

A QUESTION OF PAUL ERDŐS AND
NILPOTENT-BY-FINITE GROUPS

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Let n be a positive integer or infinity (denoted ∞), k a positive integer. We denote by $\Omega_k(n)$ the class of groups G such that, for every subset X of G of cardinality $n + 1$, there exist distinct elements $x, y \in X$ and integers t_0, t_1, \dots, t_k such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_i \in \{x, y\}$, $i = 0, 1, \dots, k$, $x_0 \neq x_1$. If the integers t_0, t_1, \dots, t_k are the same for any subset X of G , we say that G is in the class $\overline{\Omega}_k(n)$. The class $\mathcal{U}_k(n)$ is defined exactly as $\Omega_k(n)$ with the additional conditions $x_i^{t_i} \neq 1$. Let t_2, t_3, \dots, t_k be fixed integers. We denote by $\overline{\mathcal{W}}_k^*$ the class of all groups G such that for any infinite subsets X and Y there exist $x \in X$, $y \in Y$ such that $[x_0, x_1, x_2^{t_2}, \dots, x_k^{t_k}] = 1$, where $x_i \in \{x, y\}$, $x_0 \neq x_1$, $i = 2, 3, \dots, k$. Here we prove that

- (1) If $G \in \mathcal{U}_k(2)$ is a finitely generated soluble group, then G is nilpotent.
- (2) If $G \in \Omega_k(\infty)$ is a finitely generated soluble group, then G is nilpotent-by-finite.
- (3) If $G \in \overline{\Omega}_k(n)$, n a positive integer, is a finitely generated residually finite group, then G is nilpotent-by-finite.
- (4) If G is an infinite $\overline{\mathcal{W}}_k^*$ -group in which every nontrivial finitely generated subgroup has a nontrivial finite quotient, then G is nilpotent-by-finite.

1. INTRODUCTION AND RESULTS

In response to a question of Paul Erdős, B.H. Neumann proved in [19] that a group is centre-by-finite if and only if every infinite subset contains a commuting pair of distinct elements. The extension of the questions of Paul Erdős, firstly, is considered by Lennox and Wiegold [15]. Further questions of a similar nature, with different aspects, have been considered by many people (see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 23, 24, 20, 25, 26, 27]).

Our notation and terminology are standard, and can be found in [22]. For a group G , and elements $x, y, x_1, \dots, x_k \in G$ we write

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2 = x_1^{-1}x_1^{x_2}, \quad [x_1, \dots, x_k] = [[x_1, \dots, x_{k-1}], x_k]$$

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$$[x, {}_0y] = x, \quad [x, {}_ky] = [[x, {}_{k-1}y], y].$$

A group is said to be k -Engel (Engel) group if for all $x, y \in G$, $[x, {}_ky] = 1$ (respectively, there exist a positive integer t depending on x and y such that $[x, {}_ty] = 1$). The class of k -Engel (Engel) groups will be denoted by \mathcal{E}_k (respectively, \mathcal{E}). For elements x, y, z of a group G , the following commutative identities will be used frequently without special reference.

$$[xy, z] = [x, z]^y[y, z], \quad [x, yz] = [x, z][x, y]^z$$

Let k be a positive integer, n a positive integer or infinity (denoted ∞). We denote by (\mathcal{N}, n) (respectively, (\mathcal{N}_k, n)) the class of all groups G such that, for every subset X of cardinality $n + 1$, there exist distinct elements $x, y \in X$ such that $\langle x, y \rangle$ is nilpotent (respectively, nilpotent of class at most k). We also denote by $\mathcal{E}_k(n)$ (respectively, $\mathcal{E}(n)$) the class of all groups G such that, for every subset X of cardinality $n + 1$, there exist distinct elements $x, y \in X$ such that $[x, {}_ky] = 1$ (respectively, $[x, {}_ty] = 1$ for some positive integer t depending on x, y). Lennox and Wiegold [15] proved that a finitely generated soluble group $G \in (\mathcal{N}, \infty)$ if and only if G is finite-by-nilpotent. Abdollahi and Taeri [3] proved that a finitely generated soluble group $G \in (\mathcal{N}_k, \infty)$ if and only if G is a finite extension by a group in which any two generator subgroup is nilpotent of class at most k .

Let n be a positive integer or $n = \infty$. We denote by $\Omega(n)$ the class of groups G in which for every subset X of G of cardinality $n + 1$, there exist distinct elements $x, y \in X$, such that the following condition holds.

There exist a positive integer k and elements $x_0, x_1, \dots, x_k \in \{x, y\}$ with $x_0 \neq x_1$, and integers t_0, t_1, \dots, t_k , such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$.

If the integer k is the same for any subset X of G , we say that G is in the class $\Omega_k(n)$. If $G \in \Omega_k(n)$ and the integers t_0, t_1, \dots, t_k are the same for any subset X of G , we say that G is in the class $\overline{\Omega}_k(n)$. Since all torsion groups belong to $\Omega_k(n)$, we define another class of groups: the class $\mathcal{U}_k(n)$ (respectively, $\overline{\mathcal{U}}_k(n)$) is defined exactly as $\Omega_k(n)$, (respectively, $\overline{\Omega}_k(n)$) with additional conditions $x_i^{t_i} \neq 1, i = 0, 1, \dots, k$.

Also we denote by \mathcal{W}_k^* the class of all groups G such that for any infinite subsets X and Y there exist $x \in X, y \in Y$ such that $[x_0, x_1, x_2^{t_2}, \dots, x_k^{t_k}] = 1$, where t_i is an integer, $x_i \in \{x, y\}, x_0 \neq x_1, i = 2, 3, \dots, k$. If the integers t_2, t_3, \dots, t_k are the same, we obtain the class $\overline{\mathcal{W}}_k^*$. Note that

$$(\mathcal{N}_k, n) \subseteq \mathcal{E}_k(n) \subseteq \mathcal{U}_k(n) \subseteq \Omega_k(n) \subseteq \Omega_k(n + 1),$$

and

$$\overline{\mathcal{U}}_k(n) \subseteq \overline{\Omega}_k(n) \subseteq \Omega_k(\infty) \quad \text{and} \quad \overline{\mathcal{W}}_k^* \subseteq \mathcal{W}_k^* \subseteq \Omega_k(\infty).$$

Endimioni [9] proved that if $G \in (\mathcal{N}, n)$, $n \leq 3$, is a finite group, then G is nilpotent. Abdollahi [2] considered $\mathcal{E}(2)$ -groups and proved that if $G \in \mathcal{E}(2)$ is a finite group, then G is nilpotent. We generalise this result.

THEOREM 1. *Let G be a finite group with the condition $\mathcal{U}_k(2)$, then G is nilpotent.*

Note that $\mathcal{E} \subseteq \mathcal{U}(1) \subseteq \mathcal{U}(2)$. Thus Theorem 1 is a generalisation of a well known result due to Zorn (see for example [22, Theorem 12.3.4]) which states that a finite Engel group is nilpotent. Trabelsi [27] proved that a finitely generated soluble group G is nilpotent-by-finite if and only if for every pair X, Y of infinite subsets of G there exist x in X , y in Y and two positive integers $m = m(x, y)$, $n = n(x, y)$ satisfying $[x, {}_n y^m] = 1$. We generalise this result and prove that

THEOREM 2. *Let $G \in \Omega_k(\infty)$ be a finitely generated soluble group. Then G is nilpotent-by-finite.*

Longobardi and Maj [16] (see also [8]) proved that a finitely generated soluble group $G \in \mathcal{E}(\infty)$ if and only if G is finite-by-nilpotent. Theorem 2 and [2, Lemma 7] gives another proof for this result. Abdollahi [2] has proved that a finitely generated residually finite $\mathcal{E}_k(n)$ -group, n a positive integer, is finite-by-nilpotent. We consider the weaker condition $\overline{\Omega}_k(n)$ and obtain the following result.

THEOREM 3. *Let $G \in \overline{\Omega}_k(n)$ be a finitely generated residually finite group. Then G is nilpotent-by-finite.*

Recall that a group G is said to be locally graded whenever every finitely generated subgroup has a nontrivial finite quotient. We say that a group G is an \mathcal{E}_k^* -group provided that whenever X, Y are infinite subsets of G , there exist x in X and y in Y , such that $[x, {}_k y] = 1$. Note that $\mathcal{E}_k^* \subseteq \overline{\mathcal{W}}_k^*$. Puglisi and Spiezia [20] proved that every infinite locally finite or locally soluble \mathcal{E}_k^* -group is a k -Engel group. Abdollahi [1] improved this result for locally graded groups. We consider $\overline{\mathcal{W}}_k^*$.

THEOREM 4. *Let $G \in \overline{\mathcal{W}}_k^*$ be a locally graded group. Then G is nilpotent-by-finite.*

2. PROOFS

We begin by an easy lemma without proof.

LEMMA 1. *Let G be a group with A as an Abelian normal subgroup, and let g be any element of G , then for all distinct elements a and b of A we have*

$$[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = [g^{t_0}, ab^{-1}, g^{t_1}, g^{t_2}, \dots, g^{t_k}]$$

where $x_i \in \{g^a, g^b\}$, $x_0 = g^a$, $x_1 = g^b$.

LEMMA 2. *Let G be an infinite group in $\Omega_k(\infty)$, and A be a normal Abelian subgroup of G . If there exists a torsion free element g of G such that the centraliser of g^m in A , $C_A(g^m) = 1$, for all integer m , then A is finite.*

PROOF: Suppose that A is infinite. Then the set $g^A = \{g^a \mid a \in A\}$ is infinite, as $C_A(g) = 1$. Now, since $G \in \Omega_k(\infty)$, there exist distinct elements $a, b \in A$ and integers t_i such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_0 = g^a, x_1 = g^b$, and $x_i \in \{g^a, g^b\}$, $i = 0, 1, 2, \dots, k$. Thus, by Lemma 1, $[g^{t_0}, ab^{-1}, g^{t_1}, g^{t_2}, \dots, g^{t_k}] = 1$. Now, since A is normal Abelian, $u = [g^{t_0}, ab^{-1}, g^{t_1}, g^{t_2}, \dots, g^{t_k}] \in C_A(g^{t_k}) = 1$. So $u = 1$. Continuing in this way we find that $[g^{t_0}, ab^{-1}] \in C_A(g^{t_1}) = 1$. So $ab^{-1} \in C_A(g^{t_0}) = 1$, a contradiction. \square

The following lemma is proved similarly.

LEMMA 3. *Let G be a group in $\mathcal{U}_k(n)$, A be a normal Abelian subgroup of G . If there exists $g \in G$, such that $C_A(g^m) = 1$, for all integers m with $g^m \neq 1$, then $|A| \leq n$.*

Now we are ready to prove Theorem 1.

PROOF OF THEOREM 1: Suppose the assertion of the Theorem is false and choose a counter-example G of smallest order. Now G is a finite minimal nonnilpotent group. By a result of Schmidt (see [22, Theorem 9.1.9]) G is a $\{p, q\}$ -group, where p, q are distinct primes and G has a normal Sylow p -subgroup P and a cyclic Sylow q -subgroup Q . Let $x \in Q$ be an element of order q . Then, since the centre of G $Z(G) = 1$, we have $C_{Z(P)}(x) = 1$, and therefore, by Lemma 3, $|Z(P)| \leq 2$. Thus $Z(P) \leq Z(G) = 1$, a contradiction. Therefore G is nilpotent. \square

Note that $S_3 \in \mathcal{U}_2(3)$, so that the bound 2 in Theorem 1 cannot be improved.

COROLLARY 1. *Let $G \in \mathcal{U}_k(2)$ be a finitely generated soluble group. Then G is nilpotent.*

PROOF: If G is not nilpotent, then by a result of Robinson and Wehrfritz (see [22, Theorem 15.5.3]) G has a nontrivial finite nonnilpotent image. This contradicts Theorem 1. \square

A result of Kropholler states that if a finitely generated soluble group G has no section isomorphic to the restricted wreath product of a cyclic group of prime order with the infinite cyclic group, then G has finite rank. Therefore to prove Theorem 2, we first prove the following.

LEMMA 4. *Let $G = A \text{ Wr } B$ be the restricted wreath product of a cyclic group A of prime order p with the infinite cyclic group $B = \langle b \rangle$. Then $G \notin \Omega_k(\infty)$.*

PROOF: Let $R = A^B$, be the base group of $G = A \text{ Wr } B$. We shall write $r \cdot b$ for the conjugate of r under b and $r \cdot (b^\lambda + b^\mu)$ for $r \cdot b^\lambda + r \cdot b^\mu$, for all $r \in R$ and integers λ, μ . Every element of R can be expressed in the form

$$f(b) = \sum_{i=0}^{m-1} r_i \cdot b^i$$

with $r_i \in R_0$, where R_0 is the first isomorphic copy of A in R . Note that R is a free $\mathbb{F}_p\langle b \rangle$ -module with basis $\{r\}$, where $R_0 = \langle r \rangle$. Thus the above $f(b)$ can be written as

$$f(b) = r \cdot \sum_{i=0}^{m-1} s_i b^i$$

where $s_i \in \mathbb{F}_p$. Elements of G will be expressed in the form $(f(b), b^\lambda)$ with $f(b) \in R$ and $\lambda \in \mathbb{Z}$, with multiplication given by $(f(b), b^\lambda)(g(b), b^\mu) = (f(b) + g(b) \cdot b^{-\lambda}, b^{\lambda+\mu})$. Note that the conjugate $(b^{f(b)})^\lambda$ of b^λ under an element $f(b)$ of R is expressed in the form $(-f(b) + f(b) \cdot b^{-\lambda}, b^\lambda)$. Also we have the commutator identity $[f(b), b^\lambda] = f(b) \cdot (-1 + b^\lambda)$.

Now suppose, for a contradiction, that $G \in \Omega_k(\infty)$, and consider the elements $f_i(b) = r \cdot b^i, i = 1, 2, 3, \dots$. Since $G \in \Omega_k(\infty)$ there exists $i \neq j$ such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_0 = b^{f_i(b)}, x_1 = b^{f_j(b)}, x_s \in \{b^{f_i(b)}, b^{f_j(b)}\}$. Since R is a normal Abelian subgroup of G we have, by Lemma 1, that

$$[b^{t_0}, f_i f_j^{-1}, b^{t_1}, b^{t_2}, \dots, b^{t_k}] = 1,$$

or in additive notation,

$$\begin{aligned} 0 &= (f_i - f_j) \cdot (1 - b^{t_0})(-1 + b^{t_1})(-1 + b^{t_2}) \dots (-1 + b^{t_k}) \\ &= r \cdot (b^i - b^j)(1 - b^{t_0})(-1 + b^{t_1})(-1 + b^{t_2}) \dots (-1 + b^{t_k}). \end{aligned}$$

Since R is a free $\mathbb{F}_p\langle b \rangle$ -module with basis $\{r\}$, in the group ring $\mathbb{F}_p\langle b \rangle$, we have

$$(b^i - b^j)(1 - b^{t_0})(1 - b^{t_1})(1 - b^{t_2}) \dots (1 - b^{t_k}) = 0,$$

a contradiction, as the order of b is infinite. □

The proof of Theorem 2 is similar to that of [11, Theorem 2]. We include it for completeness.

PROOF OF THEOREM 2: Suppose G is not nilpotent-by-finite. Since $\Omega_k(\infty)$ is a quotient closed class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, it follows, by [21, Lemma 6.17], that we may assume that every proper quotient of G is nilpotent-by-finite. Let A be a nontrivial normal Abelian subgroup of G . Then G/A is nilpotent-by-finite. Since, by Lemma 4, G has no section isomorphic to the wreath product of a cyclic group of order prime p with the infinite cyclic group, a result of [14] shows that G has finite rank. This means that, for some positive integer t , every proper subgroup of G can be generated by at most t elements. Let T be the torsion subgroup of A . Then T is finite and so $C = C_G(T)$, the centraliser of T in G , is of finite index in G . If $T \neq 1$ then G/T is nilpotent-by-finite and thus C/T is nilpotent-by-finite. Since $T \leq Z(C)$, then $C_G(T)$ and hence G would be nilpotent-by-finite. Thus $T = 1$, and A is torsion free Abelian, and by passing to a suitable subgroup of finite index in G , if necessary, we may assume further that G/A is a finitely generated torsion

free nilpotent group. Thus there exists a finite set $T = \{t_1, \dots, t_r\}$ of elements of G such that $G = \langle A, T \rangle$, and

$$A = G_0 \leq \langle G_0, t_1 \rangle = G_1 \leq \dots \leq \langle G_{r-1}, t_r \rangle = G_r = G$$

is a central series from A to G with torsion free factors. Suppose $r = 1$ then $G = \langle A, t_1 \rangle$. If $C_A(t_1^m) = 1$, for all m , then by Lemma 2, A is finite, a contradiction. Thus there exists a positive integer m_1 , such that $C_A(t_1^{m_1}) \neq 1$. Therefore $Z(\langle A, t_1^{m_1} \rangle) \neq 1$ and hence $D = A \cap Z(\langle A, t_1^{m_1} \rangle)$ is a nontrivial normal subgroup of G . So G/D is nilpotent-by-finite and hence G is nilpotent-by-finite. Now assume that we have established the result when $r < s$, and suppose $r = s$. Then G_{s-1} is nilpotent-by-finite and $G = \langle G_{s-1}, t \rangle$. Let $H = \langle A, G_{s-1}^m \rangle$ for some suitable $m > 0$, so that H is nilpotent. Let $Y = A \cap Z(H)$, then Y is normal in $\langle H, t_s \rangle$ which is of finite index in G . Moreover $Z(\langle Y, t_s^{m_s} \rangle) \neq 1$, for some m_s , by Lemma 2. So $D_1 = Y \cap Z(\langle Y, t_s^{m_s} \rangle)$ is a nontrivial subgroup of G contained in the centre of $\langle H, t_s^{m_s} \rangle$ which is of finite index in G . We may replace $\langle H, t_s^{m_s} \rangle$ by its normal interior in G , if necessary; H still contains A and hence D_1 . Now $\langle H, t_s^{m_s} \rangle / D_1$ is nilpotent-by-finite, $D_1 \leq Z(\langle H, t_s^{m_s} \rangle)$ and $\langle H, t_s^{m_s} \rangle$ is of finite index in G , thus G is nilpotent-by-finite, a contradiction. \square

COROLLARY 2. *Let G be a finitely generated soluble group. Then $G \in \Omega(\infty)$ if and only if G is nilpotent-by-finite.*

Now, we want to consider a finitely generated residually finite group in $\bar{\Omega}_k(n)$, n a positive integer. We use a result of Wilson [28] which states that if G is a finitely generated residually finite group and N is a positive integer such that G has no section isomorphic to the twisted wreath product $A \text{ twr}_C B$, where B is finite and cyclic, A is an elementary Abelian group acted on faithfully and irreducibly by C , and $|B : C| > N$, then G is virtually a soluble minimax group. For the definition of the twisted wreath product we refer readers to Neumann [18].

Suppose that t_0, t_1, \dots, t_k are fixed integers. Recall that a group G is in the class $\bar{\Omega}_k(n)$ if for every subset X of G of cardinality $n+1$, there exist distinct elements $x, y \in X$, such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_0, x_1, \dots, x_k \in \{x, y\}$ with $x_0 \neq x_1$. In the following lemma we may assume that t_0, t_1, \dots, t_k are positive.

LEMMA 5. *Let A be a nontrivial Abelian group, $B = \langle b \rangle$ a finite cyclic group C , a subgroup of B of index m , and suppose that C acts on A . Let $W = A \text{ twr}_C B$ be the twisted wreath product of A by B with respect to the action of C on A . If $G \in \bar{\Omega}_k(n)$, n a positive integer, then $m \leq n + t_0 + t_1 + \dots + t_k$.*

PROOF: Suppose that $C = \langle b^m \rangle$ and let Y be a transversal to C in B so that $Y = \{b, b^2, \dots, b^{m-1}, 1\}$. Then $W = A \text{ twr}_C B = B \rtimes A^Y$ is the splitting extension of A^Y by B . The action of B on A^Y is given by $f^b(y) = f(y)^{c^{-1}}$, where $y' \in Y$ and $c \in C$ are unique elements such that $yb = cy'$.

Now assume, for a contradiction, that $m > n + t_0 + t_1 + \dots + t_k$ and consider $n + 1$ elements f_0, f_1, \dots, f_n so that

$$f_i(b^j) = \begin{cases} 1 & j \neq i \\ a & j = i \end{cases}, \quad j = 1, 2, \dots, m - 1,$$

where a is a fixed nontrivial element of A . Since $G \in \overline{\Omega}_k(n)$ there exists $i \neq j$ such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_0 = b^{f_i}, x_1 = b^{f_j}, x_s \in \{b^{f_i}, b^{f_j}\}$. Then, as in Lemma 4, we have $(f_i f_j^{-1})^{(1-b^{t_0})(1-b^{t_1})\dots(1-b^{t_k})} = 1$. If $i > j$, then for all $s \in \{0, 1, \dots, m\}$ we have $i + s \leq n + t_0 + t_1 + \dots + t_k \leq m - 1$, and thus $b^{i+s} \in Y$. Therefore $f_j(b^{i+s}) = 1$, as $i + s > j$ and

$$(f_i f_j^{-1})^{b^s} (b^i) = f_i^{b^s} (b^i) (f_j^{b^s} (b^i))^{-1} = f_i(b^{i+s}) (f_j(b^{i+s}))^{-1} = \begin{cases} 1 & s \neq 0 \\ a & s = 0. \end{cases}$$

Hence $(f_i f_j^{-1})^{(1-b^{t_0})(1-b^{t_1})\dots(1-b^{t_k})} (b^i) = a$, a contradiction. In the same way, if $i < j$ we get a contradiction. □

Now we are in the position to prove Theorem 3.

PROOF OF THEOREM 3: By Lemma 5, G has no section isomorphic to $W = A \text{ twr}_C B$, where A is elementary Abelian, B is finite cyclic, and C is a subgroup of B which acts faithfully irreducibly on A , such that $|B : C| > n + t_0 + t_1 + \dots + t_k$. Thus by a result of Wilson [28], G is virtually a soluble minimax. Hence there exist a normal subgroup H of G with finite index, such that H is a soluble minimax group. By Theorem 2, H is nilpotent-by-finite. Hence there exists a normal nilpotent subgroup K of H such that H/K is finite. Therefore K is finitely generated nilpotent with finite index in G , and G is nilpotent-by-finite. □

Let t_0, t_1, \dots, t_k be fixed integers. We denote by $\overline{\Omega}_k^*$ the class of all groups G such that for any infinite subsets X and Y there exist $x \in X, y \in Y$ such that $[x_0^{t_0}, x_1^{t_1}, x_2^{t_2}, \dots, x_k^{t_k}] = 1$, where $x_i \in \{x, y\}, x_0 \neq x_1, i = 0, 1, \dots, k$. Note that $\overline{\mathcal{W}}_k^* \subseteq \overline{\mathcal{W}}_k \subseteq \overline{\Omega}_k^*$. The proof of the following lemma is easy and hence it is omitted.

LEMMA 6. *Let N be an infinite normal subgroup of a $\overline{\Omega}_k^*$ -group G , then $G/N \in \overline{\Omega}_k(1)$. In particular if G is an infinite residually finite $\overline{\Omega}_k^*$ -group, then $G \in \overline{\Omega}_k(1)$.*

A result of Wilson (see [28, Theorem 2]) states that a finitely generated residually finite k -Engel group is nilpotent. As a consequence of Theorems 1 and 3 we can generalise this fact.

COROLLARY 3. *Let G be an infinite finitely generated residually finite $\overline{\mathcal{U}}_k(2)$ -group, and k be a positive integer. Then G is nilpotent.*

PROOF: By Theorem 3, there exists a normal nilpotent subgroup H of finite index in G . Now $G/H \in \overline{\mathcal{U}}_k(2)$. Thus, by Theorem 1, G/H is nilpotent, and G is soluble. Now G is a finitely generated soluble $\overline{\mathcal{U}}_k(2)$ -group, and by Theorem 1, is nilpotent. □

Recall that a group $G \in \mathcal{W}_k^*$ if for any infinite subsets X and Y there exist $x \in X, y \in Y$ such that $[x_0, x_1, x_2^{t_2}, \dots, x_k^{t_k}] = 1$, where t_i is an integer, $x_i \in \{x, y\}, x_0 \neq x_1, i = 2, 3, \dots, k$. If the integers t_2, t_3, \dots, t_k are the same, we say that $G \in \overline{\mathcal{W}}_k^*$. Following [12] we say that a group G is restrained if $\langle x \rangle^{(y)} = \langle x^{y^i} \mid i \text{ an integer} \rangle$ is finitely generated for all x, y in G . If there is a bound for the number of generators of $\langle x \rangle^{(y)}$, then we call G strongly restrained.

LEMMA 7. *Let G be a group in \mathcal{W}_k^* . Then G is restrained.*

PROOF: Let $x, y \in G$. We want to show that $\langle x \rangle^{(y)}$ is finitely generated. The result is clear if the order of y is finite. So assume that y is of infinite order. Consider the sets $X = \{x^{y^n} \mid n \text{ an integer}\}$ and $Y = \{y^m \mid m \text{ a positive integer}\}$. If X is finite then $\langle x \rangle^{(y)} = \langle X \rangle$ is finitely generated, as required. So we may assume that X is infinite. Since $G \in \mathcal{W}_k^*$, there exist integers i, j such that $[x_0, x_1, x_2^{t_2}, \dots, x_k^{t_k}] = 1$ where $x_0 \neq x_1, x_s \in \{x^{y^i}, y^j\}$. Hence $[z_0, z_1, z_2^{t_2}, \dots, z_k^{t_k}] = 1$, where $z_0 = x$ and $z_1 = y^j$ or $z_0 = y^j$ and $z_1 = x$, and $z_2, \dots, z_k \in \{x, y^j\}$.

Suppose that $z_{i_1} = z_{i_2} = \dots = z_{i_s} = y^j$, and $z_t = x$, for all $t \neq i_r$. Let $T = \{t_{i_1}, t_{i_2}, \dots, t_{i_s}\}$, and denote by $T^{(r)}$ the set of all sums of r distinct elements of T , and denote by $S^{(r)}$ the set of all sums of r distinct elements of $T \cup \{1\}$. Then it is easy to see that

$$\langle z_0, [z_0, z_1], [z_0, z_1, z_2^{t_2}], \dots, [z_0, z_1, z_2^{t_2}, \dots, z_k^{t_k}] \rangle = \langle x, x^{y^j}, x^{y^{j^r}}; r \in \bigcup_{i=1}^k T^{(i)} \cup \bigcup_{i=1}^k S^{(i)} \rangle.$$

Therefore $\langle x \rangle^{(y)} \leq \langle x^{y^{j^r}}; |r| \leq k(t_{i_1} + \dots + t_{i_s} + 1) \rangle$. This completes the proof. □

COROLLARY 4. *Let G be an infinite finitely generated soluble \mathcal{W}_k^* -group. Then G is polycyclic.*

PROOF: This follows immediately from Lemma 7 and [12, Corollary 4]. □

PROOF OF THEOREM 4: As in Lemma 7, G must be strongly restrained. Thus by [12, Theorem A] G is polycyclic-by-finite, and therefore residually finite. Now, by Lemma 6, $G \in \overline{\mathcal{O}}_k(1)$, and thus, by Theorem 3, G is nilpotent-by-finite. □

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