REVERSIBLE SKEW GENERALIZED POWER SERIES RINGS

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Abstract

In this note we show that there exist a semiprime ring $R$, a strictly ordered artinian, narrow, unique product monoid $(S, \leq)$ and a monoid homomorphism $\omega : S \rightarrow \text{End}(R)$ such that the skew generalized power series ring $R[[S, \omega]]$ is semicommutative but $R[[S, \omega]]$ is not reversible. This answers a question posed in Marks et al. [‘A unified approach to various generalizations of Armendariz rings’, Bull. Aust. Math. Soc. 81 (2010), 361–397].

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1. Introduction

Let $(S, \leq)$ be a partially ordered set. Then $(S, \leq)$ is called artinian if every strictly decreasing sequence of elements of $S$ is finite, and $(S, \leq)$ is called narrow if every subset of pairwise order-incomparable elements of $S$ is finite. A monoid $S$ equipped with an order $\leq$ is called an ordered monoid if for any $s_1, s_2, t \in S$, $s_1 \leq s_2$ implies $s_1 t \leq s_2 t$ and $t s_1 \leq t s_2$. Moreover, if $s_1 < s_2$ implies $s_1 t < s_2 t$ and $t s_1 < t s_2$, then $(S, \leq)$ is said to be strictly ordered. Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. For $s \in S$, let $\omega_s$ denote the image of $s$ under $\omega$. Let $A$ be the set of all functions $f : S \rightarrow R$ such that the support $\text{supp}(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(x, y) \in \text{supp}(f) \times \text{supp}(g) : s = xy\}$$

is finite. Thus one can define the product $fg : S \rightarrow R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(x, y) \in X_s(f, g)} f(x) \omega_x(g(y))$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, $A$ becomes a ring, called the ring of skew generalized power series. This research was in part supported by a grant from IPM (No. 89160029). This research was partially supported by the Center of Excellence for Mathematics, University of Isfahan.

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power series with coefficients in $R$ and exponents in $S$, denoted by $R[[S, \omega, \leq]]$ (or by $R[[S, \omega]]$ when there is no ambiguity concerning the order) (for more details, see [2]). Special cases of the skew generalized power series construction include skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings and Mal’cev–Neumann Laurent series rings.

Let $R$ be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \rightarrow \text{End}(R)$ a monoid homomorphism. A ring $R$ is called $(S, \omega)$-Armendariz if whenever $fg = 0$ for $f, g \in R[[S, \omega]]$, then $f(s)\omega_t(g(t)) = 0$ for all $s, t \in S$. Marks et al. in [1] introduced and investigated the notion of $(S, \omega)$-Armendariz rings and studied some property of this class of rings.

A ring $R$ is called reduced if it contains no nonzero nilpotent elements, reversible if for all $a, b \in R$, $ab = 0$ implies $ba = 0$, and semicommutative if $ab = 0$ implies $aRb = 0$ for each $a, b \in R$. It is known that each reduced ring is reversible and each reversible ring is semicommutative, but the converse not true in general. Marks et al. in [1] characterized when a skew generalized power series ring is reduced or semicommutative, and obtained a partial characterization for it to be reversible. They proved that for a strictly ordered monoid $(S, \leq)$, a monoid homomorphism $\omega : S \rightarrow \text{End}(R)$ and an $(S, \omega)$-Armendariz $S$-compatible ring $R, R[[S, \omega]]$ is semicommutative if and only if $R$ is semicommutative. They also showed that for a strictly ordered monoid $(S, \leq)$ which is an artinian, narrow, unique product (a.n.u.p., see [1, Definition 4.11]) and a monoid homomorphism $\omega : S \rightarrow \text{End}(R), R[[S, \omega]]$ is reduced if and only if $R$ is semiprime and the ring $R[[S, \omega]]$ is reversible. Marks et al. in [1] posed the following question (Question 4.14): ‘Suppose $R$ is a semiprime ring, $(S, \leq)$ is a strictly ordered a.n.u.p. monoid and $\omega : S \rightarrow \text{End}(R)$ is a monoid homomorphism. If the skew generalized power series ring $R[[S, \omega]]$ is semicommutative, must $R[[S, \omega]]$ be reversible (and therefore reduced)?’.

In this note we provide a semiprime ring $R$, a strictly ordered a.n.u.p. monoid $(S, \leq)$ and a monoid homomorphism $\omega : S \rightarrow \text{End}(R)$ such that the skew generalized power series ring $R[[S, \omega]]$ is semicommutative but $R[[S, \omega]]$ is not reversible. This gives a negative answer to the question posed by Marks et al. We also prove that for a semiprime ring $R$, a strictly ordered a.n.u.p. monoid $(S, \leq)$ and a monoid homomorphism $\omega : S \rightarrow \text{End}(R), R[[S, \omega]]$ is reversible if and only if $R[[S, \omega]]$ is semicommutative and $\omega_s$ is injective for each $s \in S$.

## 2. Main results

Let $R$ be a ring and $\alpha$ be a ring endomorphism. We denote by $R[x; \alpha]$ the skew polynomial ring whose elements are the polynomials over $R$, addition is defined as usual, and multiplication is subject to the relation $xa = \alpha(x)a$ for any $a \in R$.

**Example 2.1.** Let $K$ be a field, $R = K[x]$, $S = \mathbb{N} \cup \{0\}$ with the usual addition and trivial order. $\alpha : R \rightarrow R$ given by $\alpha(f(x)) = f(0)$ is a ring homomorphism and so $\omega : S \rightarrow \text{End}(R)$ given by $\omega(1) = \alpha$ is a monoid homomorphism. We have $R[[S, \omega]] \cong R[y; \alpha]$. 

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We show that $R[y; \alpha]$ is semicommutative but not reversible. Assume that $f = f_0 + f_1 y + \cdots + f_n y^n$, $g = g_0 + g_1 y + \cdots + g_m y^m \in R[y; \alpha]$ is such that $fg = 0$. By induction on $\deg(g) = m$ we show that $f R[y; \alpha]g = 0$. If $m = 0$ then $f_n \alpha^n(g_0) = 0$. Since $R$ is a domain, we have $\alpha^n(g_0) = 0$ and so $g_0 \in (x)$, where $(x)$ is the ideal generated by $x$ in $R$. We also have $f_0 g_0 = 0$. If $g_0 = 0$ then $f R[y; \alpha]g = 0$. If $g_0 \neq 0$ then $f_0 = 0$ and so $f R[y; \alpha]g = 0$.

Now assume inductively that the assertion is true for all $g \in R[y; \alpha]$ with $\deg(g) < m$. Since $fg = 0$, we have $f_n \alpha^n(g_m) = 0$ and so $g_m \in (x)$. Also we have $f_0 g_0 = 0$. If $f_0 \neq 0$ then $g_0 = 0$ and so $f_0 g_1 = 0$. Thus $g_1 = 0$ and, by the same argument, in this case we have, for each $i$, $g_i = 0$. Then $f R[y; \alpha]g = 0$ and the result follows.

Now assume that $f_0 = 0$. Since $g_m \in (x)$ and $f_0 = 0$ then $f R[y; \alpha]g_m y^m = 0$ and so $f(g_0 + g_1 y + \cdots + g_{m-1} y^{m-1}) = 0$. By the induction hypothesis, $f R[y; \alpha]g = 0$ and the result follows. In $R[y; \alpha]$ we have $yx = \alpha(x)y = 0$ but $xy \neq 0$. Thus $R[y; \alpha]$ is not reversible.

Let $R$ be a semiprime ring. In the next theorem we provide a necessary and sufficient condition for a skew generalized power series ring $R[[S, \omega]]$ to be reversible.

**Theorem 2.2.** Let $R$ be a semiprime ring, $(S, \leq)$ a strictly ordered a.n.u.p. monoid and $\omega : S \to \text{End}(R)$ a monoid homomorphism. Then $R[[S, \omega]]$ is reversible if and only if $R[[S, \omega]]$ is semicommutative and $\omega_s$ is injective for each $s \in S$.

**Proof.** If $R[[S, \omega]]$ is reversible, then by [1, Theorem 4.12] $\omega_s$ is injective for each $s \in S$. Now assume that $R[[S, \omega]]$ is semicommutative and $\omega_s$ is injective for each $s \in S$. Let $s \in S$, $\omega_s \in \text{End}(R)$ and $a \in R$ such that $\omega_s(a) = 0$. Since $R$ is semiprime and semicommutative, then $R$ is a reduced ring and so $\omega_s(a)a = 0$. Thus by [1, Lemma 4.4], we have $\omega_s(a)\omega_s(a) = 0$. Then $a^2 = 0$ and so $a = 0$. Thus for each $s \in S$, $\omega_s$ is a rigid endomorphism. Then, by [1, Theorem 4.12], $R[[S, \omega]]$ is reversible and the result follows. □

**References**


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