LOCAL-GLOBAL PRINCIPLE FOR THE FINITENESS AND ARTINIANNES OF GENERALISED LOCAL COHOMOLOGY MODULES

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Abstract

Let $S$ be a Serre subcategory of the category of $R$-modules, where $R$ is a commutative Noetherian ring. Let $a$ and $b$ be ideals of $R$ and let $M$ and $N$ be finite $R$-modules. We prove that if $N$ and $H^i_a(M, N)$ belong to $S$ for all $i < n$ and if $n \leq \ell\text{-grad}(a, b, N)$, then $\text{Hom}_R(R/b, H^n_a(M, N)) \in S$. We deduce that if either $H^i_a(M, N)$ is finite or $\text{Supp} H^i_a(M, N)$ is finite for all $i < n$, then $\text{Ass} H^n_a(M, N)$ is finite. Next we give an affirmative answer, in certain cases, to the following question. If, for each prime ideal $p$ of $R$, there exists an integer $n_p$ such that $b^{n_p}H^i_p(M_p, N_p) = 0$ for every $i$ less than a fixed integer $t$, then does there exist an integer $n$ such that $b^nH^i_a(M, N) = 0$ for all $i < t$? A formulation of this question is referred to as the local-global principle for the annihilation of generalised local cohomology modules. Finally, we prove that there are local-global principles for the finiteness and Artinianness of generalised local cohomology modules.


Keywords and phrases: (generalised) local cohomology module, finiteness, Artinianness, local-global principle, filter regular sequence.

1. Introduction

Throughout this paper $R$ denotes a commutative Noetherian ring with identity and $a, b, c$ are ideals of $R$. We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the set of positive and nonnegative integers, respectively. The notion of generalised local cohomology functors was introduced by Herzog, in [9], over a local ring and then continued by Suzuki in [18]. Later this concept was studied by Bijan-Zadeh, in [1], over any commutative Noetherian ring. For each integer $i$, the $i$th generalised local cohomology functor $H^i_a(\cdot, \cdot)$ is defined by

$$H^i_a(M, N) = \lim_{\longrightarrow} \text{Ext}^i_R(M/a^nM, N)$$

for all $R$-modules $M$ and $N$. Clearly, this notion is a generalisation of the usual local cohomology functor [4]. On the other hand, the concept of a filter regular sequence
has been studied in [12, 15, 17, 21] and has led to some interesting results. We denote the common length of all maximal \( a \)-filter regular \( M \)-sequences contained in \( b \) by \( f\text{-grad}(a, b, M) \) and call it the \( a \)-filter grade of \( b \) on \( M \). We briefly recall, in Section 2, the concept of a filter regular sequence and basic properties of \( f\text{-grad}(a, b, M) \), but refer the reader to [8, 19] for more details. It is clear that an \( R \)-filter regular \( M \)-sequence is just a weak \( M \)-sequence [2] and \( f\text{-grad}(R, b, M) = \text{grad}(b, M) \). If \((R, m)\) is a local ring, then \( f\text{-grad}(m, b, M) \) is just the well-known notion \( f\text{-depth}(b, M) \); see [11] for some characterisations of \( f\text{-depth}(b, M) \). Filter regular sequences were employed in [19] to establish some finiteness results on usual local cohomology modules. In this paper we use those sequences to obtain some finiteness and Artinianness results on generalised local cohomology modules.

Recall that a class \( S \) of \( R \)-modules is a Serre subcategory of the category of \( R \)-modules if it is closed under taking submodules, quotients and extensions. In Theorem 2.2, for finite \( R \)-modules \( M \) and \( N \), we prove that if \( N \) and \( H^i_a(M, N) \) belong to \( S \) for all \( i < n \) and \( n \leq f\text{-grad}(a, b, M) \), then \( \text{Hom}_R(R/b, H^0_a(M, N)) \in S \). We deduce that if either \( H^i_a(M, N) \) is finite or \( \text{Supp} H^i_a(M, N) \) is finite for all \( i < n \), then \( \text{Ass} H^i_a(M, N) \) is finite. In a certain case, when \( M = R \), this is the main result of [13]. Therefore Theorem 2.2 provides a generalisation of the main result of [13]. Notice that \( \text{Ass} H^i_a(M, N) \) is not finite in general; see, for example, [10, 16].

Let \( M, N \) be finite \( R \)-modules. As a generalisation of the \( b \)-finiteness dimension \( f^b_a(N) \) of \( N \) with respect to \( a \), we define
\[
f^b_a(M, N) = \inf \{ i \in \mathbb{N}_0 \mid b \not\in 0 :_R H^i_a(M, N) \}
\]
and denote \( f^b_a(M, N) \) by \( f_a(M, N) \). In fact, by Proposition 3.1,
\[
f_a(M, N) = \inf \{ i \in \mathbb{N}_0 \mid H^i_a(M, N) \text{ is not finite} \}
\]
In Section 3 we give some properties of \( f^b_a(M, N) \). In particular, we prove that \( f^b_a(N) \leq f^b_a(M, N) \). We present an example to show that the above inequality may be strict (Example 3.6). Thus the result [5, Proposition 2.10] of Chu is not correct. Moreover, Example 3.6 shows that the result [5, Lemma 2.9] is no longer true.

The local-global principle for the finiteness of local cohomology modules, investigated by Faltings in [6, 7], states that, for all nonnegative integers \( r \), \( f_a(N) > r \) if and only if \( f_{aR_p}(N_p) > r \) for all \( p \in \text{Spec}(R) \). Also we say that Faltings’ local-global principle for the annihilation of local cohomology modules holds at level \( r \) if
\[
f_a(N) > r \iff f_{aR_p}(N_p) > r \quad \text{for all } p \in \text{Spec}(R)
\]
is true for all finite \( R \)-modules \( N \) and all ideals \( a, b \). Raghavan proved, in [14], that the local-global principle for the annihilation of local cohomology modules holds at level 1, while Brodmann et al. proved it is true at level 2 [3, Theorem 2.6]. As a generalisation of this, we say that Faltings’ local-global principle for the annihilation of generalised local cohomology modules holds at level \( r \) if
\[
f^b_a(M, N) > r \iff f^b_{aR_p}(M_p, N_p) > r \quad \text{for all } p \in \text{Spec}(R)
\]
(\( \dagger \))
is true for all finite $R$-modules $M, N$ and all ideals $a, b$. We show, in Proposition 4.2, that the local-global principle for the annihilation of generalised local cohomology modules holds at levels 0, 1, 2. Now let $b \subseteq a$. Then we prove the following statements, in Theorem 4.4.

(i) $f_a(M, N) \geq \inf \{ f_a^b(M, N), \text{f-grad } (a, b, N) + 1 \}$. In particular, $f_a(M, N) = f_a^b(M, N)$ whenever $f_a^b(M, N) \leq \text{f-grad } (a, b, N) + 1$.

(ii) Assume that $r \leq \text{f-grad } (a, b, N) + 1$. Then

$$f_a^b(M, N) > r \iff f_{aR^p}(M, N) > r$$

for all $p \in \text{Spec}(R)$.

(iii) If $\text{Supp } N/bN \subseteq V(a)$, then the statement

$$f_a^b(M, N) > r \iff f_{aR_p}(M, N) > r$$

holds for all $r$.

(iv) Faltings’ local-global principle for the finiteness of generalised local cohomology modules holds. In other words, for any positive integer $r$, $H^i_{aR_p}(M, N)$ is finite for all $i \leq r$ and for all $p \in \text{Spec } R$ if and only if $H^i_{a}(M, N)$ is finite for all $i \leq r$.

Finally, in Theorem 5.3, for finite $R$-modules $M$ and $N$ and for a positive integer $n$, we prove that $H^i_{a}(M, N)$ is Artinian for all $i < r$ if and only if $H^i_{aR_p}(M, N)$ is Artinian for all $i < r$ and for all $p \in \text{Spec } R$. We observe that this result improves the main result of [20].

2. Preliminary results

We first recall some basic properties of filter regular sequences. The reader is referred to [8] for more details. Assume that $M$ and $N$ are finite $R$-modules. We say that a sequence $x_1, \ldots, x_n$ of elements of $R$ is an $a$-filter regular $M$-sequence if $x_i \notin p$ for all

$$p \in \text{Ass}(M/(x_1, \ldots, x_{i-1})M) \setminus V(a)$$

and for all $i = 1, \ldots, n$. If, in addition, $x_1, \ldots, x_n \in b$, then we say that $x_1, \ldots, x_n$ is an $a$-filter regular $M$-sequence in $b$. There exists an $a$-filter regular $M$-sequence in $b$ of infinite length if and only if $\text{Supp } M/bM \not\subseteq V(a)$. Now assume that $\text{Supp } M/bM \not\subseteq V(a)$. Then we denote the common length of all maximal $a$-filter regular $M$-sequences contained in $b$ by $\text{f-grad } (a, b, M)$ and we call it the $a$-filter grade of $b$ on $M$. We set $\text{f-grad } (a, b, M) = \infty$ whenever $\text{Supp } M/bM \not\subseteq V(a)$. Also, notice that

$$\text{f-grad } (a, b, M) = \inf \{ i \in \mathbb{N}_0 \mid \text{Supp } \text{Ext}_R^i(R/b, M) \not\subseteq V(a) \}$$

$$= \inf \{ i \in \mathbb{N}_0 \mid \text{Supp } H^i_b(M) \not\subseteq V(a) \},$$

$$\text{f-grad } (a, \text{Ann } N, M) = \inf \{ i \in \mathbb{N}_0 \mid \text{Supp } \text{Ext}_R^i(N, M) \not\subseteq V(a) \},$$

$$\text{f-grad } (a, b + \text{Ann } N, M) = \inf \{ i \in \mathbb{N}_0 \mid \text{Supp } H^i_b(N, M) \not\subseteq V(a) \}.$$
Since \( f\text{-grad}(R, b, M) = \text{grad}(b, M) \), we have the following well-known properties ([1, Proposition 5.5], [4, Theorem 6.2.7]):

\[
\text{grad}(b, M) = \inf\{i \in \mathbb{N}_0 \mid \text{Ext}_R^i(R/b, M) \neq 0\} = \inf\{i \in \mathbb{N}_0 \mid H^i_b(M) \neq 0\}
\]

and

\[
\text{grad}(b + \text{Ann } N, M) = \inf\{i \in \mathbb{N}_0 \mid H^i_b(N, M) \neq 0\}.
\]

If \((R, m)\) is a local ring, then \(f\text{-grad}(m, b, M)\) is just the well-known notion \(f\text{-depth}(b, M)\); see [11] for some properties of \(f\text{-depth}(b, M)\). The following lemma is of assistance in the proof of the next theorem.

**Lemma 2.1.** Let \(S\) be a Serre subcategory of the category of \(R\)-modules, \(M\) be a finite \(R\)-module and \(N \in S\). Then \(\text{Ext}_R^i(M, N) \in S\) for all \(i \in \mathbb{N}_0\).

**Proof.** Since \(\text{Ext}_R^i(M, N)\) is a subquotient of \(N^\alpha\) for some \(\alpha \in \mathbb{N}_0\), the result is clear. \(\square\)

**Theorem 2.2.** Let \(S\) be a Serre subcategory of the category of \(R\)-modules. Let \(n \in \mathbb{N}_0\) and let \(M\) and \(N\) be finite \(R\)-modules such that \(N\) and \(H^i_b(M, N)\) belong to \(S\) for all \(i < n\). If \(f\text{-grad}(a, b, N) \geq n\), then \(\text{Hom}_R(R/b, H^i_a(M, N)) \in S\). In particular, \(\text{Hom}_R(R/b, H^i_a(M, N)) \in S\) whenever \(\text{Supp } N/bN \subseteq V(a)\).

**Proof.** We prove the assertion by induction on \(n\). Since \(H^i_a(M, N) \cong \text{Hom}_R(M, \Gamma_a(N))\), the result is clear for \(n = 0\) by Lemma 2.1. Assume that \(n > 0\) and that the result has been proved for \(n - 1\). Let \(f\text{-grad}(a, b, N) \geq n\) and suppose that \(x \in b\) is an \(a\)-filter regular \(N\)-sequence. The exact sequence

\[
0 \to \Gamma_a(N) \to N \to N/\Gamma_a(N) \to 0
\]

induces the long exact sequence

\[
\cdots \to H^i_a(M, \Gamma_a(N)) \xrightarrow{f^i} H^i_a(M, N) \to H^i_a(M, N/\Gamma_a(N)) \to H^{i+1}_a(M, \Gamma_a(N)) \to \cdots.
\]

Since, by [23, Lemma 1.1],

\[
H^i_a(M, \Gamma_a(N)) \cong \text{Ext}_R^i(M, \Gamma_a(N)) \quad \text{for all } i \in \mathbb{N}_0,
\]

we use Lemma 2.1 and the above long exact sequence to see that \(H^i_a(M, N) \in S\) if and only if \(H^i_a(M, N/\Gamma_a(N)) \in S\). Also \(N/\Gamma_a(N) \in S\) and \(f\text{-grad}(a, b, N) = f\text{-grad}(a, b, N/\Gamma_a(N))\). On the other hand, since \(\text{im } f^n \in S\), the induced exact sequence

\[
0 \to \text{Hom}_R(R/b, \text{im } f^n) \to \text{Hom}_R(R/b, H^i_a(M, N)) \to \text{Hom}_R(R/b, H^i_a(M, N/\Gamma_a(N)))
\]

yields \(\text{Hom}_R(R/b, H^i_a(M, N)) \in S\) whenever \(\text{Hom}_R(R/b, H^i_a(M, N/\Gamma_a(N))) \in S\). Thus we can replace \(N\) by \(N/\Gamma_a(N)\) and, without loss of generality, assume that \(\Gamma_a(N) = 0\); and hence \(x\) is a nonzero divisor on \(N\). Next, consider the exact sequence

\[
0 \to N \xrightarrow{x} N \to N/xN \to 0
\]
which induces the long exact sequence
\[ \cdots \to H_i^J(M, N) \xrightarrow{\partial} H_i^I(M, N) \to H_i^J(M, N/xN) \to H_{i+1}^I(M, N) \xrightarrow{\partial} \cdots. \]

Now we may use the above sequence in conjunction with the hypothesis to deduce that \( H_i^J(M, N/xN) \in \mathcal{S} \) for all \( i < n - 1 \). Also it is easy to see that \( f\text{-grad} (a, b, N/xN) = f\text{-grad} (a, b, N) - 1 \). Therefore, by induction, \( \text{Hom}_R(R/b, H_{a-n}^I(M, N/xN)) \in \mathcal{S} \). Next, we use the exact sequence
\[ 0 \to H_{a-n}^I(M, N) / xH_{a-n}^I(M, N) \to H_{a-n}^I(M, N/xN) \to 0 ; \]

we use the exact sequence
\[ 0 \to H_{a-n}^I(M, N) / xH_{a-n}^I(M, N) \to H_{a-n}^I(M, N/xN) \to 0 ; \]
to obtain the exact sequence
\[ \text{Hom}_R(R/b, H_{a-n}^I(M, N/xN)) \to \text{Hom}_R(R/b, H_{a-n}^I(M, N)) \]
\[ \to \text{Ext}_R^1(R/b, H_{a-n}^I(M, N)/xH_{a-n}^I(M, N)) \]

which in turn, by Lemma 2.1, yields \( \text{Hom}_R(R/b, H_{a-n}^I(M, N)) \in \mathcal{S} \). This completes the inductive step. Finally, since the hypothesis \( \text{Supp} N/bN \subseteq V(a) \) implies \( f\text{-grad} (a, b, N) = \infty \), the last assertion follows immediately from the first one. \( \square \)

Let \( M \) be an \( R \)-module. \( M \) is called an FSF module if there is a finite submodule \( N \) of \( M \) such that the support of the quotient module \( M/N \) is finite. If \( M \) is an FSF module, then \( \text{Ass} M \) is finite and the category of FSF \( R \)-modules is a Serre subcategory of the category of \( R \)-modules [13, Proposition 2.2].

By applying the above theorem to the category of FSF \( R \)-modules we have the following corollary which recovers the main result of [13] which has been proved for ordinary local cohomology modules.

**Corollary 2.3.** Let \( M, N \) be finite \( R \)-modules and let \( n \in \mathbb{N}_0 \) be such that either \( H^0_a(M, N) \) is finite or \( \text{Supp} H^0_a(M, N) \) is finite for all \( i < n \). Then \( \text{Ass} H^0_{n-1}(M, N) \) is finite.

### 3. Finiteness properties of generalised local cohomology modules

Let \( M \) be a finite \( R \)-module. Following [4, Proposition 9.1.2] and [6, Lemma 3], the finiteness dimension \( f_a(M) \) of \( M \) relative to \( a \) is defined as follows:
\[ f_a(M) = \inf \{ i \in \mathbb{N}_0 \mid H_i^0(M) \text{ is not finite} \} \]
\[ = \inf \{ i \in \mathbb{N}_0 \mid a \not\subseteq \sqrt{(0 :_R H_i^0(M))} \}. \]

As a generalisation, the \( b \)-finiteness dimension \( f_{a}^b(M) \) of \( M \) relative to \( a \) is defined by
\[ f_{a}^b(M) = \inf \{ i \in \mathbb{N}_0 \mid b \not\subseteq \sqrt{(0 :_R H_i^0(M))} \}. \]

We now extend this definition to generalised local cohomology modules.

**Proposition 3.1.** Let \( M, N \) be finite \( R \)-modules and \( n \in \mathbb{N}_0 \). Then the following statements are equivalent:
(i) $H^i_a(M, N)$ is finite for all $i < n$;
(ii) $a \subseteq \sqrt{0 : R H^i_a(M, N)}$ for all $i < n$.

**Proof.** (i) $\Rightarrow$ (ii) is obvious. For (ii) $\Rightarrow$ (i), we use induction on $n$. When $n = 1$, there is nothing to prove. Now let $n > 1$ and suppose that the result has been proved for smaller values of $n$. By the inductive assumption, $H^i_a(M, N)$ is finite for $i = 0, \ldots, n - 2$. Also, by hypothesis, $\alpha' H^{n-1}_a(M, N) = 0$ for some $r \in \mathbb{N}$, so that, in view of Theorem 2.2, $0 : H^{n-1}_a(M, N) \alpha' = H^{n-1}_a(M, N)$ is finite. This completes the induction. □

**Definition 3.2.** Let $M$ and $N$ be finite $R$-modules. We define the b-finiteness dimension $f^b_a(M, N)$ of $M, N$ relative to $a$ by

$$f^b_a(M, N) = \inf\{i \in \mathbb{N} \mid b \not\subseteq \sqrt{0 : R H^i_a(M, N)}\}.$$

Notice that, by Proposition 3.1,

$$f^b_a(M, N) = \inf\{i \in \mathbb{N} \mid H^i_a(M, N) \text{ is not finite}\}.$$

We denote $f^b_a(M, N)$ by $f_a(M, N)$.

For $y \in R$, set $S = \{ y^n : n \geq 0 \}$. In the next lemma, for an $R$-module $M$, we denote $S^{-1}M$ by $M_y$. The following two lemmas are needed in the proof of the next proposition.

**Lemma 3.3.** Let $M, N$ be finite $R$-modules and $x \in R$. Then we have the following long exact sequence

$$\ldots \to H^i_{a + Rx}(M, N) \to H^i_a(M, N) \to H^i_{aRx}(M_x, N_x) \to H^{i+1}_{a + Rx}(M, N) \to \ldots.$$

**Proof.** Let $E^\bullet$ be an injective resolution of $N$. Then $E^\bullet_x$ is an injective resolution of $R_x$-module $N_x$. The split exact sequence

$$0 \to \Gamma_{a + Rx}(E^\bullet) \to \Gamma_a(E^\bullet) \to \Gamma_a(E^\bullet_x) \to 0$$

of complexes [4, Lemma 8.1.1] induces the exact sequence

$$0 \to \text{Hom}_R(M, \Gamma_{a + Rx}(E^\bullet)) \to \text{Hom}_R(M, \Gamma_a(E^\bullet)) \to \text{Hom}_R(M, \Gamma_a(E^\bullet_x)) \to 0$$

of complexes. On the other hand, we have the following natural isomorphism of complexes:

$$\text{Hom}_R(M, \Gamma_a(E^\bullet_x)) \cong \text{Hom}_R(M, \text{Hom}_{R_x}(R_x, \Gamma_a(R_x)(E^\bullet_x)))$$

$$\cong \text{Hom}_{R_x}(M \otimes_R R_x, \Gamma_a(R_x)(E^\bullet_x))$$

$$\cong H^0_{aR_x}(M_x, E^\bullet_x).$$

Hence the above exact sequence of complexes induces the following long exact sequence of homology modules:

$$\cdots \to H^i(\Gamma_{a + Rx}(M, E^\bullet)) \to H^i(\Gamma_a(M, E^\bullet)) \to H^i(\Gamma_{aRx}(M_x, E^\bullet_x))$$

$$\to H^{i+1}(\Gamma_{a + Rx}(M, E^\bullet)) \to \cdots.$$
Lemma 3.4 (see [4, Lemma 9.1.1]). Let $M \to N \to L$ be an exact sequence of $R$-modules such that $a \subseteq \sqrt{(0 :_RM)}$ and $a \subseteq \sqrt{(0 :_RL)}$. Then $a \subseteq \sqrt{(0 :_RN)}$.

Proposition 3.5. Let $M, N, L, K$ be finite $R$-modules.

(i) Let $R'$ be a second commutative ring and let $f : R \to R'$ be a flat homomorphism of rings. Then

$$f^b_a(M, N) \leq f^b_{aR'}(M \otimes_R R', N \otimes_R R').$$

In particular, if $S$ is a multiplicatively closed subset of $R$, then

$$f^b_a(M, N) \leq f^{S^{-1}b}_{aS^{-1}}(S^{-1}M, S^{-1}N).$$

(ii) $f^b_a(M, N) = f^b_{\sqrt{a}}(M, N) = f^b_{\sqrt{a}}(M, N)$.

(iii) Let $x \in R$. Then

$$f^b_a(M, N) = \inf\{f^b_{a+Rx}(M, N), f^b_{aRx}(M, N, N_x)\}.$$

(iv) Let $a \subseteq c$. Then

$$f^b_a(M, N) \leq f^b_c(M, N) \quad \text{and} \quad f^b_b(M, N) \leq f^b_c(M, N).$$

(v) Let $b \subseteq c$. Then

$$f^b_a(M, N) = f^b_a(M, N/\Gamma_c(N)).$$

In particular,

$$f^b_a(M, N) = f^b_a(M, N/\Gamma_b(N)).$$

(vi) Let $0 \to L \to M \to N \to 0$ be an exact sequence. Then

$$f^b_a(K, L) \geq \inf\{f^b_a(K, M), f^b_a(K, N) + 1\},$$

$$f^b_a(K, M) \geq \inf\{f^b_a(K, L), f^b_a(K, N)\},$$

$$f^b_a(K, N) \geq \inf\{f^b_a(K, L) - 1, f^b_a(K, M)\}$$

and

$$f^b_a(L, K) \geq \inf\{f^b_a(M, K), f^b_a(N, K) - 1\},$$

$$f^b_a(M, K) \geq \inf\{f^b_a(L, K), f^b_a(N, K)\},$$

$$f^b_a(N, K) \geq \inf\{f^b_a(L, K) + 1, f^b_a(M, K)\}.$$
(ix) Let $0 \to L \to M \to N \to 0$ be an exact sequence. Then

$$f^b_a(M, K) = \inf \{ f^b_a(L, K), f^b_a(N, K) \}.$$ 

(x) There exists a prime ideal $\mathfrak{p}$ in $\text{Min Supp } M$ such that $f^b_a(M, N) = f^b_a(R/\mathfrak{p}, N)$, and hence

$$f^b_a(M, N) = \inf \{ f^b_a(R/\mathfrak{p}, N) \mid \mathfrak{p} \in \text{Supp } M \}.$$ 

PROOF. (i) If $b \subseteq \sqrt{(0 :_R H^i_a(M, N))}$, then $b^i H^i_a(M, N) = 0$ for some $r \in \mathbb{N}$. Therefore

$$b^i R^i H^i_a \otimes R' (M \otimes R' N \otimes R' N) \cong b^i H^i_a(M, N) \otimes_R R' = 0.$$ 

(ii) Let $E^*$ be an injective resolution of $N$. Then, in view of [18, Proposition 2.1],

$$H^i_a(M, N) = H^i_v(\hom_{\mathbb{R}M}(M, (\mathbb{E}^*)) = H^i_v(\hom_{\mathbb{R}M}(M, (\mathbb{E}^*))) = H^i_v(M, N).$$ 

(iii) This follows from Lemmas 3.3, 3.4 and (i).

(iv) It follows from the definition that $f^b_a(M, N) \leq f^a_b(M, N)$. Also, since $R$ is Noetherian, we can use (iii) to obtain $f^b_a(M, N) \leq f^b_c(M, N)$.

(v) Since $c^i \Gamma_i(N) = 0$ for some $n \in \mathbb{N}$, we have $c^i H^i_a(M, \Gamma_i(N)) = 0$ for all $i \in \mathbb{N}_0$. Therefore $b \subseteq c \subseteq \sqrt{(0 :_R H^i_a(M, \Gamma_i(N))))$ for all $i$. Now the exact sequence

$$0 \to \Gamma_i(N) \to N \to N/\Gamma_i(N) \to 0$$

induces the long exact sequence

$$\cdots \to H^i_a(M, \Gamma_i(N)) \to H^i_a(M, N) \to H^i_a(M, N/\Gamma_i(N)) \to H^{i+1}_a(M, \Gamma_i(N)) \to \cdots .$$

Therefore, by Lemma 3.4, $b \subseteq \sqrt{(0 :_R H^i_a(M, N))}$ if and only if

$$b \subseteq \sqrt{(0 :_R H^i_a(M, N/\Gamma_i(N))}).$$

(vi) We may consider the long exact sequence

$$\cdots \to H^i_a(N, K) \to H^i_a(M, K) \to H^i_a(L, K) \to H^{i+1}_a(N, K) \to \cdots ,$$

which is obtained in [5, Lemma 2.4], and the long exact sequence

$$\cdots \to H^i_a(K, L) \to H^i_a(K, M) \to H^i_a(K, N) \to H^{i+1}_a(K, L) \to \cdots$$

and apply Lemma 3.4 to establish the assertion.

(vii) We prove, by induction on $r \in \mathbb{N}_0$, that, for any finite $R$-module $M$, if $\text{Supp } M \subseteq \text{Supp } N$ and $r \leq f^b_a(N, K)$, then $r \leq f^b_a(M, K)$. If $r = 0$ there is nothing to prove. Now suppose that $r > 0$ and assume that the assertion holds for smaller values of $r$. Suppose that $\text{Supp } M \subseteq \text{Supp } N$ and $r \leq f^b_a(N, K)$. By Gruson's theorem [22, Theorem 4.1], there exists a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M.$$
of submodules of $M$ such that $M_i/M_{i-1}$ is a homomorphic image of a direct sum of finitely many copies of $N$ for all $i = 1, \ldots, n$. On the other hand, by (vi),

$$f_a^b(M, K) \geq \inf\{f_a^b(M_1/M_0, K), \ldots, f_a^b(M_n/M_{n-1}, K)\}.$$ 

Therefore it is enough to prove that $r \leq f_a^b(M, K)$ in the case where $n = 1$. Now there exists an exact sequence

$$0 \to L \to N^\alpha \to M \to 0,$$

for some $\alpha \in \mathbb{N}$. Since $\text{Supp} \ L \subseteq \text{Supp} \ N$, the induction hypothesis implies that $r - 1 \leq f_a^b(L, K)$. Therefore, by (vi),

$$r \leq \inf\{f_a^b(L, K) + 1, f_a^b(N, K)\} \leq f_a^b(M, K).$$

(viii), (ix) and (x) are immediate by (vii). □

Next, we provide an example to show that the inequality in (vii) and (viii) may be strict.

**Example 3.6.** Let $(R, m)$ be a Gorenstein local ring with dimension $d > 0$ and $M$ be a finite $R$-module. Then $H^i_m(R) = E(R/m)$ if $i = d$ and 0 otherwise. Further, $H^i_m(R) = E(R/m)$ is not finite [4, Corollary 7.3.3], so $f_m(R) = d$. Now let $E^*$ be a minimal injective resolution of $R$. Then

$$H^i_m(M, R) = H^i(\text{Hom}_R(M, \Gamma_m(E^*))) = \begin{cases} \text{Hom}_R(M, E(R/m)) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$$

In particular,

$$H^i_m(R/m, R) = \begin{cases} R/m & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$$

and $f_m(R/m, R) = \infty > f_m(R)$. Moreover, this example shows that the following statements of Chu are not true.

(i) [5, Lemma 2.9]. Let $N$ be a finitely generated $R$-module and $M$ a nonzero cyclic $R$-module. Let $t$ be a positive integer and let $I$ be an ideal of $R$. If $H^i_I(N)$ is finitely generated for all $i < t$, then $H^i_I(N)$ is finitely generated if and only if $\text{Hom}_R(M, H^i_I(N))$ is finitely generated.

(ii) [5, Proposition 2.10]. Let the situation be as in (i). Then $H^i_I(N)$ is finitely generated for all $i < t$ if and only if $H^i_I(M, N)$ is finitely generated for all $i < t$.

## 4. Faltings’ local-global principle for the annihilation of generalised local cohomology modules

We say that the local-global principle for the annihilation of generalised local cohomology modules holds at level $r$ if the statement

$$f_a^b(M, N) > r \iff f_a^{b_R_p}(M_p, N_p) > r \quad \text{for all } p \in \text{Spec}(R)$$

holds.
is true for every choice of ideals \( a, b \) and every choice of finite \( R \)-modules \( M, N \). Since 
\[
(H^i_\alpha(M, N))_p \cong H^i_{aR_p}(M_p, N_p)
\]
for each \( p \in \text{Spec}(R) \), the above statement is equivalent to the statement
\[
f^b_\alpha(M, N) > r \iff f^b_{aR_p}(M_p, N_p) > r \quad \text{for all } p \in \text{Spec}(R).
\]

We say that the local-global principle for the annihilation of generalised local cohomology modules holds (over the ring \( R \)) if the local-global principle for the annihilation of generalised local cohomology modules holds at level \( r \) for every \( r \in \mathbb{N}_0 \). The following lemma is needed in the proof of the next proposition.

**Lemma 4.1** (see [3, Lemma 2.1] or [19, Lemma 3.1]). Let \( M \) be an \( R \)-module such that the set \( \Delta \) of all maximal members of \( \text{Ass} M \) is finite. Suppose that there exists a positive integer \( n \) such that \((a^nM)_p = 0 \) for all \( p \in \Delta \). Then \( a^nM = 0 \).

**Proposition 4.2.** The local-global principle for the annihilation of generalised local cohomology modules holds at levels \( 0, 1, 2 \).

**Proof.** Let \( 0 \leq i \leq 1 \). Assume that \( M \) and \( N \) are finite \( R \)-modules and that 
\[
f^b_{aR_p}(M_p, N_p) > i \quad \text{for all } p \in \text{Spec}(R).
\]
Since \( H^0_\alpha(M, N) \) is finite, by Corollary 2.3, we see that \( \text{Ass} H^0_\alpha(M, N) \) is finite. Therefore there exists \( n \in \mathbb{N} \) such that 
\[
(bR_p)^nH^0_\alpha(M_p, N_p) = 0 \quad \text{for all } p \in \text{Ass} H^0_\alpha(M, N).
\]
Hence \( b^nH^0_\alpha(M, N) = 0 \) by Lemma 4.1; so the local-global principle for the annihilation of generalised local cohomology modules holds at levels \( 0, 1 \).

Now let \( f^b_{aR_p}(M_p, N_p) > 2 \) for all \( p \in \text{Spec}(R) \). The above argument shows that there exists \( r \in \mathbb{N} \) such that \( b^rH^i_\alpha(M, N) = 0 \) for \( i = 0, 1 \). Since \( f^b_\alpha(M, N) = f^b_\alpha(M, N/\Gamma_b(N)) \), we can assume without loss of generality that \( \Gamma_b(N) = 0 \); and so \( f\text{-grad}(a, b, N) \geq 1 \). Therefore, by Theorem 2.2, \( H^i_\alpha(M, N) = \text{Hom}_R(R/b', H^i_\alpha(M, N)) \) is finite. Now, we can use Corollary 2.3 and Lemma 4.1 to obtain that \( f^b_\alpha(M, N) > 2 \).

The next theorem is concerned with the local-global principle for the annihilation of generalised local cohomology modules. The following lemma is of assistance in the proof of that theorem.

**Lemma 4.3** [8, Theorem 3.1]. Let \( M, N \) be finite \( R \)-modules and let \( x_1, \ldots, x_n \) be an \( a \)-filter regular \( N \)-sequence in \( a \). Then the following statements hold.

(i) \( H^i_\alpha(M, N) \cong H^i_{(x_1,\ldots,x_n)}(M, N) \) for all \( i < n \).

(ii) If \( \text{proj dim}_R(M) = d < \infty \) and \( L \) is projective, then
\[
H^{i+n}_\alpha(M \otimes_R L, N) \cong H^i_\alpha(M, H^n_{(x_1,\ldots,x_n)}(L, N))
\]
for all \( i \geq d \).

**Theorem 4.4.** Let \( M \) and \( N \) be finite \( R \)-modules and let \( b \subseteq a \).

(i) \( f^b_\alpha(M, N) \geq \inf\{f^b_\alpha(M, N), \text{f-grad} (a, b, N) + 1\} \). In particular, \( f^b_\alpha(M, N) = f^b_\alpha(M, N) \) whenever \( f^b_\alpha(M, N) \leq \text{f-grad} (a, b, N) + 1 \).
(ii) Assume that \( r \leq \text{f-grad} (a, b, N) + 1 \). Then
\[
f^b_a(M, N) > r \iff f^b_{aR_p}(M_p, N_p) > r \quad \text{for all } v \in \text{Spec}(R).
\]

(iii) If \( \text{Supp} N/bN \subseteq V(a) \), then, for all \( r \in \mathbb{N}_0 \),
\[
f^b_a(M, N) > r \iff f^b_{aR_p}(M_p, N_p) > r \quad \text{for all } v \in \text{Spec}(R).
\]

(iv) Faltings’ local-global principle for the finiteness of generalised local cohomology modules holds, that is, for any positive integer \( r \), \( H^{r}_{aR_p}(M_p, N_p) \) is finite for all \( i \leq r \) and for all \( v \in \text{Spec}(R) \) if and only if \( H^i_a(M, N) \) is finite for all \( i \leq r \).

**Proof.**

(i) Set \( g = \text{f-grad} (a, b, N) \). If \( f^b_a(M, N) \leq g + 1 \), then, for any \( i < f^b_a(M, N) \), we have \( H^i_a(M, N) = H^i_b(M, N) \) by Lemma 4.3(i); and hence \( b \subseteq \sqrt{(0 : H^i_b(M, N))} \).

Then by Proposition 3.5(iv), \( H^i_a(M, N) \) is finite for all \( i < f^b_a(M, N) \) and hence, by Proposition 3.5(iv), \( f^b_a(M, N) = f_a(M, N) \).

Therefore we may assume that \( f^b_a(M, N) > g + 1 \). Using the same argument as above, we see that \( H^i_a(M, N) \) is finite for all \( i < g \). Therefore, by Theorem 2.2, \( \text{Hom}_R(R/b^a, H^i_a(M, N)) \) is finite for all \( \alpha \in \mathbb{N} \). On the other hand, by hypothesis \( b^a H_a^i(M, N) = 0 \) for some \( \alpha \in \mathbb{N} \). Thus \( H^i_a(M, N) \) is finite and \( f_a(M, N) \geq g + 1 \).

(ii) Suppose that \( r \leq \text{f-grad} (a, b, N) + 1 \) and \( f^b_{aR_p}(M_p, N_p) > r \) for all \( v \in \text{Spec}(R) \). If \( f^b_a(M, N) \leq r \), then by (i), \( f_a(M, N) = f^b_a(M, N) \). So by Corollary 2.3, \( \text{Ass } H^i_a(M, N) \) is finite. This is a contradiction in view of Lemma 4.1. Hence \( f^b_a(M, N) > r \).

(iii) Suppose that \( \text{Supp} N/bN \subseteq V(a) \). Then \( f\text{-grad} (a, b, N) = \infty \). Thus (iii) is an immediate consequence of (ii).

(iv) This is immediate by (iii) and Proposition 3.1.

\( \square \)

5. Local-global principle for the Artinianness of generalised local cohomology modules

Let \( M \) be a finite \( R \)-module. In [20], Tang proved that, for any integer \( n \), \( H^i_a(M) \) is Artinian for all \( i < n \) if and only if \( H^i_a(M)_v \) is Artinian for all \( i < n \) and for all \( v \in \text{Spec}(R) \). In Theorem 5.3, we establish the above result for generalised local cohomology modules. The corollary to the following theorem is needed in the proof of Theorem 5.3.

**Theorem 5.1** [8, Theorem 4.2]. Let \( \mathcal{M} \) be the set of all finite subsets of \( \text{max}(R) \). Then
\[
\sup_{A \in \mathcal{M}} \text{f-grad} \left( \bigcap_{m \in A} m, a + \text{Ann } M, N \right)
\]
\[
= \inf \{ i \in \mathbb{N}_0 | H^i_a(M, N) \text{ is not Artinian} \}
\]
\[
= \inf \{ i \in \mathbb{N}_0 | \text{Supp } H^i_a(M, N) \nsubseteq \text{max}(R) \}
\]
\[
= \inf \{ i \in \mathbb{N}_0 | \text{Supp } H^i_a(M, N) \nsubseteq A \text{ for all } A \in \mathcal{M} \}.
\]
Corollary 5.2. Let \( M \) and \( N \) be finite \( R \)-modules. Then
\[
\inf \{ i \in \mathbb{N}_0 \mid H^i_a(M, N) \text{ is not Artinian} \} = \inf \{ i \in \mathbb{N}_0 \mid \dim \Ext^i_R(M/\mathfrak{a}M, N) > 0 \}.
\]

Proof. Let \( n \in \mathbb{N}_0 \). By Theorem 5.1, \( H^i_a(M, N) \) is an Artinian \( R \)-module for all \( i \leq n \) if and only if \( n < f\text{-grad } (\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_t, \mathfrak{a} + \Ann M, N) \) for some maximal ideals \( \mathfrak{m}_1, \ldots, \mathfrak{m}_t \) of \( R \). Also, by the facts mentioned at the beginning of Section 2, this is equivalent to \( \Supp \Ext^i_R(M/\mathfrak{a}M, N) \subseteq \{ \mathfrak{m}_1, \ldots, \mathfrak{m}_t \} \) for some maximal ideals \( \mathfrak{m}_1, \ldots, \mathfrak{m}_t \) of \( R \) and for all \( i \leq n \). \( \square \)

Theorem 5.3. Let \( M, N \) be finite \( R \)-modules and let \( n \) be a positive integer. Then the following statements are equivalent.

(i) \( H^i_a(M, N) \) is Artinian for all \( i < n \).
(ii) \( H^i_{\mathfrak{a}R}(M_p, N_p) \) is Artinian for all \( i < n \) and for all \( p \in \Spec(R) \).

Proof. It is clear that \( \dim \Ext^i_R(M/\mathfrak{a}M, N) = 0 \) for all \( i < n \) if and only if \( \dim \Ext^i_R(M/\mathfrak{a}M, N)_p = 0 \) for all \( i < n \) and for all prime ideals \( p \) of \( R \). Therefore the assertion follows from Corollary 5.2. \( \square \)

Corollary 5.4. Let \( M, N \) be finite \( R \)-modules and let \( n \) be a positive integer. Then the following statements are equivalent.

(i) \( H^i_{\mathfrak{g}}(M, N) \) has finite length for all \( i < n \).
(ii) \( H^i_{\mathfrak{a}R}(M_p, N_p) \) has finite length for all \( i < n \) and for all \( p \in \Spec(R) \).

Proof. This is immediate by Theorems 5.3 and 2.2. \( \square \)

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References


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