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A QUESTION OF PAUL ERDŐS AND NILPOTENT-BY-FINITE GROUPS

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Let n be a positive integer or infinity (denoted ∞), k a positive integer. We denote by $\Omega_k(n)$ the class of groups G such that, for every subset X of G of cardinality $n+1$, there exist distinct elements $x, y \in X$ and integers $t_0, t_1, \dots, t_k$ such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_i \in \{x, y\}$, $i = 0, 1, \dots, k$, $x_0 \neq x_1$. If the integers $t_0, t_1, \dots, t_k$ are the same for any subset X of G, we say that G is in the class $\overline{\Omega}_k(n)$. The class $\mathcal{U}_k(n)$ is defined exactly as $\Omega_k(n)$ with the additional conditions $x_i^{t_i} \neq 1$. Let $t_2, t_3, \dots, t_k$ be fixed integers. We denote by $\overline{\mathcal{W}}_k^*$ the class of all groups G such that for any infinite subsets X and Y there exist $x \in X$, $y \in Y$ such that $[x_0, x_1, x_2^{t_2} \dots, x_k^{t_k}] = 1$, where $x_i \in \{x, y\}, x_0 \neq x_1$, $i = 2, 3, \dots, k$. Here we prove that

(1) If $G \in \mathcal{U}_k(2)$ is a finitely generated soluble group, then G is nilpotent.

(2) If $G \in \Omega_k(\infty)$ is a finitely generated soluble group, then G is nilpotent-by-finite.

(3) If $G \in \overline{\Omega}_k(n)$, n a positive integer, is a finitely generated residually finite group, then G is nilpotent-by-finite.

(4) If G is an infinite $\overline{\mathcal{W}}_k^*$-group in which every nontrivial finitely generated subgroup has a nontrivial finite quotient, then G is nilpotent-by-finite.

1. Introduction and results

In response to a question of Paul Erdős, B.H. Neumann proved in [**19**] that a group is centre-by-finite if and only if every infinite subset contains a commuting pair of distinct elements. The extension of the questions of Paul Erdős, firstly, is considered by Lennox and Wiegold [**15**]. Further questions of a similar nature, with different aspects, have been considered by many people (see for example [**1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 23, 24, 20, 25, 26, 27**]).

Our notation and terminology are standard, and can be found in [**22**]. For a group G, and elements $x, y, x_1, \dots, x_k \in G$ we write

$$[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2 = x_1^{-1} x_1^{x_2}, \quad [x_1, \dots, x_k] = [[x_1, \dots, x_{,k-1}], x_k]$$

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$$[x,\, _0y] = x, \quad [x,\, _ky] = \big[[x,\, _{k-1}y], y\big].$$

A group is said to be k-Engel (Engel) group if for all $x, y \in G$, $[x,\, _ky] = 1$ (respectively, there exist a positive integer t depending on x and y such that $[x,\, _ty] = 1$). The class of k-Engel (Engel) groups will be denoted by $\mathcal{E}_k$ (respectively, $\mathcal{E}$). For elements x, y, z of a group G, the following commutative identities will be used frequently without special reference.

$$[xy, z] = [x, z]^y [y, z], \quad [x, yz] = [x, z][x, y]^z$$

Let k be a positive integer, n a positive integer or infinity (denoted ∞). We denote by $(\mathcal{N}, n)$ (respectively, $(\mathcal{N}_k, n)$) the class of all groups G such that, for every subset X of cardinality $n+1$, there exist distinct elements $x, y \in X$ such that $\langle x, y\rangle$ is nilpotent (respectively, nilpotent of class at most k). We also denote by $\mathcal{E}_k(n)$ (respectively, $\mathcal{E}(n)$) the class of all groups G such that, for every subset X of cardinality $n+1$, there exist distinct elements $x, y \in X$ such that $[x,\, _ky] = 1$ (respectively, $[x,\, _ty] = 1$ for some positive integer t depending on x, y). Lennox and Wiegold **[15]** proved that a finitely generated soluble group $G \in (\mathcal{N}, \infty)$ if and only if G is finite-by-nilpotent. Abdollahi and Taeri **[3]** proved that a finitely generated soluble group $G \in (\mathcal{N}_k, \infty)$ if and only if G is a finite extension by a group in which any two generator subgroup is nilpotent of class at most k.

Let n be a positive integer or $n = \infty$. We denote by $\Omega(n)$ the class of groups G in which for every subset X of G of cardinality $n+1$, there exist distinct elements $x, y \in X$, such that the following condition holds.

There exist a positive integer k and elements $x_0, x_1, \dots, x_k \in \{x, y\}$ with $x_0 \neq x_1$, and integers $t_0, t_1, \dots, t_k$, such that $\big[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}\big] = 1$.

If the integer k is the same for any subset X of G, we say that G is in the class $\Omega_k(n)$. If $G \in \Omega_k(n)$ and the integers $t_0, t_1, \dots, t_k$ are the same for any subset X of G, we say that G is in the class $\overline{\Omega}_k(n)$. Since all torsion groups belong to $\Omega_k(n)$, we define another class of groups: the class $\mathcal{U}_k(n)$ (respectively, $\overline{\mathcal{U}}_k(n)$) is defined exactly as $\Omega_k(n)$, (respectively, $\overline{\Omega}_k(n)$) with additional conditions $x_i^{t_i} \neq 1$, $i = 0, 1, \dots, k$.

Also we denote by $\mathcal{W}_k^*$ the class of all groups G such that for any infinite subsets X and Y there exist $x \in X$, $y \in Y$ such that $\big[x_0, x_1, x_2^{t_2}, \dots, x_k^{t_k}\big] = 1$, where t_i is an integer, $x_i \in \{x, y\}, x_0 \neq x_1$, $i = 2, 3, \dots, k$. If the integers $t_2, t_3, \dots, t_k$ are the same, we obtain the class $\overline{\mathcal{W}}_k^*$. Note that

$$(\mathcal{N}_k, n) \subseteq \mathcal{E}_k(n) \subseteq \mathcal{U}_k(n) \subseteq \Omega_k(n) \subseteq \Omega_k(n+1),$$

and

$$\overline{\mathcal{U}}_k(n) \subseteq \overline{\Omega}_k(n) \subseteq \Omega_k(\infty) \quad \text{and} \quad \overline{\mathcal{W}}_k^* \subseteq \mathcal{W}_k^* \subseteq \Omega_k(\infty).$$

Endimioni [9] proved that if $G \in (\mathcal{N}, n)$, $n \leqslant 3$, is a finite group, then G is nilpotent. Abdollahi [2] considered $\mathcal{E}(2)$-groups and proved that if $G \in \mathcal{E}(2)$ is a finite group, then G is nilpotent. We generalise this result.

THEOREM 1. *Let G be a finite group with the condition $\mathcal{U}_k(2)$, then G is nilpotent.*

Note that $\mathcal{E} \subseteq \mathcal{U}(1) \subseteq \mathcal{U}(2)$. Thus Theorem 1 is a generalisation of a well known result due to Zorn (see for example [22, Theorem 12.3.4]) which states that a finite Engel group is nilpotent. Trabelsi [27] proved that a finitely generated soluble group G is nilpotent-by-finite if and only if for every pair X, Y of infinite subsets of G there exist x in X, y in Y and two positive integers $m = m(x,y), n = n(x,y)$ satisfying $[x, {}_n y^m] = 1$. We generalise this result and prove that

THEOREM 2. *Let $G \in \Omega_k(\infty)$ be a finitely generated soluble group. Then G is nilpotent-by-finite.*

Longobardi and Maj [16] (see also [8]) proved that a finitely generated soluble group $G \in \mathcal{E}(\infty)$ if and only if G is finite-by-nilpotent. Theorem 2 and [2, Lemma 7] gives another proof for this result. Abdollahi [2] has proved that a finitely generated residually finite $\mathcal{E}_k(n)$-group, n a positive integer, is finite-by-nilpotent. We consider the weaker condition $\overline{\Omega}_k(n)$ and obtain the following result.

THEOREM 3. *Let $G \in \overline{\Omega}_k(n)$ be a finitely generated residually finite group. Then G in nilpotent-by-finite.*

Recall that a group G is said to be locally graded whenever every finitely generated subgroup has a nontrivial finite quotient. We say that a group G is an $\mathcal{E}_k^*$-group provided that whenever X, Y are infinite subsets of G, there exist x in X and y in Y, such that $[x, {}_k y] = 1$. Note that $\mathcal{E}_k^* \subseteq \overline{\mathcal{W}}_k^*$. Puglisi and Spiezia [20] proved that every infinite locally finite or locally soluble $\mathcal{E}_k^*$-group is a k-Engel group. Abdollahi [1] improved this result for locally graded groups. We consider $\overline{\mathcal{W}}_k^*$.

THEOREM 4. *Let $G \in \overline{\mathcal{W}}_k^*$ be a locally graded group. Then G is nilpotent-by-finite.*

2. PROOFS

We begin by an easy lemma without proof.

LEMMA 1. *Let G be a group with A as an Abelian normal subgroup, and let g be any element of G, then for all distinct elements a and b of A we have*

$$\left[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}\right] = \left[g^{t_0}, ab^{-1}, g^{t_1}, g^{t_2}, \dots, g^{t_k}\right]$$

where $x_i \in \{g^a, g^b\}$, $x_0 = g^a, x_1 = g^b$.

Lemma 2. *Let G be an infinite group in $\Omega_k(\infty)$, and A be a normal Abelian subgroup of G. If there exists a torsion free element g of G such that the centraliser of g^m in A, $C_A(g^m) = 1$, for all integer m, then A is finite.*

Proof: Suppose that A is infinite. Then the set $g^A = \{g^a \mid a \in A\}$ is infinite, as $C_A(g) = 1$. Now, since $G \in \Omega_k(\infty)$, there exist distinct elements $a, b \in A$ and integers t_i such that $[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}] = 1$, where $x_0 = g^a, x_1 = g^b$, and $x_i \in \{g^a, g^b\}$, $i = 0, 1, 2, \ldots k$. Thus, by Lemma 1, $[g^{t_0}, ab^{-1}, g^{t_1}, g^{t_2}, \ldots, g^{t_k}] = 1$. Now, since A is normal Abelian, $u = [g^{t_0}, ab^{-1}, g^{t_1}, g^{t_2}, \ldots, g^{t_{k-1}}] \in C_A(g^{t_k}) = 1$. So $u = 1$. Continuing in this way we find that $[g^{t_0}, ab^{-1}] \in C_A(g^{t_1}) = 1$. So $ab^{-1} \in C_A(g^{t_0}) = 1$, a contradiction. ∎

The following lemma is proved similarly.

Lemma 3. *Let G be a group in $\mathcal{U}_k(n)$, A be a normal Abelian subgroup of G. If there exists $g \in G$, such that $C_A(g^m) = 1$, for all integers m with $g^m \neq 1$, then $|A| \leqslant n$.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1: Suppose the assertion of the Theorem is false and choose a counter-example G of smallest order. Now G is a finite minimal nonnilpotent group. By a result of Schmidt (see [**22**, Theorem 9.1.9]) G is a $\{p, q\}$-group, where p, q are distinct primes and G has a normal Sylow p-subgroup P and a cyclic Sylow q-subgroup Q. Let $x \in Q$ be an element of order q. Then, since the centre of G $Z(G) = 1$, we have $C_{Z(P)}(x) = 1$, and therefore, by Lemma 3, $|Z(P)| \leqslant 2$. Thus $Z(P) \leqslant Z(G) = 1$, a contradiction. Therefore G is nilpotent. ∎

Note that $S_3 \in \mathcal{U}_2(3)$, so that the bound 2 in Theorem 1 cannot be improved.

Corollary 1. *Let $G \in \mathcal{U}_k(2)$ be a finitely generated soluble group. Then G is nilpotent.*

Proof: If G is not nilpotent, then by a result of Robinson and Wehrfritz (see [**22**, Theorem 15.5.3]) G has a nontrivial finite nonnilpotent image. This contradicts Theorem 1. ∎

A result of Kropholler states that if a finitely generated soluble group G has no section isomorphic to the restricted wreath product of a cyclic group of prime order with the infinite cyclic group, then G has finite rank. Therefore to prove Theorem 2, we first prove the following.

Lemma 4. *Let $G = A \operatorname{Wr} B$ be the restricted wreath product of a cyclic group A of prime order p with the infinite cyclic group $B = \langle b \rangle$. Then $G \notin \Omega_k(\infty)$.*

Proof: Let $R = A^B$, be the base group of $G = A \operatorname{Wr} B$. We shall write $r \cdot b$ for the conjugate of r under b and $r \cdot (b^\lambda + b^\mu)$ for $r \cdot b^\lambda + r \cdot b^\mu$, for all $r \in R$ and integers λ, μ. Every element of R can be expressed in the form

$$f(b) = \sum_{i=0}^{m-1} r_i \cdot b^i$$

with $r_i \in R_0$, where R_0 is the first isomorphic copy of A in R. Note that R is a free $\mathbb{F}_p\langle b\rangle$-module with basis $\{r\}$, where $R_0 = \langle r\rangle$. Thus the above $f(b)$ can be written as

$$f(b) = r \cdot \sum_{i=0}^{m-1} s_i b^i$$

where $s_i \in \mathbb{F}_p$. Elements of G will be expressed in the form $\left(f(b), b^\lambda\right)$ with $f(b) \in R$ and $\lambda \in \mathbb{Z}$, with multiplication given by $\left(f(b), b^\lambda\right)\left(g(b), b^\mu\right) = \left(f(b) + g(b) \cdot b^{-\lambda}, b^{\lambda+\mu}\right)$. Note that the conjugate $\left(b^{f(b)}\right)^\lambda$ of b^λ under an element $f(b)$ of R is expressed in the form $\left(-f(b)+f(b)\cdot b^{-\lambda}, b^\lambda\right)$. Also we have the commutator identity $\left[f(b), b^\lambda\right] = f(b)\cdot(-1+b^\lambda)$.

Now suppose, for a contradiction, that $G \in \Omega_k(\infty)$, and consider the elements $f_i(b) = r \cdot b^i, i = 1, 2, 3, \ldots$. Since $G \in \Omega_k(\infty)$ there exists $i \neq j$ such that $\left[x_0^{t_0}, x_1^{t_1}, \ldots, x_k^{t_k}\right] = 1$, where $x_0 = b^{f_i(b)}, x_1 = b^{f_j(b)}, x_s \in \left\{b^{f_i(b)}, b^{f_j(b)}\right\}$. Since R is a normal Abelian subgroup of G we have, by Lemma 1, that

$$\left[b^{t_0}, f_i f_j^{-1}, b^{t_1}, b^{t_2}, \ldots, b^{t_k}\right] = 1,$$

or in additive notation,

$$\begin{aligned} 0 &= (f_i - f_j) \cdot (1 - b^{t_0})(-1 + b^{t_1})(-1 + b^{t_2}) \cdots (-1 + b^{t_k}) \\ &= r \cdot (b^i - b^j)(1 - b^{t_0})(-1 + b^{t_1})(-1 + b^{t_2}) \cdots (-1 + b^{t_k}). \end{aligned}$$

Since R is a free $\mathbb{F}_p\langle b\rangle$-module with basis $\{r\}$, in the group ring $\mathbb{F}_p\langle b\rangle$, we have

$$(b^i - b^j)(1 - b^{t_0})(1 - b^{t_1})(1 - b^{t_2}) \cdots (1 - b^{t_k}) = 0,$$

a contradiction, as the order of b is infinite. $\square$

The proof of Theorem 2 is similar to that of [**11**, Theorem 2]. We include it for completeness.

PROOF OF THEOREM 2: Suppose G is not nilpotent-by-finite. Since $\Omega_k(\infty)$ is a quotient closed class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, it follows, by [**21**, Lemma 6.17], that we may assume that every proper quotient of G is nilpotent-by-finite. Let A be a nontrivial normal Abelian subgroup of G. Then G/A is nilpotent-by-finite. Since, by Lemma 4, G has no section isomorphic to the wreath product of a cyclic group of order prime p with the infinite cyclic group, a result of [**14**] shows that G has finite rank. This means that, for some positive integer t, every proper subgroup of G can be generated by at most t elements. Let T be the torsion subgroup of A. Then T is finite and so $C = C_G(T)$, the centraliser of T in G, is of finite index in G. If $T \neq 1$ then G/T is nilpotent-by-finite and thus C/T is nilpotent-by-finite. Since $T \leqslant Z(C)$, then $C_G(T)$ and hence G would be nilpotent-by-finite. Thus $T = 1$, and A is torsion free Abelian, and by passing to a suitable subgroup of finite index in G, if necessary, we may assume further that G/A is a finitely generated torsion free nilpotent group. Thus there exists a finite set $T = \{t_1, \dots, t_r\}$ of elements of G such that $G = \langle A, T\rangle$, and

$$A = G_0 \leqslant \langle G_0, t_1\rangle = G_1 \leqslant \cdots \leqslant \langle G_{r-1}, t_r\rangle = G_r = G$$

is a central series from A to G with torsion free factors. Suppose $r = 1$ then $G = \langle A, t_1\rangle$. If $C_A(t_1^m) = 1$, for all m, then by Lemma 2, A is finite, a contradiction. Thus there exists a positive integer m_1, such that $C_A(t_1^{m_1}) \neq 1$. Therefore $Z\big(\langle A, t_1^{m_1}\rangle\big) \neq 1$ and hence $D = A \cap Z\big(\langle A, t_1^m\rangle\big)$ is a nontrivial normal subgroup of G. So G/D is nilpotent-by-finite and hence G is nilpotent-by-finite. Now assume that we have established the result when $r < s$, and suppose $r = s$. Then G_{s-1} is nilpotent-by-finite and $G = \langle G_{s-1}, t\rangle$. Let $H = \langle A, G_{s-1}^m\rangle$ for some suitable $m > 0$, so that H is nilpotent. Let $Y = A \cap Z(H)$, then Y is normal in $\langle H, t_s\rangle$ which is of finite index in G. Moreover $Z(\langle Y, t_s^{m_s}\rangle) \neq 1$, for some m_s, by Lemma 2. So $D_1 = Y \cap Z\big(\langle Y, t_s^{m_s}\rangle\big)$ is a nontrivial subgroup of G contained in the centre of $\langle H, t_s^{m_s}\rangle$ which is of finite index in G. We may replace $\langle H, t_s^{m_s}\rangle$ by its normal interior in G, if necessary; H still contains A and hence D_1. Now $\langle H, t_s^{m_s}\rangle/D_1$ is nilpotent-by-finite, $D_1 \leqslant Z\big(\langle H, t_s^{m_s}\rangle\big)$ and $\langle H, t_s^{m_s}\rangle$ is of finite index in G, thus G is nilpotent-by-finite, a contradiction. $\square$

Corollary 2. *Let G be a finitely generated soluble group. Then $G \in \Omega(\infty)$ if and only if G is nilpotent-by-finite.*

Now, we want to consider a finitely generated residually finite group in $\overline{\Omega}_k(n)$, n a positive integer. We use a result of Wilson [**28**] which states that if G is a finitely generated residually finite group and N is a positive integer such that G has no section isomorphic to the twisted wreath product $A \operatorname{twr}_C B$, where B is finite and cyclic, A is an elementary Abelian group acted on faithfully and irreducibly by C, and $|B : C| > N$, then G is virtually a soluble minimax group. For the definition of the twisted wreath product we refer readers to Neumann [**18**].

Suppose that $t_0, t_1, \dots, t_k$ are fixed integers. Recall that a group G is in the class $\overline{\Omega}_k(n)$ if for every subset X of G of cardinality $n+1$, there exist distinct elements $x, y \in X$, such that $[x_0^{t_0}, x_1^{t_1}, \dots, x_k^{t_k}] = 1$, where $x_0, x_1, \dots, x_k \in \{x, y\}$ with $x_0 \neq x_1$. In the following lemma we may assume that $t_0, t_1, \dots, t_k$ are positive.

Lemma 5. *Let A be a nontrivial Abelian group, $B = \langle b\rangle$ a finite cyclic group C, a subgroup of B of index m, and suppose that C acts on A. Let $W = A \operatorname{twr}_C B$ be the twisted wreath product of A by B with respect to the action of C on A. If $G \in \overline{\Omega}_k(n)$, n a positive integer, then $m \leqslant n + t_0 + t_1 + \cdots + t_k$.*

Proof: Suppose that $C = \langle b^m\rangle$ and let Y be a transversal to C in B so that $Y = \{b, b^2, \dots, b^{m-1}, 1\}$. Then $W = A \operatorname{twr}_C B = B \ltimes A^Y$ is the splitting extension of A^Y by B. The action of B on A^Y is given by $f^b(y) = f(y')^{c^{-1}}$, where $y' \in Y$ and $c \in C$ are unique elements such that $yb = cy'$.

Now assume, for a contradiction, that $m > n + t_0 + t_1 + \cdots + t_k$ and consider $n+1$ elements $f_0, f_1, \ldots, f_n$ so that

$$f_i(b^j) = \begin{cases} 1 & j \neq i \\ a & j = i \end{cases}, \quad j = 1, 2, \ldots, m-1,$$

where a is a fixed nontrivial element of A. Since $G \in \overline{\Omega}_k(n)$ there exists $i \neq j$ such that $[x_0^{t_0}, x_1^{t_1}, \cdots, x_k^{t_k}] = 1$, where $x_0 = b^{f_i}, x_1 = b^{f_j}, x_s \in \{b^{f_i}, b^{f_j}\}$. Then, as in Lemma 4, we have $(f_i f_j^{-1})^{(1-b^{t_0})(1-b^{t_1})\cdots(1-b^{t_k})} = 1$. If $i > j$, then for all $s \in \{0, 1, \ldots, m\}$ we have $i + s \leqslant n + t_0 + t_1 + \cdots + t_k \leqslant m - 1$, and thus $b^{i+s} \in Y$. Therefore $f_j(b^{i+s}) = 1$, as $i + s > j$ and

$$\left(f_i f_j^{-1}\right)^{b^s}(b^i) = f_i^{b^s}(b^i)\left(f_j^{b^s}(b^i)\right)^{-1} = f_i(b^{i+s})\left(f_j(b^{i+s})\right)^{-1} = \begin{cases} 1 & s \neq 0 \\ a & s = 0. \end{cases}$$

Hence $(f_i f_j^{-1})^{(1-b^{t_0})(1-b^{t_1})\cdots(1-b^{t_k})}(b^i) = a$, a contradiction. In the same way, if $i < j$ we get a contradiction. ∎

Now we are in the position to prove Theorem 3.

Proof of Theorem 3: By Lemma 5, G has no section isomorphic to $W = A \operatorname{twr}_C B$, where A is elementary Abelian, B is finite cyclic, and C is a subgroup of B which acts faithfully irreducibly on A, such that $|B : C| > n + t_0 + t_1 + \cdots + t_k$. Thus by a result of Wilson [**28**], G is virtually a soluble minimax. Hence there exist a normal subgroup H of G with finite index, such that H in a soluble minimax group. By Theorem 2, H is nilpotent-by-finite. Hence there exists a normal nilpotent subgroup K of H such that H/K is finite. Therefore K is finitely generated nilpotent with finite index in G, and G is nilpotent-by-finite. ∎

Let $t_0, t_1, \ldots t_k$ be fixed integers. We denote by $\overline{\Omega}_k^*$ the class of all groups G such that for any infinite subsets X and Y there exist $x \in X$, $y \in Y$ such that $[x_0^{t_0}, x_1^{t_1}, x_2^{t_2} \ldots, x_k^{t_k}] = 1$, where $x_i \in \{x, y\}, x_0 \neq x_1$, $i = 0, 1, \ldots, k$. Note that $\overline{\mathcal{W}}_k^* \subseteq \mathcal{W}_k^* \subseteq \overline{\Omega}_k^*$. The proof of the following lemma is easy and hence it is omitted.

Lemma 6. *Let N be an infinite normal subgroup of a $\overline{\Omega}_k^*$-group G, then $G/N \in \overline{\Omega}_k(1)$. In particular if G is an infinite residually finite $\overline{\Omega}_k^*$-group, then $G \in \overline{\Omega}_k(1)$.*

A result of Wilson (see [**28**, Theorem 2]) states that a finitely generated residually finite k-Engel group is nilpotent. As a consequence of Theorems 1 and 3 we can generalise this fact.

Corollary 3. *Let G be an infinite finitely generated residually finite $\overline{\mathcal{U}}_k(2)$-group, and k be a positive integer. Then G is nilpotent.*

Proof: By Theorem 3, there exists a normal nilpotent subgroup H of finite index in G. Now $G/H \in \overline{\mathcal{U}}_k(2)$. Thus, by Theorem 1, G/H is nilpotent, and G is soluble. Now G is a finitely generated soluble $\overline{\mathcal{U}}_k(2)$-group, and by Theorem 1, is nilpotent. ∎

Recall that a group $G \in \mathcal{W}_k^*$ if for any infinite subsets X and Y there exist $x \in X$, $y \in Y$ such that $[x_0, x_1, x_2^{t_2} \dots, x_k^{t_k}] = 1$, where t_i is an integer, $x_i \in \{x, y\}, x_0 \neq x_1$, $i = 2, 3, \dots, k$. If the integers $t_2, t_3, \dots, t_k$ are the same, we say that $G \in \overline{\mathcal{W}}_k^*$. Following [12] we say that a group G is restrained if $\langle x \rangle^{\langle y \rangle} = \langle x^{y^i} \mid i \text{ an integer} \rangle$ is finitely generated for all x, y in G. If there is a bound for the number of generators of $\langle x \rangle^{\langle y \rangle}$, then we call G strongly restrained.

Lemma 7. *Let G be a group in $\mathcal{W}_k^*$. Then G is restrained.*

Proof: Let $x, y \in G$. We want to show that $\langle x \rangle^{\langle y \rangle}$ is finitely generated. The result is clear if the order of y is finite. So assume that y is of infinite order. Consider the sets $X = \{x^{y^n} \mid n \text{ an integer}\}$ and $Y = \{y^m \mid m \text{ a positive integer}\}$. If X is finite then $\langle x \rangle^{\langle y \rangle} = \langle X \rangle$ is finitely generated, as required. So we may assume that X is infinite. Since $G \in \mathcal{W}_k^*$, there exist integers i, j such that $[x_0, x_1, x_2^{t_2}, \dots, x_k^{t_k}] = 1$ where $x_0 \neq x_1$, $x_s \in \{x^{y^i}, y^j\}$. Hence $[z_0, z_1, z_2^{t_2}, \dots, z_k^{t_k}] = 1$, where $z_0 = x$ and $z_1 = y^j$ or $z_0 = y^j$ and $z_1 = x$, and $z_2, \dots, z_k \in \{x, y^j\}$.

Suppose that $z_{i_1} = z_{i_2} = \dots = z_{i_s} = y^j$, and $z_t = x$, for all $t \neq i_r$. Let $T = \{t_{i_1}, t_{i_2}, \dots, t_{i_s}\}$, and denote by $T^{(r)}$ the set of all sums of r distinct elements of T, and denote by $S^{(r)}$ the set of all sums of r distinct elements of $T \cup \{1\}$. Then it is easy to see that

$$\left\langle z_0, [z_0, z_1], [z_0, z_1, z_2^{t_2}], \dots, [z_0, z_1, z_2^{t_2}, \dots, z_k^{t_k}] \right\rangle = \left\langle x, x^{y^j}, x^{y^{jr}};\ r \in \bigcup_{i=1}^{k} T^{(i)} \cup \bigcup_{i=1}^{k} S^{(i)} \right\rangle.$$

Therefore $\langle x \rangle^{\langle y \rangle} \leqslant \left\langle x^{y^{jr}};\ |r| \leqslant k(t_{i_1} + \dots + t_{i_s} + 1) \right\rangle$. This completes the proof. ∎

Corollary 4. *Let G be an infinite finitely generated soluble $\mathcal{W}_k^*$-group. Then G is polycyclic.*

Proof: This follows immediately from Lemma 7 and [**12**, Corollary 4]. ∎

Proof of Theorem 4: As in Lemma 7, G must be strongly restrained. Thus by [**12**, Theorem A] G is polycyclic-by-finite, and therefore residually finite. Now, by Lemma 6, $G \in \overline{\Omega}_k(1)$, and thus, by Theorem 3, G is nilpotent-by-finite. ∎

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