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MEASURES OF NONCOMPACTNESS IN A SOBOLEV SPACE AND INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

The aim of this paper is to introduce a new measure of noncompactness on the Sobolev space $W^{n,p}[0,T]$. As an application, we investigate the existence of solutions for some classes of functional integro-differential equations in this space using Darbo's fixed point theorem.

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1. Introduction

Sobolev spaces play a prominent role in modern analysis, in particular, in the theory of partial differential equations and its applications in mathematical physics. They form an indispensable tool in approximation theory, spectral theory and differential geometry. The theory of these spaces is also of interest in itself.

Integro-differential equations (IDE) feature in many fields of biological science, applied mathematics, physics and other disciplines, such as the theory of elasticity, biomechanics, electromagnetic, electrodynamics, fluid dynamics, heat and mass transfer and oscillating magnetic fields (see, for example, [11, 14, 16]). A range of numerical methods have been applied to the study of IDE. Some examples are the tau method, direct methods, collocation methods, Runge–Kutta methods, wavelet methods and spline approximation (see, for example, [5, 9, 10, 17, 20, 23]).

In 1930, Kuratowski [18] introduced the concept of measure of noncompactness. Later, Banaś and Goebel [7] generalised this concept axiomatically, which is more convenient in applications. The tool of measure of noncompactness has been used in the theory of operator equations in Banach spaces. The fixed point theorems derived from them have many applications. There is considerable literature devoted to this subject (see, for example, [6, 8, 12, 15, 16, 21, 22]). The principal application of measures of noncompactness in fixed point theory is through Darbo's fixed point theorem [7]. This yields a tool to investigate the existence and behaviour of solutions of many classes of integral equations such as those of Volterra, Fredholm and Uryson types (see [1, 2, 12, 13]).

Motivated by these investigations and the measures of noncompactness considered in [7], we introduce a new measure of noncompactness on the Sobolev space $W^{n,p}[0,T]$. Then we study the problem of existence of solutions of the functional integro-differential equation

$$x^{(n+1)}(t) = f\Big(t, x(\xi(t)), x'(\xi(t)), \dots, x^{(n)}(\xi(t)), \int_0^{\beta(t)} k(t,s)x(s)\,ds\Big) \tag{1.1}$$

in the Sobolev space $W^{n,p}[0,T]$ where $t \in [0,T]$. In our considerations, we apply Darbo's fixed point theorem associated with this new measure of noncompactness.

2. Preliminaries

In this section, we recall some basic facts concerning measures of noncompactness, defined axiomatically in Definition 2.1 below. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+ = [0,+\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x,r)$ denote the closed ball centred at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0,r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and Conv X the closure and the closed convex hull of X, respectively. Denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets of E.

Definition 2.1 [7]. A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is called a measure of noncompactness on E if it satisfies the following conditions:

(1) the family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$;
(2) $X \subset Y \Longrightarrow \mu(X) \le \mu(Y)$;
(3) $\mu(\overline{X}) = \mu(X)$;
(4) $\mu(\mathrm{Conv}\, X) = \mu(X)$;
(5) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$;
(6) if $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots,$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

We recall the well-known fixed point theorem of Darbo type.

Theorem 2.2 [7]. *Let Ω be a nonempty, bounded, closed and convex subset of the space E and μ a measure of noncompactness on E. Let $F : \Omega \longrightarrow \Omega$ be a continuous mapping such that there exists a constant $k \in [0,1)$ with the property*

$$\mu(FX) \le k\mu(X)$$

for any nonempty subset X of Ω. Then F has a fixed point in the set Ω.

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We introduce a measure of noncompactness on the space $L^p[0,T]$. In order to define this measure, take an arbitrary set X of $\mathfrak{M}_{L^p[0,T]}$. For $x \in X$ and $\varepsilon > 0$, set

$$\omega(x,\varepsilon) = \sup\{\|\tau_h x - x\|_p : |h| < \varepsilon\},$$
$$\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in X\},$$

where

$$\tau_h x(t) = \begin{cases} x(t+h) & 0 \le t+h \le T, \\ 0 & \text{otherwise}, \end{cases}$$

for all $t, h \in [0,T]$. Then define

$$\omega_0(X) = \lim_{\varepsilon \longrightarrow 0} \omega(X,\varepsilon).$$

The mapping $\omega_0 = \omega_0(X)$ is a measure of noncompactness on the space $L^p[0,T]$ [7].

3. Main results

In this section, we introduce a measure of noncompactness on the Sobolev space $W^{n,p}[0,T]$. The Sobolev space $W^{n,p}([0,T])$ is defined to consist of those measurable functions f which, together with all their distributional derivatives $f^{(k)}$ of order $k \le n$, belong to $L^p[0,T]$ with the norm

$$\|f\|_{n,p} = \max_{0 \le k \le n} \|f^{(k)}\|_p,$$

where $f^{(0)} = f$.

THEOREM 3.1. *Suppose $1 \le n < \infty$ and X is a bounded subset of $W^{n,p}[0,T]$. Set $X^{(0)} = X$ and $X^{(k)} = \{x^{(k)} : x \in X\}$. Then $\mu : \mathfrak{M}_{W^{n,p}[0,T]} \longrightarrow \mathbb{R}_+$ given by*

$$\mu(X) = \max_{0 \le k \le n} \omega_0(X^{(k)})$$

is a measure of noncompactness on $W^{n,p}[0,T]$.

The proof relies on the following observations.

LEMMA 3.2 [3]. *Suppose $\mu_1, \mu_2, \ldots, \mu_n$ are measures of noncompactness on Banach spaces $E_1, E_2, \ldots, E_n$ respectively. Moreover, assume that the function $F : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ is convex and $F(x_1, \ldots, x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, \ldots, n$. Then*

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

defines a measure of noncompactness on $E_1 \times E_2 \times \cdots \times E_n$, where X_i denotes the natural projection of X into E_i for $i = 1, 2, \ldots, n$.

LEMMA 3.3 [19]. *For $i = 1, 2$, let $(E_i, \|.\|_i)$ be Banach spaces and let $L : E_1 \longrightarrow E_2$ be a one-to-one, continuous linear operator of E_1 onto E_2. If μ_2 is a measure of noncompactness on E_2, define, for $X \in \mathfrak{M}_{E_1}$,*

$$\widetilde{\mu}_2(X) := \mu_2(LX).$$

Then $\widetilde{\mu}_2$ is a measure of noncompactness on E_1.

Proof of Theorem 3.1. First, consider $E = (L^p[0,T])^{n+1}$ equipped with the norm

$$\|(x_1,\ldots,x_n,x_{n+1})\| = \max_{1\le i\le n+1} \|x_i\|_p.$$

Set $F(x_1,\ldots,x_{n+1}) = \max_{1\le i\le n+1} x_i$ for any $(x_1,\ldots,x_{n+1}) \in \mathbb{R}_+^{n+1}$. All the conditions of Lemma 3.2 are satisfied, so

$$\mu_2(X) := \max_{1\le i\le n+1} \omega_0(X_i)$$

defines a measure of noncompactness on the space E, where X_i denotes the natural projection of X for $i = 1,2,\ldots,n+1$. Define the operator $L : W^{n,p}[0,T] \longrightarrow E$ by

$$L(x) = (x, x', x'', x^{(3)}, \ldots, x^{(n)}).$$

Obviously, L is a one-to-one and continuous linear operator. We show that $L(W^{n,p}[0,T])$ is closed in E. To do this, choose $\{x_n\} \subset W^{n,p}[0,T])$ such that $L(x_n)$ is a Cauchy sequence in E. Thus, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $k, m > N$,

$$\|L(x_k - x_m)\| < \varepsilon.$$

So,

$$\begin{aligned}\|x_k - x_m\|_{n,p} &= \max_{0\le i\le n} \|x_k^{(i)} - x_m^{(i)}\|_p = \|(x_k - x_m, x'_k - x'_m, \ldots, x_k^{(n)} - x_m^{(n)})\| \\ &= \|L(x_k - x_m)\| < \varepsilon.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence of $W^{n,p}[0,T]$ and there exists $x \in W^{n,p}[0,T]$ such that $x_n \longrightarrow x$. Since L is continuous, $L(x_n) \longrightarrow L(x)$. This implies that $Y = L(W^{n,p}[0,T])$ is closed. Thus, the operator $L : W^{n,p}[0,T] \longrightarrow Y$ is a one-to-one and continuous linear operator of $W^{n,p}[0,T]$ onto Y. Since Y is a closed subspace of X, μ_2 is a measure of noncompactness on Y. Hence, for $X \in \mathfrak{M}_{W^{n,p}[0,T]}$,

$$\widetilde{\mu_2}(X) = \mu_2(LX) = \max_{0\le k\le n} \omega_0(X^{(k)}) = \mu(X).$$

Now, using Lemma 3.3, the proof is complete. □

Corollary 3.4. *Let $\mathcal{F}$ be a bounded subset of $W^{n,p}[0,T]$. Then the following two conditions are equivalent:*

(i) *$\mathcal{F}$ is a totally bounded subset of $C^n[a,b]$.*

(ii) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|\tau_h f^{(k)} - f^{(k)}\|_p < \varepsilon$$

for all $0 \le k \le n$, $h \in [a,b]$ with $|h| < \delta$ and $f \in \mathcal{F}$.

Proof. Suppose $\mathcal{F}$ satisfies condition (i). Then $\mu(\mathcal{F}) = 0$ and, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\omega(\mathcal{F}^{(k)}, \delta) < \varepsilon$$

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for all $0 \le k \le n$. Thus, for any $0 \le k \le n$, $f \in \mathcal{F}$ and $h \in [a, b]$ such that $|h| < \delta$,

$$\|\tau_h f^{(k)} - f^{(k)}\|_p \le \omega(f^{(k)}, \delta) \le \omega(\mathcal{F}^{(k)}, \delta) \le \varepsilon,$$

and condition (ii) is satisfied. Conversely, assume that $\mathcal{F}$ satisfies condition (ii). Take an arbitrary $\varepsilon > 0$. By condition (ii), there exists $\delta > 0$ such that

$$\|\tau_h f^{(k)} - f^{(k)}\|_p < \varepsilon,$$

for all $0 \le k \le n$ and $h \in [0, T]$ with $|h| < \delta$, so we have

$$\max_{0\le k\le n} \omega(f^{(k)}, \delta) = \max_{0\le k\le n} \sup\{\|\tau_h f^{(k)} - f^{(k)}\|_p : |h| \le \delta\} < \varepsilon$$

for all $f \in \mathcal{F}$, and so

$$\max_{0\le k\le n} \omega(\mathcal{F}^{(k)}, \delta) \le \varepsilon.$$

Therefore, $\mu(\mathcal{F}) = 0$ and condition (i) is satisfied. □

4. Existence of solutions for some classes of integro-differential equations

In this section we study the existence of solutions for Equation (1.1).

DEFINITION 4.1. A function $f : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to have the Carathéodory property if:

(1) for all $x \in \mathbb{R}^n$, the function $t \to f(t, x)$ is measurable on $[0, T]$;
(2) for almost all $t \in [0, T]$, the function $x \to f(t, x)$ is continuous on $\mathbb{R}^n$.

LEMMA 4.2. *Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ and $1 \le p \le \infty$. If $\{f_n\}$ is convergent to $f \in L^p(\Omega))$ in the L_p-norm, then there is a subsequence $\{f_{n_k}\}$ which converges to f almost everywhere and there is $g \in L_p(\Omega)$, $g \ge 0$, such that*

$$|f_{n_k}(x)| \le g(x) \quad \textit{for almost all } x \in \Omega.$$

THEOREM 4.3 (Minkowski's inequality for integrals, [4]). *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces and f is an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$. If $f \ge 0$ and $1 \le p < \infty$, then*

$$\bigg(\int\bigg(\int f(x, y)\, d\nu(y)\bigg)^p d\mu(x)\bigg)^{1/p} \le \int\bigg(\int f(x, y)^p\, d\mu(x)\bigg)^{1/p} d\nu(y).$$

We will consider the Equation (1.1) under the following assumptions:

(i) $\xi, \beta : [0, T] \longrightarrow [0, T]$ are measurable functions.
(ii) $f : [0, T] \times \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions and there exists a function $a \in L^q[a, b]$ such that

$$|f(t, x_0, x_1, \ldots, x_{n+1})| \le a(t) \max_{0\le i\le n+1} |x_i|. \tag{4.1}$$

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(iii) $k : [0, T] \times [0, T] \longrightarrow \mathbb{R}$ is a $[0, T] \times [0, T]$-measurable function such that

$$\operatorname*{ess\,sup}_{s\in[0,T]} \int_0^T |k(t, s)|\, dt \le 1 \quad \text{and} \quad \operatorname*{ess\,sup}_{t\in[0,T]} \int_0^T |k(t, s)|\, ds \le 1.$$

(iv) $D := \max\{T^{(n+1)/p}/n!(pn + 1)^{1/p}, T^{1/p}\}\, \|a\|_q < 1$.

Remark 4.4. Under hypothesis (iii), the linear operator $K : L^p[0, T] \to L^p[0, T]$ defined by

$$(Kx)(t) = \int_0^{\beta(t)} k(t, s)x(s)\, ds$$

is a continuous linear operator and $\|Kx\|_p \le \|x\|_p$.

Theorem 4.5. *Under assumptions* (i)–(iv)*, the equation* (1.1) *has at least one solution in the space* $W^{n,p}[0, T]$.

Proof. The differential equation (1.1) has at least one solution in the space $W^{n+1,p}[0, T]$ if and only if the nonlinear integral equation

$$u(t) = p(t) + \frac{1}{n!} \int_0^t (t - s)^n f(s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Kx(s))\, ds$$

has at least one solution in the space $W^{n,p}[0, T]$ where

$$p(t) = (t - T) \sum_{k=0}^{n} \frac{x_k}{k!} t^n - \frac{t}{T} \sum_{k=0}^{n} \frac{y_k}{k!} (t - T)^n.$$

We define the operator $F : W^{n,p}[0, T] \longrightarrow W^{n,p}[0, T]$ by

$$Fx(t) = p(t) + \frac{1}{n!} \int_0^t (t - s)^n f(s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Kx(s))\, ds. \tag{4.2}$$

First, by considering the Carathéodory conditions, we infer that Fx is measurable for any $x \in W^{n,p}[0, T]$. Also, for any $t \in \mathbb{R}_+$ and $1 \le k \le n$, Fx has measurable derivative $d^k(Fx)(t)/dt^k$ of order k $(1 \le k \le n)$ given by

$$p^{(k)}(t) + \frac{1}{(n - k)!} \int_0^t (t - s)^{n-k} f(s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Kx(s))\, ds.$$

Using conditions (i)–(iv), for arbitrarily fixed $t \in [0, T]$,

$$\begin{aligned}
&\Big(\int_0^T \Big| \frac{1}{n!} \int_0^t (t - s)^n f(s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Kx(s))\, ds \Big|^p dt \Big)^{1/p} \\
&\quad\le \Big(\int_0^T \Big| \frac{1}{n!} \int_0^T \chi_{[0,t]}(s)(t - s)^n f(s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Kx(s))\, ds \Big|^p dt \Big)^{1/p} \\
&\quad\le \frac{1}{n!} \int_0^T \Big(\int_0^T |\chi_{[0,t]}(s)(t - s)^n f(s, x(\xi(s)), x'(\xi(s)), \dots, x^{(n)}(\xi(s)), Kx(s))|^p\, dt \Big)^{1/p} ds \\
&\quad\le \frac{T^{(np+1)/p}}{n!(pn + 1)^{1/p}} \int_0^T |a(s)| \max\{|x(\xi(s))|, |x'(\xi(s))|, \dots, |x^{(n)}(\xi(s))|, |Kx(s)|\}\, ds.
\end{aligned}$$

Thus, from (4.2),

$$\|Fx\|_p \leq \|p\|_p + \frac{T^{(np+1)/p}}{n!(pn+1)^{1/p}}\|a\|_q \max\{\|x(s)\|_p, \|x'(s)\|_p, \ldots, \|x^{(n)}\|_p, \|Kx\|_p\}$$

and similarly,

$$\left\|\frac{d^k(Fx)}{dt^k}\right\|_p \leq \|p\|_p + \tfrac{T^{((n-k)p+1)/p}}{(n-k)!(p(n-k)+1)^{1/p}}\|a\|_q \max\{\|x\|_p, \|x'\|_p, \ldots, \|x^{(n)}\|_p, \|Kx\|_p\}.$$

Hence,

$$\|Fx\|_w \leq \|p\|_p + \max\left\{\frac{T^{(n+1)/p}}{n!(pn+1)^{1/p}}, T^{1/p}\right\}\|a\|_q\|x\|_w. \tag{4.3}$$

From the inequality (4.3), F transforms the ball $\bar{B}_{r_0}$ into itself where $r_0 = \|p\|_p/(1-D)$.

Next, we show that the map F is continuous. Let $\{x_m\}$ be an arbitrary sequence in $W^{n,p}[0,T]$ which converges to $x \in W^{n,p}[0,T]$ in the $W^{n,p}[0,T]$-norm. Since the Volterra integral operator K generated by k maps (continuously) the space $L^p[0,T]$ into itself, Kx_m converges to Kx. From Lemma 4.2, there is a subsequence $\{x_{m_k}\}$ which converges to x almost everywhere, such that $\{x_{m_k}^{(k)}\}$ converges to $x^{(k)}$ almost everywhere for all $1 \leq k \leq n$, $\{Kx_{m_k}\}$ converges to Kx almost everywhere and there is $h \in L^p[0,T]$, $h \geq 0$, such that

$$\max\{|x_{m_k}(\xi(t))|, |x'_{m_k}(\xi(t))|, |x''_{m_k}(\xi(t))|, \ldots, |x_{m_k}^{(n)}(\xi(t))|, |Kx_{m_k}(t)|\} \leq h(t) \tag{4.4}$$

almost everywhere on $[0,T]$. Since $x_{m_k} \to x$ almost everywhere in $[0,T]$ and f satisfies the Carathéodory conditions,

$$f(s, x_{m_k}(\xi(s)), \ldots, x_{m_k}^{(n)}(\xi(s)), Kx_{m_k}(s)) \longrightarrow f(s, x(\xi(s)), \ldots, x^{(n)}(\xi(s)), Kx(s)) \tag{4.5}$$

for almost all $t \in [0,T]$. From inequalities (4.1) and (4.4),

$$|f(s, x_{m_k}(\xi(s)), \ldots, x_{m_k}^{(n)}(\xi(s)), Kx_{m_k}(s))| \leq a(s)h(s) \text{ almost everywhere on } [0,T]. \tag{4.6}$$

From Lebesgue's Dominated Convergence theorem, (4.5) and (4.6) yield

$$\begin{aligned}\int_0^t (t-s)^n f(s, x_{m_k}(\xi(s)), \ldots, x_{m_k}^{(n)}(\xi(s)), Kx_{m_k}(s))\,ds \\ \longrightarrow \int_0^t (t-s)^n f(s, x(\xi(s)), \ldots, x^{(n)}(\xi(s)), Kx(s))\,ds\end{aligned} \tag{4.7}$$

for almost all $t \in [0,T]$. Inequality (4.6) implies that

$$\begin{aligned}|F(x_{m_k})(t)| &\leq \left|\frac{1}{n!}\int_0^t (t-s)^n f(s, x_{m_k}(s), \ldots, x_{m_k}^{(n)}(s), Kx_{m_k}(s))\,ds\right| \\ &\leq \frac{1}{n!}\left|\int_0^t (t-s)^n a(s)h(s)\,ds\right|\end{aligned} \tag{4.8}$$

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for almost all $t \in [0, T]$. From the assumptions on a,

$$\begin{aligned}\Big(\int_0^T \Big|\int_0^t (t-s)^n a(s)h(s)\,ds\Big|^p dt\Big)^{1/p} &\le \int_0^T \Big(\int_0^T |(t-s)^n a(s)h(s)\,dt|^p\Big)^{1/p} ds\\ &\le \frac{T^{(np+1)/p}}{n!(pn+1)^{1/p}}\|a\|_q\|h\|_p. \qquad (4.9)\end{aligned}$$

From inequalities (4.7), (4.8) and (4.9) and Lebesgue's dominated Convergence theorem,

$$\|Fx_{m_k} - Fx\|_{L^p} \longrightarrow 0.$$

Since any sequence $\{x_m\}$ converging to x in L^p has a subsequence $\{x_{m_k}\}$ such that $Fx_{m_k} \longrightarrow Fx$ in L^p, we conclude that $F : L^p[0, T] \longrightarrow L^p[0, T]$ is a continuous operator. By a similar argument, $d^k(Fx)/dt^k : L^p[0, T] \longrightarrow L^p[0, T]$ is a continuous operator. Thus, $F : W^{n,p}[0, T] \longrightarrow W^{n,p}[0, T]$ is a continuous operator.

Finally, let X be a nonempty and bounded subset of $\bar{B}_{r_0}$ and assume that $\varepsilon > 0$ is an arbitrary constant. Let $h \in [0, T]$, with $|h| \le \varepsilon$ and $x \in X$. Set

$$g(s) = f(s, x(\xi(s)), x'(\xi(s)), \ldots, x^{(n)}(\xi(s)), Kx(s))$$

and

$$m(s) = \max\{|x(\xi(s))|, |x'(\xi(s))|, \ldots, |x^{(n)}(\xi(s))|, |Kx(s)|\}.$$

Then,

$$\begin{aligned}\|\tau_h Fx - Fx\|_p &\le \|\tau_h p - p\|_p + \frac{1}{n!}\Big(\int_0^T \Big|\int_0^t [(t-s)^n - (t+h-s)^n]g(s)\,ds\Big|^p dt\Big)^{1/p}\\ &\quad + \frac{1}{n!}\Big(\int_0^T \Big|\int_t^{t+h} (t+h-s)^n g(s)\,ds\Big|^p dt\Big)^{1/p}\\ &\le \|\tau_h p - p\|_p + \frac{1}{n!}\Big(\int_0^T \Big|\int_0^t hn(2t+h)^n a(s)m(s)\,ds\Big|^p dt\Big)^{1/p}\\ &\quad + \frac{1}{n!}\Big(\int_0^T \Big|\int_t^{t+h} (3T)^n a(s)m(s)\,ds\Big|^p dt\Big)^{1/p}\\ &\le \|\tau_h p - p\|_p + \frac{nh}{n!}\int_0^T a(s)m(s)\Big(\int_0^T |(2t+h)^n|^p dt\Big)^{1/p} ds\\ &\quad + \frac{3T^n}{n!}\int_0^T a(s)m(s)\Big(\int_0^T \chi_{[t,t+h]}(s)\,dt\Big)^{1/p} ds\\ &\le \omega(p,\varepsilon) + \frac{nhT(2T+h)^n}{n!}\|a\|_q \max\{\|x\|_p, \|x'\|_p, \ldots, \|x^{(n)}\|_p, \|Kx\|_p\}\\ &\quad + \frac{3T^{n+1}h}{n!}\|a\|_q \max\{\|x\|_p, \|x'\|_p, \ldots, \|x^{(n)}\|_p, \|Kx\|_p\}\\ &\le \omega^T(p,\varepsilon) + \Big(\frac{nT(2T+\varepsilon)^n}{n!} + \frac{3T^{n+1}}{n!}\Big)\|a\|_q r_0\varepsilon \qquad (4.10)\end{aligned}$$

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and similarly,

$$\left\|\tau_h \frac{fd^k(Fx)}{dt^k} - \frac{d^k(Fx)}{dt^k}\right\|_p \leq \omega(p^{(k)}, \varepsilon) + \left(\frac{nT(2T+\varepsilon)^{n-k}}{(n-k)!} + \frac{3T^{n-k+1}}{(n-k)!}\right)\|a\|_q r_0 \varepsilon. \quad (4.11)$$

Since x was an arbitrary element of X in (4.10) and (4.11), this yields

$$\omega(F(X), \varepsilon) \leq \omega^T(p, \varepsilon) + \left(\frac{nT(2T+\varepsilon)^n}{n!} + \frac{3T^{n+1}}{n!}\right)\|a\|_q r_0 \varepsilon$$

and

$$\omega([F(X)]^{(k)}, \varepsilon) \leq \omega(p^{(k)}, \varepsilon) + \left(\frac{nT(2T+\varepsilon)^{n-k}}{(n-k)!} + \frac{3T^{n-k+1}}{(n-k)!}\right)\|a\|_q r_0 \varepsilon$$

for all $1 \leq k \leq n$. Since $\{p\}$ is a compact set, $\omega(p, \varepsilon) \longrightarrow 0$ and $\omega(p^{(i)}, \varepsilon) \longrightarrow 0$. Therefore,

$$\omega_0(F(X)) = 0,$$
$$\omega_0([F(X)]^{(k)}) = 0$$

and, finally,

$$\max_{0\leq k\leq n} \omega_0([F(X)]^{(k)}) \leq \lambda \max_{0\leq k\leq n} \omega_0(X^{(k)}),$$

with $\lambda = 0$. From Theorem 2.2, the operator F has a fixed point x in $\bar{B}_{r_0}$ and the functional integral-differential equation (1.1) has at least one solution in $W^{n,p}[0, T]$. □

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