclude that $T_0(A) = 0$, for any $A \in A(X)$, which completes the proof of the theorem. 

PROOF OF THEOREM 1: The proof goes through using the same arguments as in the proof of the Theorem of [7], with the exception that one has to use Theorem 2 instead of the Lemma in [7].

In the proof of Theorem 2 (the relation (13)) we met an additive mapping $T : F(X) \to F(X)$ satisfying the relation

$$(m + n)T(A^2) = mAT(A) + nT(A)A$$

for all $A \in F(X)$. In the case $m = n$ this reduces to $2T(A^2) = T(A)A + AT(A)$. Vukman [7] has proved that when we have an additive mapping $T : R \to R$, where $R$ is an arbitrary 2-torsion free semiprime ring, satisfying the relation $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$, then $T$ is a left and right centraliser. These observations lead to the following conjecture.

CONJECTURE 1. Let $m$ and $n, m \neq -n$ be some nonzero integers and let $R$ be a semiprime ring with suitable torsion restrictions. Suppose there exists an additive mapping $T : R \to R$ satisfying the relation

$$(m + n)T(x^2) = mxT(x) + nT(x)x$$

for all $x \in R$. In this case $T$ is a left and right centraliser.

Our last result is related to conjecture above.

THEOREM 3. Let $m$ and $n, m \neq -n$, be some nonzero integers and let $R$ be a $|mn|$ and $|m + n|$-torsion free semiprime ring. Suppose there exists and additive mapping $T : R \to R$ satisfying the relation

$$(23) \quad (m + n)T(xy) = mxT(y) + nT(x)y,$$

for all pairs $x, y \in R$. In this case $T$ is a left and a right centraliser.

PROOF: We have the relation

$$(23) \quad (m + n)T(xy) = mxT(y) + nT(x)y,$$

for all pairs $x, y \in R$. We compute the expression $(m + n)^2T(xyx)$ in two ways. First applying the relation above

$$(m + n)^2T(xy) = m(m + n)xT(yx) + n(m + n)T(y)x + n(m + n)T(x)yx,$$

$x, y \in R$. 


Thus we have
\[(m + n)^2 T(xyx) = m^2 xy T(x) + mnx T(y)x + mn T(x)yx + n^2 T(x)yx,\]
for \(x, y \in R\). On the other hand using (23)
\[(m + n)^2 T((xy)x) = m(m + n)xy T(x) + n(m + n)T(xy)x
= m(m + n)xy T(x) + n(mx T(y) + n T(x)y)x, x, y \in R.\]
Thus we have
\[(m + n)^2 T(xyx) = m^2 xy T(x) + mnx T(y)x + mn T(x)yx + n^2 T(x)yx, x, y \in R.\]
Subtracting the relation (25) from (24) we obtain \(mn T(x)yx = xy T(x) = 0\), for all pairs \(x, y \in R\), which reduces to
\[T(x)yx - xy T(x) = 0, x, y \in R\]
since we have assumed that \(R\) is \(|mn|\)-torsion free. Putting in the above relation first \(yx\) for \(y\) then multiplying from the right side by \(x\) and subtracting the relations so obtained one from another we obtain \(xy [T(x), x] = 0\), for all pairs \(x, y \in R\). From this one obtains easily \([T(x), x] y [T(x), x] = 0\), for all pairs \(x, y \in R\). Hence it follows
\[(26) \quad [T(x), x] = 0, x \in R\]
by the semiprimeness of \(R\). The substitution \(y = x\) in (23) gives
\[(m + n)T(x^2) = mx T(x) + nT(x)x, x \in R.\]
By (26) one can then replace \(x T(x)\) by \(T(x)x\) which gives \((m + n)T(x^2) = (m + n)T(x)x\) for all \(x \in R\). Since we have assumed that \(R\) is \(|m + n|\)-torsion free, it follows that \(T(x^2) = T(x)x\) holds for all \(x \in R\). Of course, we also have \(T(x^2) = xT(x)\), for all \(x \in R\). In other words, \(T\) is a left and right Jordan centraliser. By Theorem B \(T\) is a left and a right centraliser. The proof of the theorem is complete.

We conclude with the following conjecture.

**Conjecture 2.** Let \(R\) be a semiprime ring with suitable torsion restrictions and let \(T : R \to R\) be an additive mapping satisfying the relation
\[T(x^{m+n+1}) = x^m T(x)x^n\]
for all \(x \in R\) and some integers \(m \geq 1, n \geq 1\). In this case \(T\) is a left and right centraliser.
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