Bull. Aust. Math. Soc. **86** (2012), 510–514
doi:10.1017/S0004972712000032

ON ODD PERFECT NUMBERS

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(Received 14 December 2011)

Abstract

Let q be an odd prime. In this paper, we prove that if N is an odd perfect number with $q^\alpha \parallel N$ then $\sigma(N/q^\alpha)/q^\alpha \neq p, p^2, p^3, p^4, p_1p_2, p_1^2p_2$, where p, p_1, p_2 are primes and $p_1 \neq p_2$. This improves a result of Dris and Luca ['A note on odd perfect numbers', arXiv:1103.1437v3 [math.NT]]: $\sigma(N/q^\alpha)/q^\alpha \neq 1, 2, 3, 4, 5$. Furthermore, we prove that for $K \geq 1$, if N is an odd perfect number with $q^\alpha \parallel N$ and $\sigma(N/q^\alpha)/q^\alpha \leq K$, then $N \leq 4^{K^8}$.

2010 *Mathematics subject classification*: primary 11A05; secondary 11A25.
Keywords and phrases: odd perfect number, primitive prime factor.

1. Introduction

For a positive integer N, let $\sigma(N)$ be the sum of all positive divisors of N. We call N perfect if $\sigma(N) = 2N$. It is well known that an even integer N is perfect if and only if $N = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both primes. It is not known whether or not odd perfect numbers exist. If such a number N exists, it must have the form $N = p^\alpha q_1^{2\beta_1} \cdots q_t^{2\beta_t}$, where $p, q_1, \ldots, q_t$ are primes and $p \equiv \alpha \equiv 1 \pmod 4$. This was proved by Euler in 1849. Recently, Ochem and Rao [6] showed that there is no odd perfect number below 10^{1500}. Moreover, it has been proved that an odd perfect number must have at least nine distinct prime factors (see [5]).

Suppose that N is a perfect number with $q^\alpha \parallel N$, where q is prime and $q^\alpha \parallel N$ means that $q^\alpha \mid N$ and $q^{\alpha+1} \nmid N$. Since $\sigma(N) = 2N$,

$$\sigma\Big(\frac{N}{q^\alpha}\Big)\sigma(q^\alpha) = \sigma(N) = 2N = 2q^\alpha \frac{N}{q^\alpha}. \tag{1.1}$$

Since $(q^\alpha, \sigma(q^\alpha)) = 1$,

$$q^\alpha \,\Big|\, \sigma\Big(\frac{N}{q^\alpha}\Big),$$

This work was supported by the National Natural Science Foundation of China (Grant No. 11071121) and the Project of Graduate Education Innovation of Jiangsu Province (CXZZ11-0868).

and $\sigma(N/q^\alpha)/q^\alpha$ is a divisor of $2N$. If N is an even perfect number with $q^\alpha \parallel N$, then $\sigma(N/q^\alpha)/q^\alpha = 1$ or 2. If N is an odd perfect number and $q^\alpha \parallel N$, then by (1.1), $4 \nmid \sigma(N/q^\alpha)/q^\alpha$.

In the following, we always assume that q is an odd prime. Recently, Dris and Luca [3] posed a new approach to research on odd perfect numbers and proved the following results.

THEOREM A. *If N is an odd perfect number with $q^\alpha \parallel N$, then $\sigma(N/q^\alpha)/q^\alpha \notin \{1, 2, 3, 4, 5\}$.*

THEOREM B. *For every fixed $K > 5$, there are only finitely many odd perfect numbers N such that, for some prime power $q^\alpha \parallel N$, $\sigma(N/q^\alpha)/q^\alpha < K$. All such N are bounded by some effectively computable number depending on K.*

For a positive integer n with the standard factorisation $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ ($\alpha_i > 0$, $i = 1, 2, \ldots, s$), let $\Omega(n) = \alpha_1 + \cdots + \alpha_s$ and $\omega(n) = s$.

In this paper, we improve the above results by proving the following theorems.

THEOREM 1.1. *Suppose that N is an odd perfect number with $q^\alpha \parallel N$. Let $m = \sigma(N/q^\alpha)/q^\alpha$. Then*

$$\Omega(m) + \omega(m) \ge \omega(N) - \log_2 \omega(N),$$

where $\log_2$ means the logarithm to base 2.

From $\omega(N) \ge 9$ and Theorem 1.1, we immediately have the following corollary.

COROLLARY 1.2. *Suppose that N is an odd perfect number with $q^\alpha \parallel N$. Let $m = \sigma(N/q^\alpha)/q^\alpha$. Then $\Omega(m) + \omega(m) \ge 6$. That is,*

$$m \ne p,\ p^2,\ p^3,\ p^4,\ p_1p_2,\ p_1^2p_2,$$

where p, p_1, p_2 are primes and $p_1 \ne p_2$.

THEOREM 1.3. *Suppose that $K \ge 1$ and N is an odd perfect number. If $q^\alpha \parallel N$ with $\sigma(N/q^\alpha)/q^\alpha \le K$, then $N \le 4^{K^8}$.*

REMARK 1.4. From a detailed proof of Theorem 1.3, we can in fact show that $N \le 4^{K^\theta}$, where $\theta = \log 4/\log 3 + o(1)$.

2. Preliminary lemmas

Suppose that N is an odd perfect number, so $\sigma(N) = 2N$. Write

$$N = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_s^{\lambda_s} q^\alpha,$$

where the primes $p_1, p_2, \ldots, p_s, q$ are distinct odd numbers and not necessarily ordered increasingly. Let

$$\sigma(p_i^{\lambda_i}) = \begin{cases} m_i q^{\beta_i} & i = 1, 2, \ldots, k, \\ q^{\beta_i} & i = k+1, \ldots, s, \end{cases} \tag{2.1}$$

https://doi.org/10.1017/S0004972712000032 Published online by Cambridge University Press

where $m_i \geq 2$, and $q \nmid m_i$ for $i = 1, 2, \ldots, k$. We put $m = m_1 m_2 \cdots m_k$ and $t = \omega_0(m)$; $\omega_0(m)$ is the number of distinct odd prime factors of m. It is clear that $k \leq \Omega(m)$.

Since $\sigma(N) = 2N$,

$$\sigma(p_1^{\lambda_1}) \cdots \sigma(p_s^{\lambda_s})\sigma(q^\alpha) = 2N = 2p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_s^{\lambda_s} q^\alpha.$$

That is,

$$mq^{\beta_1+\beta_2+\cdots+\beta_s}\sigma(q^\alpha) = 2p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_s^{\lambda_s} q^\alpha.$$

By $q \nmid \sigma(q^\alpha)$ and $q \nmid m$, we have $\alpha = \beta_1 + \beta_2 + \cdots + \beta_s$. Hence,

$$m\sigma(q^\alpha) = m\frac{q^{\alpha+1} - 1}{q - 1} = 2p_1^{\lambda_1} \cdots p_k^{\lambda_k} p_{k+1}^{\lambda_{k+1}} \cdots p_s^{\lambda_s} = \frac{2N}{q^\alpha}. \tag{2.2}$$

By (1.1) and (2.2),

$$m = \frac{2N}{q^\alpha \sigma(q^\alpha)} = \frac{\sigma(N/q^\alpha)}{q^\alpha}.$$

Definition 2.1. A prime factor p of $a^n - 1$ is called primitive if $p \nmid a^j - 1$ for all $0 < j < n$.

Our proofs are based on the following lemmas.

Lemma 2.2 [1, 2, 7]. *Let a and n be integers greater than 1. There exists a primitive prime factor of $a^n - 1$, except precisely in the following cases:* (i) $n = 2$, $a = 2^\beta - 1$, *where* $\beta \geq 2$; (ii) $n = 6$, $a = 2$.

Lemma 2.3 [3]. *Let λ, α, β be positive integers, and p, q be primes such that*

$$\frac{p^{\lambda+1} - 1}{p - 1} = q^\beta, \quad p^\lambda \,\Big|\, \frac{q^{\alpha+1} - 1}{q - 1}.$$

Then $p^{\lambda-1} \mid \alpha + 1$.

Let $d(\alpha + 1)$ denote the number of positive divisors of $\alpha + 1$.

Lemma 2.4. *Let N be an odd perfect number with $q^\alpha \parallel N$. Then $d(\alpha + 1) \leq \omega(N)$.*

Proof. Let $n_1, n_2, \ldots, n_w$ be all of the distinct divisors of $\alpha + 1$ which are larger than 1. If $2 \mid \alpha + 1$, then by Euler's result we have $q \equiv \alpha \equiv 1 \pmod 4$. Thus, by Lemma 2.2, for each $1 \leq i \leq w$, there exists a primitive prime factor q_i of $q^{n_i} - 1$. Since $2 \mid q - 1$ and $n_1, n_2, \ldots, n_w$ are distinct and larger than 1, we know that $q_1, \ldots, q_w$ are distinct odd primes. Noting that $n_1, n_2, \ldots, n_w$ are divisors of $\alpha + 1$,

$$q^{n_i} - 1 \mid q^{\alpha+1} - 1, \quad 1 \leq i \leq w.$$

Hence,

$$q_1 \cdots q_w \,\Big|\, \frac{q^{\alpha+1} - 1}{q - 1}.$$

By (2.2), we have $d(\alpha + 1) = w + 1 \leq s + 1 = \omega(N)$. □

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3. Proofs of theorems

Proof of Theorem 1.1. If $(m,\ p_{k+1}\cdots p_s)=p_{k+1}\cdots p_s$, then $s-k\le t$. So

$$k+t\ge s=\omega(N)-1.$$

Since $k+t\le\Omega(m)+\omega(m)$ and $\omega(N)\ge 2$,

$$\Omega(m)+\omega(m)\ge\omega(N)-1\ge\omega(N)-\log_2\omega(N).$$

If $(m,p_{k+1}\cdots p_s)\ne p_{k+1}\cdots p_s$, without loss of generality, we may assume that

$$\frac{p_{k+1}\cdots p_s}{(m,p_{k+1}\cdots p_s)}=p_{l+1}\cdots p_s,\quad k\le l<s. \tag{3.1}$$

By (2.2) and (3.1),

$$p_{l+1}^{\lambda_{l+1}}\cdots p_s^{\lambda_s}\mid\sigma(q^\alpha).$$

Using (2.1) and Lemma 2.3,

$$p_i^{\lambda_i-1}\mid\alpha+1,\quad i=l+1,\dots,s.$$

So

$$p_{l+1}^{\lambda_{l+1}-1}\cdots p_s^{\lambda_s-1}\mid\alpha+1.$$

Since $\sigma(p_i^{\lambda_i})=q^{\beta_i}$ and $2\nmid q$, we have that λ_i is even for $k+1\le i\le s$. Thus, $\lambda_i\ge 2$ for $l+1\le i\le s$ and then $p_{l+1}\cdots p_s\mid\alpha+1$.

Case 1. $2\nmid m$. By (2.2) and

$$\frac{q^{\alpha+1}-1}{q-1}=q^\alpha+\cdots+q+1\equiv\alpha+1\pmod 2,$$

we have $2\mid\alpha+1$. Thus, $2p_{l+1}\cdots p_s\mid\alpha+1$. By Lemma 2.4,

$$2^{s-l+1}\le d(\alpha+1)\le\omega(N).$$

That is,

$$s-l+1\le\log_2\omega(N).$$

Thus

$$l\ge\omega(N)-\log_2\omega(N).$$

By $\omega_0(m)=t$ and (3.1), we have $l-k\le t$. So $l\le k+t\le\Omega(m)+\omega(m)$. Hence,

$$\Omega(m)+\omega(m)\ge\omega(N)-\log_2\omega(N).$$

Case 2. $2\mid m$. Since $p_{l+1}\cdots p_s\mid\alpha+1$,

$$2^{s-l}\le d(\alpha+1)\le\omega(N).$$

So

$$l\ge s-\log_2\omega(N)=\omega(N)-\log_2\omega(N)-1.$$

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In a similar manner to case 1,

$$l \le k + t \le \Omega(m) + \omega_0(m) = \Omega(m) + \omega(m) - 1.$$

Hence

$$\Omega(m) + \omega(m) \ge \omega(N) - \log_2 \omega(N).$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.3. Since $m = m_1 m_2 \cdots m_k = \sigma(N/q^\alpha)/q^\alpha \le K$, we have $\omega(m) \le \Omega(m) \le \log K/\log 2$. By Theorem 1.1,

$$\frac{2\log K}{\log 2} \ge \Omega(m) + \omega(m) \ge \omega(N) - \log_2 \omega(N).$$

Since $\omega(N) \ge 9$,

$$\log_2 \omega(N) \le \tfrac{1}{2}\omega(N).$$

Thus, $\omega(N) \le 4\log K/\log 2$. Using a famous result of Heath-Brown [4],

$$N < 4^{4^{\omega(N)}} \le 4^{4^{4\log K/\log 2}} = 4^{K^8}.$$

This completes the proof of Theorem 1.3. □

Acknowledgement

We are grateful to the referee for his/her helpful comments.

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https://doi.org/10.1017/S0004972712000032 Published online by Cambridge University Press