ON THE POWERS OF SOME TRANSCENDENTAL NUMBERS

ARTŪRAS DUBICKAS

We construct a transcendental number \( \alpha \) whose powers \( \alpha^n, n = 1, 2, 3, \ldots \), modulo 1 are everywhere dense in the interval \([0,1]\). Similarly, for any sequence of positive numbers \( \delta = (\delta_n)_{n=1}^\infty \), we find a transcendental number \( \alpha = \alpha(\delta) \) such that the inequality \( \{\alpha^n\} < \delta_n \) holds for infinitely many \( n \in \mathbb{N} \), no matter how fast the sequence \( \delta \) converges to zero. Finally, for any sequence of real numbers \( (r_n)_{n=1}^\infty \) and any sequence of positive numbers \( (\delta_n)_{n=1}^\infty \), we construct an increasing sequence of positive integers \( (g_n)_{n=1}^\infty \) and a number \( \alpha > 1 \) such that \( ||\alpha^{g_n} - r_n|| < \delta_n \) for each \( n \geq 1 \).

1. INTRODUCTION

Throughout this paper, we shall denote by \( \{x\} \), \( [x] \) and \( ||x|| \) the fractional part of a real number \( x \), the integral part of \( x \), and the distance from \( x \) to the nearest integer, respectively. Clearly, \( x = [x] + \{x\} \) and \( ||x|| = \min (\{x\}, 1 - \{x\}) \). By \( \mathbb{N} \) and \( \mathbb{Q} \) we denote the set of positive integers and the set of rational numbers, respectively.

Let \( \alpha > 1 \) be a real number. Koksma [7] proved that for almost all \( \alpha > 1 \) the fractional parts \( \{\alpha^n\}_{n=1}^\infty \) are uniformly distributed in the interval \([0,1]\). However, for most specific \( \alpha \), the distribution of the sequence \( \{\alpha^n\}_{n=1}^\infty \) is an open question. The “exceptional” \( \alpha \) in this respect (in the sense that for them the distribution of the sequence \( \{\alpha^n\}_{n=1}^\infty \) in \([0,1]\) is quite well-known) are Pisot and Salem numbers. See, for instance, Salem’s book [14] and some recent papers on this kind of problems [3, 5, 6, 9, 17]. In general, the problem of the distribution of the fractional parts \( \{\alpha^n\}_{n=1}^\infty \) goes back to Weyl [16]. Later, some unsolved problems about the distribution of the powers of the number \( \alpha = 3/2 \) were raised by Vijayaraghavan [15] and Mahler [11]. The current status of these problems is described in a recent review of Adhikari and Rath [1].

Since we shall be concerned with Pisot numbers later on, let us recall that a real algebraic integer \( \alpha > 1 \) is called a Pisot number if its conjugates over \( \mathbb{Q} \), except for \( \alpha \) itself, all lie in the open unit disc \( |z| < 1 \). For each Pisot number \( \alpha \), we have \( ||\alpha^n|| \to 0 \) as \( n \to \infty \) (see also [5, 6, 9] for some related problems). In contrast, for a Salem number \( \alpha \), by a result of Pisot and Salem [13], the sequence \( \{\alpha^n\}_{n=1}^\infty \) is everywhere dense in \([0,1]\),
but not uniformly distributed in \([0, 1]\). Hence, for every \(\alpha\) which is an \(m\)th root of a Salem number with some \(m \in \mathbb{N}\), the sequence \(\{\alpha^n\}_{n=1}^{\infty}\) is also everywhere dense in \([0, 1]\).

However, if \(\alpha > 1\) is an algebraic number which is neither a Pisot number nor a root of a Salem number, then the distribution of the sequence \(\{\alpha^n\}_{n=1}^{\infty}\) is not known. Moreover, if \(\alpha\) is a transcendental number, say, \(\alpha = e, \pi, \log 3\) or similar, then it is not even known whether the sequence \(\{\alpha^n\}_{n=1}^{\infty}\) has just one or more than one limit point. One of the results of Pisot [12] implies, for example, that there are arbitrarily large numbers \(\alpha\) for which \(\{\alpha^n\}_{n=1}^{\infty}\) has just one or more than one limit point. One of the results of Pisot [12] implies, for example, that there are arbitrarily large numbers \(\alpha\) for which \(\{\alpha^n\}_{n=1}^{\infty}\) has just one or more than one limit point. One of the results of Pisot [12] implies, for example, that there are arbitrarily large numbers \(\alpha\) for which \(\{\alpha^n\}_{n=1}^{\infty}\) has just one or more than one limit point. One of the results of Pisot [12] implies, for example, that there are arbitrarily large numbers \(\alpha\) for which \(\{\alpha^n\}_{n=1}^{\infty}\) has just one or more than one limit point. One of the results of Pisot [12] implies, for example, that there are arbitrarily large numbers \(\alpha\) for which \(\{\alpha^n\}_{n=1}^{\infty}\) has just one or more than one limit point.

Moreover, if \(\alpha\) is a transcendental number, say, \(\alpha = \sqrt{2}, \pi\log 3\) or similar, then it is not even known whether the sequence \(\{\alpha^n\}_{n=1}^{\infty}\) is dense in \([0, 1]\). Curiously, but except for an unpublished manuscript of Lerma [8] which gives a (quite complicated) construction of some \(\alpha > 1\) whose powers are uniformly distributed in \([0, 1]\) it seems like that there is no method known which would allow the explicit construction of a transcendental number \(\alpha\) whose powers modulo 1 are everywhere dense in \([0, 1]\), although, by the above mentioned result of Koksma, almost all transcendental numbers have this property. We thus begin with the following construction of \(\alpha\) by a recurrent sequence similar to [2]. For such \(\alpha\), the sequence \(\{\alpha^n\}_{n=1}^{\infty}\) is everywhere dense, because its subsequence \(\{\alpha^n\}_{n=1}^{\infty}\) is everywhere dense.

**Theorem 1.** Let \((r_n)_{n=1}^{\infty}\) be a sequence of real numbers in \([0, 1]\) which is everywhere dense in \([0, 1]\) such that \(r_n = 0\) for infinitely many indices \(n\). Suppose that \(x_1 := 1\) and \(x_n := 1 + [(x_{n-1} + r_{n-1})^n - r_n]\) for \(n \geq 2\). Then the limit \(\alpha := \lim_{n \to \infty} (x_n + r_n)^{1/n!} > 1\) exists, it is a transcendental number, and the sequence \(\{\alpha^n\}_{n=1}^{\infty}\) is everywhere dense in \([0, 1]\).

We can take, for instance, \(r_n\) to be the \(n\)th term of the sequence of blocks of Farey fractions that are separated by one zero

\[
1/2, 0, 1/3, 2/3, 0, 1/4, 3/4, 0, 1/5, 2/5, 3/5, 4/5, 0, 1/6, 5/6, 0, \ldots
\]

The problem of the distribution of the sequence \(\{\alpha^n\}_{n=1}^{\infty}\) in \([0, 1]\) is related to a purely diophantine problem of how close the elements of this sequence are to 0 and 1. Recently, Corvaja and Zannier [4] generalised an old result of Mahler [10] and proved that if \(\alpha > 1\) is an algebraic number such that, for some positive \(\delta < 1\), the inequality \(|\alpha^n| < (1 - \delta)^n\) has infinitely many solutions in positive integers \(n\) then \(\alpha^n\) is a Pisot number for some \(m \in \mathbb{N}\). Earlier, Mahler proved this result for rational numbers \(\alpha\) using a version of Roth's theorem. In principle, using some properties of Pisot numbers, one can derive our next theorem from [4]. However, since the condition on \(\delta\) is much stronger than the one considered in [4], we shall give a simple direct proof without using the results of [4].
Some transcendental numbers

**Theorem 2.** Let \( \alpha \) be a real number and let \( (\delta_n)_{n=1}^{\infty} \) be a sequence of positive numbers satisfying \( \lim_{n \to \infty} \delta_n^{1/n} = 0 \). If the inequality \( \|\alpha^n\| < \delta_n \) has infinitely many solutions in \( n \in \mathbb{N} \) then either \( \alpha \) is a transcendental number or \( \alpha^m \) is an integer for some \( m \in \mathbb{N} \).

In addition, it is shown in [4] that there exists a transcendental number \( \alpha > 1 \) such that \( \|\alpha^n\| < 2^{-n} \) for infinitely many \( n \in \mathbb{N} \). In this direction, for any sequence \( \delta = (\delta_n)_{n=1}^{\infty} \) of positive numbers, we construct a transcendental number \( \alpha = \alpha(\delta) \) such that the inequality \( \|\alpha^n\| < \delta_n \) holds for infinitely many \( n \in \mathbb{N} \), no matter how fast the the sequence \( \delta \) converges to 0.

**Theorem 3.** Let \( \delta = (\delta_n)_{n=1}^{\infty} \) be a sequence of positive numbers. Set \( x_1 := 1 \) and \( x_n := x_{n-1}^{u_n} + 1 \) for \( n \geq 2 \), where \( u_1, u_2, u_3, \ldots \) are some positive integers depending on \( \delta \) (see the proof how). Then the limit \( \alpha := \lim_{n \to \infty} x_n^{1/(u_1 u_2 \cdots u_n)} > 1 \) exists, it is a transcendental number, and the inequality \( \{\alpha^n\} < \delta_n \) holds for infinitely many \( n \in \mathbb{N} \).

In fact, not only zero but also any given sequence can be "copied" by some powers of \( \alpha \) modulo 1 with any prescribed accuracy. In our final theorem, we do not bother about the arithmetical nature of the limit \( \alpha \). (One can easily ensure that the number \( \alpha \) in Theorem 4 below is transcendental, for example, by adding infinitely many "extra terms" \( r_n = 0 \) and by increasing the "gaps" between consecutive \( q_n \)'s if necessary.) Also, we replace \( 1 + \lfloor x \rfloor \) by the ceiling function \( \lceil x \rceil \) and construct the approximants to \( \alpha \) directly rather than via integer parts of their powers as in Theorems 1 and 3. More precisely, we show that, for any sequence of real numbers \( (r_n)_{n=1}^{\infty} \), there is a number \( \alpha > 1 \) whose powers \( \alpha^{q_n} \), where \( q_n \) are some positive integers, tend to the numbers \( r_n \) (with respect to the metric \( \| \cdot \| \)) with any prescribed rate.

**Theorem 4.** Let \( \delta = (\delta_n)_{n=1}^{\infty} \) be a sequence of positive numbers, and let \( (r_n)_{n=1}^{\infty} \) be a sequence of real numbers. Suppose that \( y_0 > 2 \) and \( y_n := (\lceil y_{n-1}^n \rceil + r_n)^{1/q_n} \) for \( n \geq 1 \), where \( q_1 < q_2 < q_3 < \ldots \) are any positive integers satisfying \( q_{n+1} > q_n + \log_2(1/\delta_n) + 3 \) for \( n \geq 1 \). Then the limit \( \alpha := \lim_{n \to \infty} y_n \geq 2 \) exists, and, for this \( \alpha \), the inequality \( \|\alpha^{q_n} - r_n\| < \delta_n \) holds for each \( n \in \mathbb{N} \).

In particular, Theorem 4 implies that, for any sequence of real numbers \( (r_n)_{n=1}^{\infty} \) and any sequence of positive integers \( q_1 < q_2 < q_3 < \ldots \) satisfying \( \lim_{n \to \infty} (q_{n+1} - q_n) = \infty \), there is an \( \alpha > 2 \) such that \( \lim_{n \to \infty} \|\alpha^{q_n} - r_n\| = 0 \). Also, setting \( \delta_n = \epsilon \) for \( n \in \mathbb{N} \), taking \( q_n = m n \) for \( n \in \mathbb{N} \) with some fixed \( m \geq \log_2(1/\epsilon) + 3 \), and writing \( \alpha \) for \( \alpha^m \), we deduce the following corollary:

**Corollary 5.** Let \( (r_n)_{n=1}^{\infty} \) be a sequence of real numbers. Then, for any \( \epsilon > 0 \), there is an \( \alpha > 1 \) such that \( \|\alpha^n - r_n\| < \epsilon \) for each \( n \in \mathbb{N} \).

The construction itself and all of the proofs in this paper are similar to those in [2]. In the next section, we first give a self-contained proof of Theorem 2 and then derive from it an auxiliary lemma. The proofs of Theorems 1 and 3 given in Section 3 are based on the lemma. In Section 4 we shall prove Theorem 4.
2. ON THE APPROXIMATION OF THE POWERS OF A NUMBER

Proof of Theorem 2: If $|\alpha| < 1$ then $||\alpha^n|| = |\alpha|^n$ for each $n \geq n_1(\alpha)$, so $|\alpha| = ||\alpha^n||^{1/n} < \delta_1^n$ has infinitely many solutions in $n \in \mathbb{N}$ only if $\alpha = 0$. For $\alpha = \pm 1$, the claim is also trivial. So, without loss of generality, we can assume that $|\alpha| > 1$.

Let $I$ be the infinite set of indices $n$ for which $||\alpha^n|| < \delta_n$. Suppose that $\alpha$ is an algebraic number, say, of degree $d$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ over $\mathbb{Q}$. Let also $a_d \in \mathbb{N}$ be the leading coefficient of the minimal polynomial of $\alpha$ over $\mathbb{Q}$. Put $x_n := [\alpha^n + 1/2]$. Consider the product $P_n := a_d^n \prod_{j=1}^d (\alpha_j^n - x_n)$. It is a rational integer.

If $P_n = 0$, then $\alpha_j^n = x_n$ for some index $j$. By considering any automorphism of the normal extension $\mathbb{Q}(\alpha_1, \ldots, \alpha_d)/\mathbb{Q}$ which maps $\alpha_j \mapsto \alpha$ and using the fact that $x_n$ is an integer, we obtain that $\alpha^n = x_n$. This implies that $\alpha^n$ is an integer for some $m \in \mathbb{N}$. If $P_n \neq 0$, then $|P_n| > 1$. For each $n \in I$, we have $|\alpha^n - x_n| < \delta_n$. Hence

$$a_d^n \prod_{j=1}^d |\alpha_j^n - x_n| > |P_n| \geq 1.$$  

Putting $c := \max_{1 \leq j \leq d} |\alpha_j|$ and using $|x_n| \leq |\alpha|^n + 1/2 < c^n + 1$, we obtain that

$$1 < a_d^n \prod_{j=2}^d |\alpha_j^n - x_n| = a_d^n \prod_{j=2}^d (\alpha_j^n - x_n) \leq a_d^n (2c^n + 1)^{d-1} \leq \delta_n b^n,$$

where $b$ is a positive constant depending on $\alpha$ only (and not on $n$). Hence $1/b < \delta_1^n$ for every $n \in I$. This is a contradiction with $\lim_{n \to \infty} \delta_1^n = 0$, which implies that $\alpha$ is a transcendental number.

Lemma 6. Let $(r_n)_{n=1}^\infty$ be an arbitrary sequence of real numbers in $[0, 1)$ satisfying $r_n = 0$ for infinitely many indices $n$. Suppose that $x_1 := 1$ and

$$x_n := 1 + [(x_{n-1} + r_{n-1})^{v_n} - r_n]$$

for $n \geq 2$, where $v_1 = 1, v_2, v_3, \ldots$ are positive integers. Then

$$\alpha := \lim_{n \to \infty} (x_n + r_n)^{1/(v_1 v_2 \ldots v_n)}$$

is a transcendental number greater than 1 and

$$x_n + r_n < \alpha^{v_1 \ldots v_n} < x_n + r_n + (x_n + r_n)^{-v_n + 1}$$

for each $n \geq 2$.

Proof: Observe that the sequence $(x_n + r_n)^{1/(v_1 \ldots v_n)}$ is increasing. Indeed, by the definition of $x_n$,

$$x_n + r_n = 1 + [(x_{n-1} + r_{n-1})^{v_n} - r_n] + r_n > (x_{n-1} + r_{n-1})^{v_n}.$$
Next, we shall show that the sequence \((x_n + r_n + (x_n + r_n)^{-v_{n+1}})^{1/(n!v_1...v_n)}\) is decreasing. To prove this, we need to show that

\[
x_n + r_n + (x_n + r_n)^{-v_{n+1}} < (x_{n-1} + r_{n-1} + (x_{n-1} + r_{n-1})^{-(n-1)v_n})^{v_{n+1}}.
\]

Indeed, using \(x_n + r_n \leq 1 + (x_{n-1} + r_{n-1})^{v_n}\) and \(v_n \geq 1\), we deduce that, for each \(n \geq 3\),

\[
(x_{n-1} + r_{n-1} + (x_{n-1} + r_{n-1})^{-(n-1)v_n})^{v_{n+1}}
\]
\[
\geq (x_{n-1} + r_{n-1})^{v_n} + n v_n (x_{n-1} + r_{n-1})^{v_n-1 - (n-1)v_n}
\]
\[
\geq (x_{n-1} + r_{n-1})^{v_n} + n v_n \geq (x_{n-1} + r_{n-1})^{v_n} + 3
\]
\[
\geq x_n + r_n + 2 > x_n + r_n + (x_n + r_n)^{-v_{n+1}}.
\]

It follows that the sequences \(x_n^{1/(n!v_1...v_n)}\), \(n = 1, 2, \ldots\), (which is increasing) and \((x_n + r_n + (x_n + r_n)^{-v_{n+1}})^{1/(n!v_1...v_n)}\), \(n = 2, 3, \ldots\), (which is decreasing) tend to certain limits, say, \(\alpha\) and \(\gamma\), respectively, as \(n\) tends to infinity. Obviously, \(\alpha \leq \gamma\), so

\[
x_n + r_n < \alpha^{n!v_1...v_n} \leq \gamma^{n!v_1...v_n} < x_n + r_n + (x_n + r_n)^{-v_{n+1}}
\]

for each \(n \geq 2\). Note that, since the right hand side is at most \(x_n + r_n + 1\), we have \(\alpha = \gamma\) (although we shall not need it). It is clear that \(\alpha > 1\).

Next, we shall prove that the number \(\alpha\) is transcendental. Let \(I\) be the infinite set of indices \(n\) for which \(r_n = 0\). Denote \(V_n := n!v_1\ldots v_n\). We have \(x_n < \alpha^{V_n} < x_n + x_n^{v_{n+1}} \leq x_n + x_n^n \leq x_n + 1\). Fix \(\beta \in (1, \alpha)\). Then \(\alpha^{V_n} - 1 > \beta^{V_n}\) for each sufficiently large \(n\). Hence \(||\alpha^{V_n}|| < x_n^{-n} < (\alpha^{V_n} - 1)^{-n} < \beta^{-n}\) for each sufficiently large \(n \in I\). By Theorem 2, either \(\alpha\) is a transcendental number or \(\alpha^m \in \mathbb{N}\) for some \(m \in \mathbb{N}\). However, if \(\alpha^m\) is an integer, then \(\alpha^{V_n}\) must be an integer too for every \(n \geq m\), because \(V_n = n!v_1\ldots v_n\) is divisible by \(m\). This is, however, not the case, because \(\alpha^{V_n} \in (x_n, x_n + 1)\) for \(n \geq 2\). Consequently, \(\alpha\) is a transcendental number.

3. PROOFS OF THEOREMS 1 AND 3

PROOF OF THEOREM 1: Let us apply the lemma for \(v_1 = v_2 = v_3 = \cdots = 1\). The lemma implies that \(\alpha := \lim_{n \to \infty} x_n^{1/n!}\) is a transcendental number greater than 1 and \(x_n + r_n < \alpha^{n!} < x_n + r_n + x_n^{-n}\) for \(n \geq 2\).

Fix \(y \in (0, 1)\). In order to prove that \(y\) is a limit point of the sequence \(\{\alpha^{n!}\}_{n=1}^\infty\), it is sufficient to show that, for any positive number \(\varepsilon\) satisfying \(\varepsilon < 1 - y\), there is an \(n \in \mathbb{N}\) such that \(\{\alpha^{n!}\} \in (y, y + \varepsilon)\). Indeed, the interval \((y, y + \varepsilon/2)\) contains infinitely many \(r_n's\). Let \(I\) be the set of corresponding \(n's\). We claim that \(\{\alpha^{n!}\} \in (y, y + \varepsilon)\) for all sufficiently large \(n \in I\). For this, it is sufficient to show that

\[
x_n + r_n < \alpha^{n!} < x_n + y + \varepsilon.
\]
Indeed, adding two inequalities \( y < r_n \) and \( x_n + r_n < \alpha_n^{\ast} \), we immediately get the first inequality \( x_n + y < \alpha_n^{\ast} \). The second inequality, namely, \( \alpha_n^{\ast} < x_n + y + \varepsilon \) would follow from the inequalities \( r_n < y + \varepsilon/2 \) (which holds by the definition of \( I \)) and \( \alpha_n^{\ast} < x_n + r_n + \varepsilon/2 \). From \( \alpha_n^{\ast} < x_n + r_n + x_n^{-n} \), we see that the required inequality holds if \( x_n > 2/\varepsilon \). This is indeed the case, because \( x_n > \alpha_n^{\ast} - r_n - 1 \), so \( x_n \rightarrow \infty \) as \( n \rightarrow \infty \). Finally, since the sequence \( \{\alpha_n^{\ast}\}_{n=1}^{\infty} \) is everywhere dense in \((0,1)\), it is everywhere dense in \([0,1]\). \( \square \)

PROOF OF THEOREM 3: This time, we shall apply the lemma with \( r_1 = r_2 = r_3 = \cdots = 0 \) and with \( u_n = n v_n \). Here, \( v_n, n = 1, 2, \ldots \), are some positive integers to be chosen later. Then the lemma implies that \( \alpha := \lim_{n \rightarrow \infty} \alpha^{1/(n v_1 \ldots v_n)} \) is a transcendental number and

\[
x_n < \alpha^{n v_1 \ldots v_n} < x_n + x_n^{-n v_n + 1}.
\]

Fix any \( \beta \in (1, \alpha) \). For each \( n \) large enough, say \( n \geq n_1 \), we have \( x_n > \alpha^{n v_1 \ldots v_n} - 1 > \beta^{n v_1 \ldots v_n} \). Hence \( \log x_n > n! v_1 \ldots v_n \log \beta \). The inequality \( \{\alpha^N\} < \delta_N \) holds for every number \( N = n! v_1 \ldots v_n \) provided that \( x_n^{n v_{n+1}} < \delta_N \), that is, \( n v_{n+1} \log x_n > \log(1/\delta_N) \). So we can simply put \( v_1 = \cdots = v_{n_1} = 1 \) and, for each \( n \geq n_1 \), take any positive integer \( v_{n+1} \) greater than \( \log(1/\delta_{n v_1 \ldots v_n})/(n! v_1 \ldots v_n n \log \beta) \), which is always possible. \( \square \)

In particular, let us consider the sequence \( x_1 := 1 \) and \( x_{n+1} := x_n^2 + 1 \) for each \( n \geq 1 \). As above, the sequence \( x_n^{1/2^n}, n = 1, 2, \ldots \), is increasing, whereas the sequence \( (x_n + 1/(2x_n))^{1/2^n}, n = 1, 2, \ldots \), is decreasing. They both thus tend to the same limit \( \xi \). Since the inequality

\[
\{\xi^{2^n}\} < 1/(2x_n) < 1/(2(\xi^{2^n} - 1)) < (1/\xi)^{2^n}
\]

holds for all sufficiently large \( n \), the theorem of Corvaja and Zannier [4] implies that either the number \( \xi \) is transcendental or there is an \( m \in \mathbb{N} \) such that \( \xi^m \) is a Pisot number. The second possibility seems very unlikely. We thus conclude this section with the following transcendence type problem: prove that the number \( \xi \) is transcendental.

4. PROOF OF THEOREM 4

Without loss of generality we may assume that \( r_n \in [0,1) \) for each \( n \geq 1 \). Also, we can assume that \( \delta_n \leq 1/2 \), so \( q_{n+1} - q_n \geq 4 \). Since

\[
y_n = ((y_{n-1}^{q_n}) + r_n)^{1/q_n} \geq (y_{n-1}^{q_n} + r_n)^{1/q_n} \geq y_{n-1},
\]

the sequence \( \{y_n\}_{n=1}^{\infty} \) is non-decreasing. Also, \( y_n^{q_n} - r_n \) is an integer, so that \( \{y_n^{q_n}\} = r_n \) for every \( n \in \mathbb{N} \).

From \( y_{n-1}^{q_n} < y_n^{q_n} + 1 \) and \( r_n < 1 \), we have

\[
y_n/y_{n-1} < (1 + 2y_{n-1}^{-q_n})^{1/q_n} < 1 + 2/(q_n y_n^{q_n}) \geq r_n < 1/2.
\]

\[\text{https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972700039782}\]
Hence \( y_n - y_{n-1} < 2/(q_n y_{n-1}^{q_n}) \). Adding \( n \) such inequalities (for \( y_n - y_{n-1} \), for \( y_{n-1} - y_{n-2} \), \ldots, for \( y_1 - y_0 \)) and using \( y_j \geq y_0 \) for \( j = 1, 2, \ldots, n-1 \), we obtain that \( y_n - y_0 \) is bounded from above by \( 2/(q_1 y_0^{q_1-2}(y_0 - 1)) \), so the limit \( \alpha := \lim_{n \to \infty} y_n \) exists. Obviously, it is greater than or equal to \( y_0 \geq 2 \).

Next, we shall estimate the quotient \((y_{k+1}/y_k)^{q_n}\) for \( k \geq n \). Since \( q_n/q_{k+1} < 1 \) and \( y_k \geq 2 \), we have
\[
(y_{k+1}/y_k)^{q_n} < (1 + 2y_k^{-q_{k+1}})^{q_n/q_{k+1}} < 1 + 2q_n/(q_{k+1}y_k^{q_{k+1}+1}) < 1 + 2/y_k^{q_{k+1}+1} \leq 1 + y_k^{-q_{k+1}+1}.
\]
It follows that, for every fixed \( n \in \mathbb{N} \),
\[
(\alpha/y_n)^{q_n} = \prod_{k=n}^{\infty} (y_{k+1}/y_k)^{q_n} < \prod_{k=n}^{\infty} (1 + y_k^{-q_{k+1}+1}).
\]

In order to estimate the product \( \prod_{k=n}^{\infty} (1 + \tau_k) \), where \( \tau_k := y_k^{-q_{k+1}+1} \), we shall first bound it as \( \exp\left(\sum_{k=n}^{\infty} \tau_k\right) \) and then use the inequality \( \exp(\tau) < 1 + 2\tau \), because the sum \( \tau = \sum_{k=n}^{\infty} \tau_k \) turns out to be bounded by 1. Indeed, using the inequality \( y_k \geq y_n \geq 2 \), we obtain that
\[
\tau = \sum_{k=n}^{\infty} y_k^{-q_{k+1}+1} \leq \frac{1}{y_n^{q_{n+1}-2}(y_n - 1)} \leq y_n^{-q_{n+1}+2}
\]
(which is at most 1), hence \((\alpha/y_n)^{q_n} < 1 + 2/y_n^{q_{n+1}-2} \leq 1 + 1/y_n^{q_{n+1}-3} \). Therefore \( 0 \leq \alpha^{q_n} - y_n^{q_n} < 1/y_n^{q_{n+1}-q_n-3} \leq 1/2^{q_{n+1}-q_n-3} \). Using \( \{y_n^{q_n}\} = r_n \), we conclude that \( \|\alpha^{q_n} - r_n\| < 2^{-q_{n+1}+q_n+3} \) for each \( n \in \mathbb{N} \). The right hand side of this inequality does not exceed \( \delta_n \) provided that \( q_{n+1} \geq q_n + \log_2(1/\delta_n) + 3 \). This completes the proof of Theorem 4.

If the sequence \( \{q_n\}_{n=1}^{\infty} \) is not growing very fast, then the arithmetical nature of the limit obtained by this kind of iterations seems to be quite mysterious even in the simplest case \( r_1 = r_2 = r_3 = \cdots = 0 \) and \( q_n = n \). For instance, let us start with \( y_1 \in (1, \sqrt{2}] \), and consider the sequence \( (y_n)_{n=1}^{\infty} \) obtained by the following iterations
\[
y_n := \left[y_{n-1}^{n}\right]^{1/n}
\]
for \( n \geq 2 \). Then \( y_2 = 2^{1/2}, y_3 = 3^{1/3}, y_4 = 5^{1/4}, y_5 = 8^{1/5}, y_6 = 13^{1/6}, \ldots \). By the same argument as above, the limit \( \zeta := \lim_{n \to \infty} y_n \) exists: prove that \( \zeta \) is a transcendental number.

REFERENCES


Department of Mathematics and Informatics
Vilnius University
Naugarduko 24
Vilnius LT-03225
Lithuania

e-mail: arturas.dubickas@mif.vu.lt