SKELETON C*-SUBALGEBRAS

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ABSTRACT. We study skeleton C*-subalgebras of a given C*-algebra. We show that if A is a unital (non-unital but σ-unital) simple C*-algebra, M is any unital (non-unital) matroid C*-algebra, then A contains a skeleton C*-subalgebra B with a quotient which is isomorphic to M. Other results for skeleton C*-subalgebras are also obtained. Applications of these results to the structure of quasi-multipliers and perturbations of C*-algebras are given.

1. Introduction. Matrix algebras $\mathcal{M}_n$, the C*-algebras of $n \times n$ matrices over $\mathbb{C}$, and $\mathcal{K}$, the C*-algebra of compact operators on an infinite dimensional, separable Hilbert space are often called elementary C*-algebras for the obvious reasons. Matroid C*-algebras may be viewed as a generalization of elementary C*-algebras. Though non-elementary matroid C*-algebras are quite different (they are antiliminal, for instance) from elementary ones, they inherit many properties from elementary C*-algebras. They are “matroid”. Next, of course, are (simple) AF C*-algebras. The class of AF C*-algebras is one of the best understood classes of C*-algebras. They have a rich but manageable structure of projections and provide many interesting and important examples. The reason that AF C*-algebras are better understood is that they are approximately finite dimensional and therefore “matrix-like”.

In [20] and [25], fundamental approximate identities were studied. For example, S. Zhang ([25]) showed that every $\sigma$-unital (non-unital) simple C*-algebra with real rank zero has a fundamental approximate identity. The existence of such an approximate identity provides some “matrix-like” structure inside the C*-algebra. For example, we showed in [20] that a C*-algebra with fundamental approximate identity has a “skeleton” algebra with a quotient isomorphic to $\mathcal{K}$. In this note we introduce formally the concept of “skeleton”:

DEFINITION 1.1. Let A be a C*-algebra. A C*-subalgebra B is called a skeleton C*-subalgebra if the hereditary C*-subalgebra generated by B is A.

It should be noted that if A is unital, A has a skeleton C*-subalgebra which is isomorphic to $\mathbb{C}$. Therefore, we do not search for a trivial skeleton but for a rich skeleton with nice properties. We will show that if A is a $\sigma$-unital (non-unital) simple C*-algebra then for any unital (non-unital) matroid C*-algebra $\mathcal{M}$, there is a skeleton C*-subalgebra B of A such that B has a quotient which is isomorphic to $\mathcal{M}$. This shows that every $\sigma$-unital

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simple $C^*$-algebra has a “matrix-like” structure. For a simple $C^*$-algebra $A$ with real rank zero, stable rank one and unperforated $K_0(A)$, there is a simple $AF$ skeleton $B$ of $A$ such that $K_0(A) = K_0(B)$ and for every projection $p$ in $A$ there is a projection $q$ in $B$ such that $p \sim q$ (in the sense of Murray and von Neumann). Applications of these results are given in Section 3.

1.2. Let $A$ be a $C^*$-algebra, $a, b \in A$. We write (see [10]) $a \prec b$ if there are $x, y \in A$ such that $a = xby$. If $a, b \in A_+$, $a \prec b$, then, by [10, 1.7], there is $z \in A$ such that $z^*z = a$, $zz^* \in \text{Her}(b)$, the hereditary $C^*$-algebra generated by $b$.

1.3. Given $\varepsilon > 0$, let $f_\varepsilon$ be the continuous function on $\mathbb{R}$ defined by

$$f_\varepsilon(t) = \begin{cases} 0 & t \leq \frac{\varepsilon}{2} \\ \frac{2}{\varepsilon} (t - \frac{\varepsilon}{2}) & \frac{\varepsilon}{2} < t < \varepsilon \\ 1 & t \geq \varepsilon. \end{cases}$$

1.4. Given $z$ in $A$ with polar decomposition (in $A^{**}$) $z = u|z|$ and $\varepsilon > 0$ we know from [10, 1.3] that $uf_\varepsilon(|z|)$ is in $A$. For any $x \in \text{Her}(|z|)$,

$$\|uf_\varepsilon(|z|)x - ux\| \leq \|f_\varepsilon(|z|)x - x\| \to 0$$

as $\varepsilon \to 0$. Therefore, $ux \in A$ for any $x \in \text{Her}(|z|)$.

In fact the mapping $\varphi$ defined by

$$\varphi(x) = uxu^*$$

is an isomorphism from $\text{Her}(|z|)$ onto $\text{Her}(|z^*|)$ (see [10, 1.7]). If $a, b \in A_+$, we write $a \sim_\phi b$ if there is $z \in A$ such that $z^*z = a$, $zz^* = b$. If $a \sim_\phi b$, then there is a partial isometry $u \in A^{**}$, where $z = u|z|$ is the polar decomposition, such that the mapping $\varphi = uaxu^*$ is an isomorphism from $\text{Her}(a)$ onto $\text{Her}(b)$. Moreover, if $a' \in \text{Her}(a)_+$, $b' \in \text{Her}(b)_+$ are such that $\varphi(a') = b'$ then $[u(a')^{1/2}]u(a')^{1/2} = b'$ and $[u(a')^{1/2}]^*[u(a')^{1/2}] = b'$. Therefore, $a' \sim_\phi b'$. We write $a \prec_\phi b$ if there is $b' \in \text{Her}(b)$ such that $a \sim_\phi b'$. Clearly, the relation $\sim_\phi$ is transitive and the relation $\preceq_\phi$ is an equivalence relation.

1.5. There is another relation “$\sim_\tau$” introduced by G.K. Pedersen [20, 5.26]. (See also [13] for the case of infinite sums.) If $x, y \in A_+$, we write $x \sim_\tau y$ if there are $z_i \in A$, $i = 1, 2, \ldots, n$, such that $x = \sum_{i=1}^n z_i^*z_i$, $y = \sum_{i=1}^n z_iz_i^*$, and write $x \leq_\tau y$ if there is $y' \in A_+$ such that $x \sim_\tau y', y' \leq y$. If $A$ has a trace $\tau$, then $\tau(x) = \tau(y)$ if $x \sim_\tau y$. (See [20, 5.26] or [13].)

1.6. We will use the notation $\mathcal{P}(A)$ for the Pedersen ideal of the $C^*$-algebra $A$. 

[Note: The text continues with further mathematical content, extending the discussion on $C^*$-algebras and related concepts.]
2. Scaling Approximate Identities and Skeleton C*-subalgebras.

**Lemma 2.1.** Let $A$ be a C*-algebra, $a$ and $b$ two positive elements in $\mathcal{P}(A)$. If $\varepsilon > 0$ is such that $f_\varepsilon(b)$ generates $\mathcal{P}(A)$ as an ideal, then there are $a_1, a_2, \ldots, a_n \in P(A)_+$ such that

$$a = \sum_{i=1}^{n} a_i, \quad a_2 \sim_T a_3 \sim_T \cdots \sim_T f_{\varepsilon/2}(b),$$

$$a_1 \sim_\phi a_i, \quad i = 1, 2, \ldots, n$$

and $a_1 \sim_\phi f_{\varepsilon/2}(b)$.

**Proof.** There are $x_i, y_i \in A, i = 1, 2, \ldots, m$ such that

$$a = \sum_{i=1}^{m} x_i f_\varepsilon(b)y_i \leq \frac{1}{2} \left( \sum_{i=1}^{m} x_i f_\varepsilon(b)x_i^* + \sum_{i=1}^{m} y_i^* f_\varepsilon(b)y_i \right).$$

We may write $a \leq \sum_{i=1}^{m} r_i$, where $0 \leq r_i < f_\varepsilon(b)$. It follows from [22, 1.4.10] that there are $z_i \in A$ such that $a = \sum_{i=1}^{m} z_i z_i^*$ and $z_i z_i^* \leq r_i, i = 1, 2, \ldots, n$. Therefore we may write, by 1.4,

$$a = \sum_{i=1}^{n} a_i \text{ such that } a_i \sim_\phi b_i, \quad b_i \in \text{Her}(f_\varepsilon(b)), \quad i = 1, 2, \ldots, n.$$  

We will adjust the $b_i$'s and $a_i$'s so that

$$b_1 \leq b_1 \leq \cdots \leq b_n \leq f_{\varepsilon/2}(b),$$

$$a_i \sim_T b_i, \quad a_1 \sim_\phi a_i,$$

$i = 2, 3, \ldots, n$ and $a_1 \sim_\phi b_1$.

We use induction on $n$. If $n = 2$, $a = a_1 + a_2, a_i \sim_\phi b_i$, and $b_i \in \text{Her}(f_\varepsilon(b)), i = 1, 2$.

Since

$$b_1 \leq (f_{\varepsilon/2}(b) - b_2) + b_2,$$

applying [22, 1.4.10] we obtain $c_1, c_1', d_1, d_1'$ such that $c_1 \sim_\phi d_1, c_1' \sim_\phi d_1', b_1 = c_1 + c_1'$, $d_1 \leq f_{\varepsilon/2}(b) - b_1$, and $d_1' \leq d_2$. Set $b_1' = d_1'$ and $b_2' = b_2 + d_1$. Then $b_1' \leq b_2' \leq f_{\varepsilon/2}(b)$.

Since $a_1 \sim_\phi b_1$, there are $t_1, t_1' \geq 0$ such that $a_1 = t_1 + t_1', t_1 \sim_\phi c_1$, and $t_1' \sim_\phi c_1'$. Set $a_1' = t_1'$ and $a_2' = a_2 + t_1$. Then $a = a_1' + a_2', a_1' \sim_\phi b_1', a_2' \sim_T b_2'$.

Now assume that $a = \sum_{i=1}^{m} a_i$,

$$b_1 \leq b_3 \leq \cdots \leq b_n \leq f_{\varepsilon/2}(b),$$

$$a_i \sim_T b_i, \quad a_2 \sim_\phi a_i, \quad i = 3, 4, \ldots, n,$$

$$a_2 \sim_\phi b_2 \text{ and } b_1 \leq f_{\varepsilon/2}(b).$$

Since $b_1 \leq (f_{\varepsilon/2}(b) - b_n) + b_n$, applying [22, 1.4.10] we obtain $c_n, c_n', d_n, d_n' \geq 0$ such that

$$b_1 = c_n + c_n', \quad c_n \sim_\phi d_n, \quad c_n' \sim_\phi d_n'$$

$$d_n \leq f_{\varepsilon/2}(b) - b_n \text{ and } d_n' \leq b_n.$$
Set \( b'_1 = d'_n, b' = b_n + d_n \). Then \( b'_1 \leq b'_n \leq f_{e/2}(b) \). There are \( t_n, t'_n \geq 0 \) such that \( t_n + t'_n = a_1, t_n \sim_{c_n} t'_n \sim_{c'_n} \). Set \( d'_1 = t'_n, d'_n = a_n + t'_n \). Then \( d'_n \sim_T b'_n \leq f_{e/2}(b), d'_1 \sim_{c'_1} b'_1 \leq f_{e/2}(b) \), and \( a = d'_1 + \sum_{i=2}^{n-1} a_i + (a_n + a'_n) \) and
\[
 b_2 \leq b_3 \leq \cdots \leq b'_n = b_n + d_n, \quad b'_1 \leq b'_n.
\]

Repeating this argument with \( d'_1, b'_1 \) and \( b_{n-1} \), we get \( d'_1 = t_{n-1} + t'_n, t_{n-1} \sim_{c_{n-1}} d_{n-1} \leq b_{n-1} - b_{n-1}, t'_n \sim_{c'_n} d'_n \leq b_{n-1}, t_{n-1}, d_{n-1}, d'_n \geq 0 \). Set \( b''_1 = d''_{n-1}, b''_{n-1} = b_{n-1} + d_{n-1} \), \( a'' = t_{n-1}, a'_{n-1} = t_{n-1} + a_{n-1} \). Then
\[
 b''_1 \leq b''_{n-1} \leq b_n \leq b'_n \leq f_{e/2}(b),
 b_2 \leq b_3 \leq \cdots \leq b_{n-1} \leq b'_n \leq b'_n,
 a = a'' + \sum_{i=2}^{n-3} a_i + (a_{n-1} + t_{n-1}) + (a_n + t_n),
\]
and \( a'' \sim_{c''} b''_1, a'_{n-1} = a_{n-1} + t_{n-1} \sim_T b'_{n-1}, a'_n \sim_{c'_n} b'_n \).

Proceeding in this way, we can write
\[
 a_1 = \sum_{i=1}^{n} t_i, \quad t_i \geq 0, \quad t_i \sim_{c_i} d_i \leq b_{i+1} - b_i, \quad 2 \leq i \leq n
\]
\[
 (b_{n+1} = f_{e/2}(b)), \quad t_1 \sim_{c_1} d_1 \leq b_2, b'_1 = b_1 + d_1, \quad 2 \leq i \leq n.
\]
We have
\[
 a = t_1 + \sum_{i=2}^{n} (a_i + t_i) \quad \text{and} \quad b'_1 = b_1 + d_1, \quad 2 \leq i \leq n.
\]
Set \( b'_1 = d'_1 \); then
\[
 b'_1 \leq b'_2 \leq \cdots \leq b'_n \leq f_{e/2}(b)
\]
and \( a'_i = a_i + t_i \sim_T b'_i, 2 \leq i \leq n, a'_1 \sim_{c'_1} b'_1 \).

This completes the proof. \( \square \)

**Definition 2.2.** Let \( A \) be a \( \sigma \)-unital \( \mathbb{C}^* \)-algebra and \( \{ e_n \} \) be an approximate identity for \( A \). Denote \( e_n - e_{n-1} \) by \( g_n (e_0 = 0) \). If there is a sequence of positive numbers \( \{ \varepsilon_k \} \) and a subsequence of positive integers \( \{ n(k) \} \) such that
\begin{enumerate}
\item \( f_{\varepsilon_k}(g_{n(k)}) \sim_T g_n \sim_T g_{n+1} \) for \( n(k) < n < n(k + 1) \),
\item \( g_{n(k)} \sim_{c_{n(k)}} g_{n(k-1)} \) for \( n(k) = n(k-1) < n < n(k) \) and \( g_{n(k)} \leq f_{\varepsilon_{n-1}}(g_{n(k-1)}) \),
\item \( g_{n(2k-1)} \perp g_n \) if \( n > n(2k-1) \) or \( n < n(2k-3) \),
\item \( g_{n(2k)}(\sum_{n(k)-1}^{n(k)} g_i) \leq g_{n(2k)}(\sum_{n(k)-1}^{n(k)} g_i)g_{n(2k)} = g_{n(2k)} \),
\end{enumerate}
where \( k = 1, 2, \ldots \), then we say that \( \{ e_n \} \) is a scaling approximate identity.

It should be noted that if \( \{ e_n \} \) is a fundamental approximate identity, then \( \{ e_n \} \) is a scaling approximate identity.
THEOREM 2.3. If $A$ is a $\sigma$-unital (non-unital) simple C*-algebra, then $A$ contains a scaling approximate identity.

PROOF. Let $a$ be a strictly positive element of $A$. By taking a proper sequence of continuous functions $h_n$, we can construct (by taking $e'_n = h_n(a)$) an approximate identity $\{e'_n\}$ such that for each $n$, there is $0 \leq a_n \leq e_n - e_{n+1}$ ($e_0 = 0$), $a_n(e_n - e_{n-1}) = (e_n - e_{n-1})a_n = a_n$, $a_n \neq 0$, and $a_n \perp e_m - e_{m-1}$ if $n \neq m$. Moreover, $e_n \in P(A)$. Set $g_n = e_n - e_{n-1}$, $b_n = g_n - a_n$, $n = 1, 2, \ldots$. Applying Lemma 2.1, we obtain

$$b_2 = r_{2,1} + r_{2,2} + \ldots + r_{2,m(2)}$$

such that

$$0 \leq r_{2,i+1} \leq f_{e_1}(a_1) \leq f_{e_1}(g_1), \quad r_{2,m(2)} \leq f_{e_1}(r_{2,1})$$

for some $1 > \varepsilon_1 > 0$ and $r_{2,i} \neq 0, i = 1, 2, \ldots, m(2) - 1$, for some $1 > \varepsilon_1 > 0$ and $r_{2,i} \neq 0, i = 1, 2, \ldots, m(2)$.

We also obtain

$$a_2 = r_{2,m(2)+1} + \ldots + r_{2,m(2)\cdot m'(2)}$$

such that

$$0 \leq r_{2,m(2)+i+1} \leq f_{e_{1+i}}(r_{2,m(2)}), \quad r_{2,m(2)+i} \leq f_{e_i}(r_{2,m(2)}),$$

for some $1 > \varepsilon_2 > 0$, and $r_{2,m(2)+i} \neq 0, i = 1, 2, \ldots, m'(2)$.

Repeating this process, we get a sequence of nonzero positive elements as follows:

$$b_3 = r_{3,1} + r_{3,2} + \ldots + r_{3,m(3)},$$

$$a_3 = r_{3,m(3)+1} + \ldots + r_{3,m(3)\cdot m'(3)},$$

$$\ldots$$

$$b_k = r_{k,1} + r_{k,2} + \ldots + r_{k,m(k)},$$

$$a_k = r_{k,m(k)+1} + \ldots + r_{k,m(k)\cdot m'(k)},$$

$$\ldots$$

such that

$$r_{k,i+1} \leq f_{e_{k-1}}(r_{k-1,m(k-1)+m'(k-1)}),$$

$$r_{k,m(k)} \leq f_{e_i}(r_{k-1,m(k-1)+m'(k-1)}),$$

$$r_{k,m(k)} \leq f_{e_i}(r_{k-1,m(k-1)+m'(k-1)}),$$

for some $1 > \varepsilon_{k-1} > 0$, and

$$r_{k,m(k)+i+1} \leq f_{e_{k+i}}(r_{k,m(k)}),$$

$$r_{k,m(k)+m'(k)} \leq f_{e_i}(r_{k,m(k)+i}), i = 1, 2, \ldots, m'(k) - 1,$$

$$r_{k,m(k)+m'(k)} \leq f_{e_i}(r_{k,m(k)}),$$

for some $1 > \varepsilon_{k-1} > 0$, and
for some $1 > \varepsilon_{2k} > 0$.

Now set

$e_1 = e'_1$, $e_2 = e'_1 + r_{2,1}$, $e_3 = e'_1 + r_{2,1} + r_{2,2}$, 

$\ldots$

$e_{m(2)+1} = e'_1 + r_{2,1} + \cdots + r_{2,m(2)} = e'_1 + b_2$, 

$e_{m(2)+2} = e'_1 + b_2 + r_{2,m(2)+1}$, 

$\ldots$

$e_{m(2)+m(2)+1} = e'_1 + b_2 + r_{2,m(2)+1} + \cdots + r_{2,m(2)+m(2)} = e'_1 + b_2 + a_2 = e'_2$, 

$\ldots$

$e_{1+m(k)+\sum_{n=2}^{m(k)}(m(n)+m'(n))} = e'_{k-1} + \sum_{n=1}^{m(k)} r_{k,n} = e'_{k-1} + b_k$, 

$e_{2+m(k)+\sum_{n=2}^{m(k)}(m(n)+m'(n))+1} = e'_{k-1} + b_k + r_{k,m(k)+1}$, 

$\ldots$

$e_{1+\sum_{n=1}^{m(n)+m'(n)}(m(n)+m'(n))} = e'_{k-1} + b_k + a_k = e'_k$, 

$\ldots$

Take $n(1) = 1$, $n(2) = 1 + m(2)$, $n(3) = 1 + m(2) + m'(2)$, \ldots, $n(2k) = 1 + m(2) + \cdots + m(k-1)$, and $n(2k+1) = 1 + m(2) + \cdots + m(k-1) + m'(k-1)$, $k = 1, 2, \ldots$. From the construction one can check easily that $\{e_n\}$, $\{n_k\}$ and $\{\varepsilon_k\}$ satisfy the conditions (i) to (iv) in 2.2. 

\textbf{THEOREM 2.4.} Let $A$ be a $C^*$-algebra with a scaling approximate identity $\{e_n\}$. Then $A$ has a skeleton $C^*$-subalgebra $B$ such that $B$ has a quotient which is isomorphic to $K$.

\textbf{PROOF.} We will keep the notation of 2.2.

We first claim that there are $g_k^{(i)} \geq 0$, $1 \leq i \leq k$ and $u_k^{(i)}$, $1 \leq i \leq k-1$, $k = 1, 2, \ldots$, such that

1. $g_k^{(i)} \leq g_l^{(i)}$, if $l \leq k$, $g_k^{(i)} \in \text{Her}(f_{\varepsilon_{k+1}}(g_{m(2)})$, 

2. $(u_k^{(i)})(u_k^{(i)})^* = g_k^{(i)}$, $(u_k^{(i)})^* (u_k^{(i)}) = f_{\sigma_k}(g_{m(2)})$, 

where $\sigma_k = \frac{1}{2} \varepsilon_{2k}$, $k = 1, 2, \ldots$.

We will prove the claim by induction on $k$. Assume that the claim is true for all $k' \leq k$.

Since $g_{m(2(k+1))} \leq \phi f_{\varepsilon_{2(k+1)}}(g_{2k})$, there is $u_{k+1}^{(k)}$ in $A$ such that

$(u_{k+1}^{(k)})(u_{k+1}^{(k)})^* = f_{\sigma_{k+1}}(g_{m(2(k+1))})$, 

$(u_{k+1}^{(k)})(u_{k+1}^{(k)})^* = g_{k+1}^{(k)} \in \text{Her}(f_{\varepsilon_{2k}}(g_{m(2k)}))$.

Define $u_{k+1}^{(i)} = u_k^{(i)}u_{k+1}^{(k)}$, $1 \leq i < k$.

Then

$(u_{k+1}^{(i)})(u_{k+1}^{(i)})^* = (u_{k+1}^{(k)})(u_{k+1}^{(k)})^* (u_k^{(i)})(u_k^{(i)})$, 

$= (u_{k+1}^{(k)})^* f_{\sigma_k}(g_{m(2k)})(u_{k+1}^{(k)})$. 

\b
Since $f_{2s}(g_{n(2k)})$ is a unit for $\text{Her}(f_{2s}(g_{n(2k)}))$, 
\[ (u_{k+l}^{(i)})^* (u_{k+l}^{(i)}) = f_{2s+1}(g_{n(2(k+1))}). \]

Set 
\[ g_{k+l}^{(i)} = (u_{k+l}^{(i)})^* (u_{k+l}^{(i)}) = (u_{k}^{(i)}) g_{k+l}^{(k)} (u_{k}^{(i)})^*, \]

$1 \leq i < k$; then $g_{k+l}^{(i)} \in \text{Her}(f_{2s}(g_{n(2k)}))$. This completes the proof of the claim.

Let $\chi(k(t))$ denote the characteristic function of the set $[0, \varepsilon_{2k}]$. Set $p_{k}^{(k-1)} = \chi(g_{n(2k)})$ and $p_{k}^{(i)} = u_{k}^{(i)} (u_{k}^{(i)})^*$. Then $p_{k}^{(i)}$ are closed projections (with respect to $A$) in $A^{**}$. Set $g_{k+l} = \sum_{i=1}^{k} p_{k}^{(i)}$; $g_{k+l}^{(i)}$ is also a closed projection in $A^{**}$. Let $B_{2}$ be the $C^{*}$-subalgebra generated by $\{u_{2}^{(i)}, e_{n_{2}}, \ldots, B_{k+1} \}$ the $C^{*}$-subalgebra generated by $\{B_{k}, u_{k}^{(k)}, e_{n(2(k+1))} - e_{n(2k)} \}$. Notice that $g_{k+l}^{(i)}$ commutes with $e_{n(2(i+1))} - e_{n(2i)}$ and $u_{i+1}^{(i)}$, $1 \leq i \leq k$. It is a routine exercise that $g_{k+l}^{(i)}$ is isomorphic to $M_{k} (k \geq 2)$.

Now for fixed $m$, for $k \geq m$, 
\[ e_{n(2m)} g_{k+l}^{(i)} = e_{k}^{(i)} e_{n(2m)} = \sum_{i=1}^{m} p_{k}^{(i)}. \]

So $\{e_{n(2m)} g_{k+l}^{(i)} \} (k \geq m)$ is a decreasing sequence of closed projections in $A^{**}$. So $\{e_{n(2m)}^{(i)} \}$ converges strongly to a positive element $q_{m}$ in $A^{**}$. Hence $q_{m}$ is an upper semi-continuous function or the quasi-state space of $A$ (see [20, 3.11]). By a standard compactness argument, $q_{m} \neq 0$, and hence $q_{m}$ is a nonzero projection in $A^{**}$. Now $\{q_{m} \}$ is an increasing sequence of projections, so $q_{m} \not\rightarrow q$ for some nonzero projection $q$ in $A^{**}$. Furthermore, $q_{m} \rightarrow q$ strongly.

Since $g_{k+l}^{(i)}$ commutes with every element of $B_{i}$, $2 \leq i \leq k$, we conclude that $q$ commutes with every element of $B_{i}$, $2 \leq i$. It is routine to check that $B_{i}$ is isomorphic to $M_{i}$, $i \geq 2$. Denote by $B$ the $C^{*}$-subalgebra generated by $\{B_{i} : i = 2, 3, \ldots \}$; then $q$ commutes with every element of $B$. Thus there is *-homomorphism from $B$ onto $B_{q}$. Moreover, one can easily check that $B_{q}$ is isomorphic to $K$.

**LEMMA 2.5.** Let $A$ be a non-elementary simple $C^{*}$-algebra and $a$ be a nonzero positive positive element of $\text{P}(A)$. Then for every $k$, there is a skeleton $C^{*}$-subalgebra $B$ of $\text{Her}(a)$ and a closed projection $p$ in $A^{**}$ such that $p$ commutes with each element in $B$ and such that $pB$ is isomorphic to $M_{k}$.

**PROOF.** Since $A$ is simple, so also is $\text{Her}(a)$. If $sp(a)$ is finite, then $\text{Her}(a)$ has an identity $e$. There is a positive element $b$ in $\text{Her}(a)$ with infinitely many points in $sp(b)$. So $sp(e + b)$ has infinitely many points. Since $\text{Her}(e + b) = \text{Her}(a)$, we may assume that $sp(a)$ has infinitely many points. There are continuous functions $h_{1}, h_{2}, \ldots, h_{k}$ and $h_{1}', h_{2}', \ldots, h_{k}'$ on $sp(a)$ such that

\[ a \leq \sum_{i=1}^{k} h_{i}(a), \]

\[ h_{i}'(a) \perp h_{j}(a) \text{ if } j \neq i, \]

\[ h_{i}'(a) h_{j}(a) = h_{i}(a) h_{j}'(a) = h_{i}'(a), \]

\[ \|h_{i}'(a)\| = \|h_{i}(a)\| = 1, \quad i = 1, 2, \ldots, k. \]
Repeated application of [10, 1.8] shows that there are nonzero elements $b_i \in A_{h_i(a)}$ such that

$$b_1 \supset b_2 \supset \ldots \supset b_k$$

(see [10, 2.3]); we may assume that $0 \leq b_i \leq 1$, and that $\|b_i\| = 1$, $i = 1, 2, \ldots, k$. Take $b_1 = b'_1$ and apply [10, 1.7] repeatedly; we obtain $b_i \in A_{h_i(a)}$, $\|b_i\| = 1$, and $z_i \in \text{Her}(a)$ such that

$$z_i^* z_i = b_1, \quad z_i z_i^* = b_i, \quad i = 2, \ldots, k.$$ 

There are $u_i \in \text{Her}(a)$ such that

$$u_i^* u_i = f_{1/8}(b_1),$$

$$u_i u_i^* = f_{1/8}(b_i)$$

(see 1.4), $i = 1, \ldots, k$.

Let $\chi_{1/4}(t)$ denote the characteristic function of $[1/4, 1]$. Set $p_i = \chi_{1/4}(b_i)$ and $p = \sum_{i=1}^{k} p_i$. Then $p$ is a closed projection in $A^{**}$. It is easy to see that $p$ commutes with $h_i(a)$, $i = 1, 2, \ldots, k$, and commutes with $u_i$, $i = 1, \ldots, k$.

Let $B$ denote the $C^*$-subalgebra generated by

$$\{ h_i(a), i = 1, 2, \ldots, k, u_i, i = 2, 3, \ldots, k \}.$$ 

Then $\sum_{i=1}^{k} h_i(a) \geq a$. So $B$ is a skeleton $C^*$-subalgebra of $\text{Her}(a)$. Moreover, $p$ commutes with each element of $B$. It is a routine exercise that $pB$ is isomorphic to $M_k$. 

**Theorem 2.6.** Let $A$ be a $\sigma$-unital, non-unital, non-elementary simple $C^*$-algebra. Then for any non-unital matroid $C^*$-algebra $\mathcal{M}$, there is a skeleton $C^*$-subalgebra $B$ of $A$ such that $B$ has a quotient isomorphic to $\mathcal{M}$.

**Proof.** As in the proof of 2.3, there is an approximate identity $\{ e_n \}$ for $A$ satisfying the following conditions:

(i) $e_n e_m = e_m e_n = e_n$, if $n > m$;

(ii) there are $a_n$ in $A$ such that $0 \leq a_n \leq e_n - e_{n-1}(e_0 = 0)$ and $a_n(e_m - e_{m-1}) = (e_m - e_{m-1})a_n = 0$ if $m \neq n$;

(iii) $(e_n - e_{n-1})(e_m - e_{m-1}) = 0$ if $|n - m| \geq 2$ and $\|e_n\| = 1$.

Suppose that $0 < q(1) < r(1) \leq q(2) < r(3) \leq \cdots$ is a sequence of integers such that $\mathcal{M}$ is the following inductive limit:

$$\mathcal{M}_{q(1)} \xrightarrow{f_{q(1)r(1)}} \mathcal{M}_{r(1)} \xrightarrow{q(2)r(2)} \mathcal{M}_{r(2)} \xrightarrow{q(2)r(3)} \mathcal{M}_{r(3)} \longrightarrow \cdots.$$ 

Here $r(n) \mid q(n + 1)$, and $f_{mn}$ is the homomorphism consisting of adding $n - m$ rows and columns of zeros to each matrix in $\mathcal{M}_m$, and $g_{mn} = 1 \otimes 1_p$, i.e.

$$g_{mn}(x) = \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix}_{p \times p}.$$
where $p = \frac{m}{n}$. We set $s(n) = \frac{q(n+1)}{r(n)}$, $t(n) = r(n) - q(n)$ and $g_n = e_n - e_{n-1}$, $n = 1, 2, \ldots$.

As in Lemma 2.5, there are $d_1^{(1)}, d_2^{(1)}, \ldots, d_q^{(1)}$ and $u_2, u_3, \ldots, u_q^{(1)}$ in $\text{Her}(a_1)$ such that $0 \leq d_i^{(1)}$, $\|d_i^{(1)}\| = 1$, $d_i^{(1)} \perp d_j^{(1)}$ if $i \neq j$, and

$$
\begin{align*}
    u_i^{(1)} & = f_1/g(d_i^{(1)}), \\
    u_i^{(1)} & = f_1/g(d_i^{(1)}).
\end{align*}
$$

Moreover, if we take $e'_1 = \Sigma_i^{q(1)} p_i^{(1)}$, where $p_i^{(1)} = \chi_{1/4}(d_i^{(1)})$, and $B_1$ the $C^*$-subalgebra generated by $\{g_1, u_i, \; i = 2, 3, \ldots, q(1)\}$, then $e'_1$ commutes with each element of $B_1$ and $e'_1 B_1$ is isomorphic to $\mathcal{M}_{q(1)}$.

Repeated application of [10, 1.8] shows that there are elements $d_i^2 \in \text{Her}(a_{i+1})$, $(i = 1, 2, \ldots, t(1))$ such that

$$
\begin{align*}
f_{1/2}(d_i^{(1)}) & > d_1^2 > d_2^2 > \cdots > d_t^{(1)}
\end{align*}
$$

(see [10, 2.3]). As in the proof of Lemma 2.5, there are $d_i^{(2)} \in \text{Her}(a_{i+1})$ with $d_i^{(2)} \geq 0$, $\|d_i^{(2)}\| = 1$, $i = 1, 2, \ldots, t(1)$, $d_i^{(2)} \in \text{Her}(f_{1/2}(d_i^{(1)}))$ with $d_i^{(2)} \leq 0$, $\|d_i^{(2)}\| = 1$, and $u_i^{(2)} \in A$ such that

$$
\begin{align*}
    (u_i^{(2)})^*(u_i^{(2)}) & = f_{1/2}(d_i^{(2)}), \\
    (u_i^{(2)})^*(u_i^{(2)}) & = f_{1/2}(d_i^{(2)}),
\end{align*}
$$

$i = 2, 3, \ldots, t(1) + 1$. Set

$$
\begin{align*}
    u_j^{(i+1)} & = u_j^{(i)} u_{i+1}^{(i+1)}, \quad j = 2, 3, \ldots, q(1), \\
    p_i^{(2)} & = \chi_{1/4}(d_i^{(2)}), \quad i = 1, 2, \ldots, t(1) + 1.
\end{align*}
$$

Then

$$
\begin{align*}
    (u_i^{(2)} p_i^{(2)})^*(u_i^{(2)} p_i^{(2)}) & = p_i^{(2)}, \\
    (u_i^{(2)} p_i^{(2)})^*(u_i^{(2)} p_i^{(2)}) & = p_i^{(2)},
\end{align*}
$$

$i = 2, 3, \ldots, t(1) + 1$, and

$$
\begin{align*}
    (u_j^{(2)} p_j^{(2)})^*(u_j^{(2)} p_j^{(2)}) & = p_j^{(2)}(u_{i+1}^{(i+1)})(u_{j-1}^{(i+1)})(u_{j+1}^{(i+1)}) p_j^{(2)} = p_j^{(2)}, \\
    (u_j^{(2)} p_j^{(2)})^*(u_j^{(2)} p_j^{(2)}) & = p_j^{(2)} \leq p_j^{(1)}, \quad j = t(1) + 2, \ldots, r(1),
\end{align*}
$$

where $p_j^{(2)}$ are closed projections. Let $e'_2 = \Sigma_j^{q(1)} p_j^{(2)}$, then $e'_2 p_j^{(1)} = p_j^{(1)} e'_2 = p_j^{(2)}$, $j = t(1) + 2, \ldots, r(1)$, $e'_2 e'_1 = e'_1 e'_2 = \Sigma_j^{q(1)} p_j^{(2)}$ and $e'_2$ commutes with each element of $B_2$, where $B_2$ is the $C^*$-subalgebra generated by $\{B_j, u_i^{(2)}, \; i = 1, 2, \ldots, t(1) + 1, \; e_1 e_{n+1} - e_1\}$. It is a routine exercise to check that $e'_2 B_2$ is isomorphic to $\mathcal{M}_{q(1)}$. Moreover, $e'_2 B_1$ is isomorphic to $\mathcal{M}_{q(1)}$. If we identify $e'_2 B_2$ with $\mathcal{M}_{q(1)}$, and $e'_2 B_1$ with $\mathcal{M}_{q(1)}$, then the isomorphism $e'_1 B_1 \rightarrow e'_2 e'_1 B_1 = e'_2 B_1$ gives the homomorphism $f_{q(1)} e_1$ from $\mathcal{M}_{q(1)}$ into $\mathcal{M}_{q(1)}$, and we may write $e'_1 B_1 \rightarrow e'_2 B_2$.

We assume that there are $C^*$-subalgebras $B_1, B_2, \ldots, B_m$, ... satisfying the following:
(1) $B_{2n}$ is generated by $\{B_{2n-1}, e_{\sum_{i=1}^{r(n)+1}}^{(2n)}, u_i^{(2n)} : i = 2, 3, \ldots, t(n) + 1\}$;

(2) there are $d_i^{(2n)} \geq 0, \|d_i^{(2n)}\| = 1, i = 1, 2, \ldots, t(n) + 1$, such that

$$d_i^{(2n)} \in \text{Her}(a_{\sum_{j=1}^{i-1} r(j)+i}), \quad i = 1, 2, \ldots, t(n),$$

$$d_i^{(2n)} \in \text{Her}(f_{1/2}(d_i^{(2n-1)})), \quad i = 2, 3, \ldots, t(n) + 1;$$

(3) if we set $u_j^{(2n)} = u_j^{(2n)} u_j^{(2n+1)}, j = 2, 3, \ldots, q(n)$ and $p_i^{(2n)} = \chi_{1/4}(d_i^{(2n)}), i = 1, 2, \ldots, t(n) + 1$, then

$$\left(u_i^{(2n)} p_i^{(2n)}\right)^* \left(u_i^{(2n)} p_i^{(2n)}\right) = p_i^{(2n)},$$

$$\left(u_i^{(2n)} p_i^{(2n)}\right)^* \left(u_i^{(2n)} p_i^{(2n)}\right) = p_i^{(2n)},$$

$$i = 2, 3, \ldots, t(n) + 1,$$

where $p_j^{(2n)} \leq p_{j-q(2)}^{(2n-1)}$ are closed projections, $j = t(n) + 2, \ldots, r(n)$, and $p_1^{(2n)} \leq p_1^{(2n-1)}$;

(4) if we set $e_{2n}^{(2n)} = \sum_{j=1}^{r(n)} p_j^{(2n)}$ then $e_{2n}^{(2n)} e_{2n-1}^{(2n-1)} = e_{2n}^{(2n)} e_{2n-1}^{(2n-1)} = \sum_{j=t(n)+1}^{r(n)} p_j^{(2n)}$ and $e_{2n}^{(2n)}$ commutes with each element of $B_{2n}, e_{2n}^{(2n)} B_{2n}$ is isomorphic to $\mathcal{M}_{r(n)}$ and $e_{2n}^{(2n)} B_{2n-1} = e_{2n}^{(2n)} e_{2n-1}^{(2n-1)} B_{2n-1}$ is isomorphic to $\mathcal{M}_{r(n)}$;

(5) if we identify $e_{2n}^{(2n)} B_{2n}$ with $\mathcal{M}_{r(n)}$ and $e_{2n-1}^{(2n-1)} B_{2n-1}$ with $\mathcal{M}_{r(n)}$, the isomorphism $e_{2n}^{(2n-1)} B_{2n-1}$ to $e_{2n}^{(2n)} B_{2n}$ given by $x \mapsto e_{2n}^{(2n)} x (x \in e_{2n}^{(2n-1)} B_{2n-1})$ gives the homomorphism:

$$\mathcal{M}_{r(n)} \xrightarrow{\text{iso}} \mathcal{M}_{r(n)};$$

(6) $B_{2n+1}$ is generated by $\{B_{2n}, u_i^{(2n+1)} : i = 2, 3, \ldots, s(n)\}$;

(7) $u_i^{(2n+1)} \in \text{Her}(f_{1/2}(d_i^{(2n+1)})), i = 2, \ldots, s(n)$, and there are

$$d_i^{(2n+1)} \in \text{Her}(f_{1/2}(d_i^{(2n)})), \quad d_i^{(2n+1)} \geq 0, \|d_i^{(2n+1)}\| = 1, d_i^{(2n+1)} \perp d_j^{(2n+1)},$$

$i \neq j, i, j = 1, 2, \ldots, s(n)$, such that

$$\left(u_i^{(2n+1)}\right)^* \left(u_i^{(2n+1)}\right) = f_{1/4}(d_i^{(2n+1)}),$$

$$\left(u_i^{(2n+1)}\right)^* \left(u_i^{(2n+1)}\right) = f_{1/4}(d_i^{(2n+1)}),$$

$i = 2, 3, \ldots, 2(n);$
(8) if we set
\[ u_j^{(2n+1)} = u_j^{(2n)} u_j^{(2n-1)}, \quad j = 2, 3, \ldots, s(n), \]
\[ p_i^{(2n+1)} = \chi_{1/4}(d_i^{(2n+1)}), \quad i = 1, 2, \ldots, s(n), \]
then
\[ (u_j^{(2n+1)} p_i^{(2n+1)})^* (u_j^{(2n+1)} p_i^{(2n+1)}) = p_i^{(2n+1)}, \]
\[ (u_j^{(2n+1)} p_i^{(2n+1)}) (u_j^{(2n+1)} p_i^{(2n+1)})^* = p_i^{(2n+1)}, \]
i = 2, 3, \ldots, s(n), and
\[ j = s(n) + 1, \ldots, s(n+1), \]
where \( p_j^{(2n+1)} \) are closed projections;

(9) if we set \( e_i^{2n+1} = \sum_j p_j^{(2n+1)} \), then \( e_i^{2n+1} e_i^{2n} = e_i^{2n} e_i^{2n+1} = e_i^{2n+1} \), \( e_i^{2n+1} \) commutes with each element of \( B_{2n+1} \). Moreover, \( e_i^{2n+1} B_{2n+1} \) is isomorphic to \( M_{q(n+1)} \) and \( e_i^{2n+1} B_{2n} \) is isomorphic to \( M_{q(n)} \).

(10) if we identify \( e_i^{2n+1} B_{2n+1} \) with \( M_{q(n+1)} \) and \( e_i^{2n} B_{2n} \) with \( M_{q(n)} \), then the isomorphism from \( e_i^{2n} B_{2n} \) to \( e_i^{2n+1} B_{2n+1} \) given by \( x \rightarrow e_i^{2n+1} x \), \( x \in e_i^{2n} B_{2n} \), gives the homomorphism:
\[ e_i^{2n} B_{2n} \xrightarrow{e_i^{2n+1}(x)} e_i^{2n+1} B_{2n+1}. \]

If \( m = 2n \), then, as in Lemma 2.5, there are \( d_i^{(m+1)}, \ldots, d_i^{(m+1)} \)
and
\[ u_i^{(m+1)} \in \text{Her}(f_{1/2}(d_i^{(m)})), \]
i = 2, 3, \ldots, s(n), such that \( d_i^{(m+1)} \geq 0, ||d_i^{(m+1)}|| = 1, d_i^{(m+1)} \perp d_j^{(m+1)} \) if \( i \neq j \)
and
\[ (u_i^{(m+1)})^* u_i^{(m+1)} = f_{1/8}(d_i^{(m+1)}), \]
\[ (u_i^{(m+1)})^* (u_i^{(m+1)}) = f_{1/8}(d_i^{(m+1)}), \]
i = 2, 3, \ldots, s(b). Set \( B_{m+1} \) equal to the \( C^* \)-subalgebra generated by
\[ \{ B_{m+1}, u_i^{(m+1)}, \quad i = 2, 3, \ldots, s(n) \} \]
and
\[ p_i^{(m+1)} = \chi_{1/4}(d_i^{(m+1)}), \quad i = 1, 2, \ldots, s(n), \]
\[ u_j^{(m+1)} = u_k^{(m)} u_j^{(m+1)}, \quad j = 2, 3, \ldots, s(n), \]
\[ u_j^{(m+1)} = u_k^{(m)} u_j^{(m+1)}, \quad j = 2, 3, \ldots, s(n), k = 2, 3, \ldots, s(n). \]
Then
\[ (u_i^{(m+1)} p_i^{(m+1)})^* (u_i^{(m+1)} p_i^{(m+1)}) = p_i^{(m+1)}, \]
\[ (u_i^{(m+1)} p_i^{(m+1)}) (u_i^{(m+1)} p_i^{(m+1)})^* = p_i^{(m+1)}, \]
\[ i = 2, 3, \ldots, s(n), \text{and} \]
\[ (u_{j+1}^{(m+1)})^* (u_{j+1}^{(m+1)}) = (u_{k}^{(m)})^* (u_{k}^{(m)}) (u_{j}^{(m+1)}) (u_{j}^{(m+1)}) = p_{1}^{(m+1)}, \]
\[ (u_{j+1}^{(m+1)})^* (u_{j+1}^{(m+1)}) = (u_{j}^{(m+1)})^* (u_{j}^{(m+1)}) = p_{1}^{(m+1)}, \]
\[ j = 2, 3, \ldots, s(n), k = 2, 3, \ldots, s(n), \]

where \( p_{1}^{(m+1)} \) are closed projections. If we set \( e'_{m+1} = \sum_{j=1}^{s(n)} p_{j}^{(m+1)} \), then \( e'_{m+1} e'_{m} = e'_{m+1} e'_{m+1} = e'_{m+1} \) commutes with each element in \( B_{m+1} \). It is a routine exercise to check that \( e'_{m+1} B_{m+1} \) is isomorphic to \( \mathcal{M}_{q(n+1)} \). Moreover, if we define a map \( \varphi \) from \( e'_{m} B_{m} \) onto \( e'_{m+1} B_{m+1} \) by \( \varphi(x) = e'_{m+1} x \) then the map is an isomorphism. If we identify \( e'_{m+1} B_{m+1} \) with \( \mathcal{M}_{q(n+1)} \) and \( e'_{m} B_{m} \) with \( \mathcal{M}_{(n+1)} \), the map \( \varphi \) gives the homomorphism:

\[ e'_{m} B_{m} \xrightarrow{\psi_{q(n+1)}} e'_{m+1} B_{m+1}. \]

If \( m = 2n - 1 \), repeated application of \([10, 1.8]\) shows that there are elements \( d'_{i} \in \text{Her}(a_{\sum_{i=1}^{s(n+1)} \rho_{j+1+i}}), i = 1, 2, \ldots, t(n) \), such that

\[ f_{1/2}(d'_{i}^{(m)}) > d'_{2} > d'_{2} > \cdots > d'_{t(n)} \]

(see \([10, 2.3]\)). As in the proof of Lemma 2.5, there are

\[ d'_{i}^{(m+1)} \in \text{Her}(a_{\sum_{i=1}^{s(n+1)} \rho_{j+1+i}}), \]
\[ d_{n(n+1)}^{(m+1)} \in \text{Her}(f_{1/2}(d'_{i}^{(m)})), \]

\[ d_{i}^{(m+1)} \geq 0, \| d_{i}^{(m+1)} \| = 1, i = 1, 2, \ldots, t(n) + 1, \text{and } u_{i}^{(m+1)} \in A \text{ such that} \]

\[ (u_{i}^{(m+1)})^* (u_{i}^{(m+1)}) = f_{1/8}(d_{i}^{(m+1)}) \]

and

\[ (u_{i}^{(m+1)}) (u_{i}^{(m+1)}) = f_{1/8}(d_{i}^{(m+1)}), \]

\[ i = 2, 3, \ldots, t(n) + 1. \]

Set

\[ u_{j+1}^{(m+1)} = u_{j}^{(m)} u_{j+1}^{(m+1)}, j = 2, 3, \ldots, q(n), \]

and

\[ p_{j}^{(m+1)} = \chi_{1/4}(d_{i}^{(m+1)}), i = 1, 2, \ldots, t(n) + 1. \]
Then
\[
(u_i^{(m+1)} p_i^{(m+1)})^* (u_i^{(m+1)} p_i^{(m+1)}) = p_i^{(m+1)},
\]
\[
(u_j^{(m+1)} p_j^{(m+1)})^* (u_j^{(m+1)} p_j^{(m+1)}) = p_j^{(m+1)}, \quad i = 2, 3, \ldots, t(n) + 1.
\]
Moreover,
\[
(u_j^{(m+1)} p_j^{(m+1)})^* (u_j^{(m+1)} p_j^{(m+1)}) = p_j^{(m+1)},
\]
where \(p_j^{(m+1)}\) are closed projections and \(p_j^{(m+1)} \leq p_j^{(m)}, \quad j = 1, 2, \ldots, q(n).
\]
Set \(e_{m+1} = \sum_{j=1}^{n} p_j^{(m+1)}\), then
\[
e_{m+1}^j p_j^{(m)} = p_j^{(m)} e_{m+1} = p_j^{(m+1)}, \quad j = 1, 2, \ldots, q(n),
\]
and \(e_{m+1}^j\) commutes with each element in \(B_{m+1}\), where \(B_{m+1}\) is the \(C^*\)-subalgebra generated by
\[
\{ B_m, u_i^{(m+1)} \}, \quad i = 1, 2, \ldots, t(n) + 1, \quad e_{\sum_{j=1}^{t(n)+1} - e_1}.
\]
It is a routine exercise to check that \(e_{m+1}^j B_{m+1}\) is isomorphic to \(M_{t(n)}\). Moreover, if we define a map \(\varphi\) from \(e_{m+1}^j B_{m+1}\) onto \(e_{m+1}^j B_m\) by \(\varphi(x) = e_{m+1}^j x\) then the map is an isomorphism. If we identify \(e_{m+1}^j B_{m+1}\) with \(M_{t(n)}\) and \(e_{m}^j B_m\) with \(M_{q(n)}\), the map \(\varphi\) gives the homomorphism:
\[
e_{m+1}^j B_m \xrightarrow{f_{q(n)}} e_{m+1}^j B_{m+1}.
\]
For fixed \(n\), \(\{ e_n^j e_m^j \}\) is a decreasing sequence of closed projections \((m \geq n)\). So \(\{ e_n^j e_m^j \}\) converges strongly to a positive element \(q_m\) in \(A^{**}\). Hence \(q_m\) is an upper semi-continuous function on the quasi-state space of \(A\) (see [20, 3.11]). By a standard compactness argument, \(q_m \neq 0\), and hence \(q_m\) is a nonzero projection in \(A^{**}\). Now \(\{ q_m \}\) is an increasing sequence of projections, and so \(q_m \not\rightarrow q\) for some nonzero projection \(q\) in \(A^{**}\). Furthermore, \(e_m^j \rightarrow q\) strongly.

Since \(e_m^j\) commutes with every element in \(B_i\), \(1 \leq i \leq m\), we conclude that \(q\) commutes with every element of \(B_m\). It is then easy to see that \(q B_m\) is isomorphic to \(e_m^j B_m\). If \(B\) denotes the \(C^*\)-subalgebra generated by \(\{ B_m, m = 1, 2, \ldots, m\}\), then \(q\) commutes each element of \(B\). So there is a homomorphism from \(B\) onto \(q B\). By the construction of \(\{ B_m \}\), it is easily checked that \(q B\) is the norm closure of the following inductive limit:
\[
q B_1 \xrightarrow{\overline{r_{1}^{(n)}}} q B_2 \xrightarrow{\overline{r_{2}^{(n)}}} q B_3 \xrightarrow{\overline{r_{3}^{(n)}}} q B_4 \xrightarrow{\overline{r_{4}^{(n)}}} q B_5 \rightarrow \cdots
\]
Therefore $qB$ is isomorphic to $M$. Since $\sum_{k=1}^{n} t(k) \to \infty$ as $n \to \infty$, and $e_{\sum_{k=1}^{n} t(k)+1} \in B$, $B$ is a skeleton $C^*$-subalgebra of $A$. This completes the proof. 

We also have the following:

**Theorem 2.7.** Let $A$ be a unital and non-elementary simple $C^*$-algebra. Then for any unital matroid $C^*$-algebra $M$, there is a skeleton $C^*$-subalgebra $B$ of $A$ such that $B$ has a quotient which is isomorphic to $M$.

2.8. Real rank of a $C^*$-algebra has been defined by L. G. Brown and G. K. Pedersen in [5]. A $C^*$-algebra is said to have real rank zero if the invertible selfadjoint elements are norm dense in $A_{s.a.}$. A $C^*$-algebra has real rank zero if and only if the elements in $A_{s.a.}$ with finite spectra are dense in $A_{s.a.}$, and if and only if $A$ has (HP), i.e. every hereditary $C^*$-subalgebra of $A$ has an approximate identity consisting of projections (see [5, 2.6]). Trivial examples of $C^*$-algebras with real rank zero are von Neumann algebras and AF $C^*$-algebras.

**Theorem 2.9.** Let $A$ be a separable $C^*$-algebra with real rank zero and stable rank one. If $K_0(A)$ is unperforated, then there is a skeleton $C^*$-subalgebra $B$ of $A$ such that $B$ is an AF $C^*$-algebra with $K_0(B) = K_0(A)$. Moreover, for every projection $p$ in $A$, there is a projection $q$ in $B$ such that $p$ is equivalent (in the sense of Murray and von Neumann) to $q$.

**Proof.** $K_0(A)$ is a countable, unperforated ordered group. It follows from [2, 6.5.1] and [24, 1.6] that $K_0(A)$ has the Riesz interpolation property (see [15, A3.1]). Therefore, $K_0(A)$ is a dimension group ([15, 3.1]). In other words,

$$K_0(A) = \lim_{\to} \left\{ \mathbb{Z}(r_n), \varphi_n \right\}.$$ 

Suppose that $\{ e_n \}$ is an approximate identity for $A$ consisting of projections. Set

$$p_1 = e_1, \quad p_n = e_n - e_{n-1}, \quad n = 2, 3, \ldots.$$ 

If $A$ is unital, we assume that $p_1 = 1$, $p_n = 0$ if $n > 1$. Without loss of generality, we may assume that $[p_1] \in \mathbb{Z}(r^n)$ and $[p_1] = (k(1), k(2), \ldots, k(r_1))$, where $k(i)$ is a nonzero integer. Suppose that $[q_1] = (1, 0, \ldots, 0)$. Then $[q_1] \leq [p_1]$. Since $A$ has cancellation (see [2, 6.5.1]), $p_1 \sim q_1$ (in the sense of Murray and von Neumann). Therefore there is a projection $q_{1,1}^{(1)} \leq p_1$ such that

$$q_{1,1}^{(1)} \in [q_{1,1}^{(1)}] \text{ and } [p_1 - q_{1,1}^{(1)}] = (k(1) - 1, k(2), \ldots, k(r_1)).$$

Recursively, we can construct projections

$$q_{ij}^{(1)} \leq p_1,$$

$1 \leq j \leq k(i), i = 1, 2, \ldots, r(1)$, such that

$$q_{ij}^{(1)} \perp q_{ij}^{(1)} \text{ if } i \neq i' \text{ or } j \neq j',$$

$$q_{ij}^{(1)} \sim q_{ij}^{(1)} \text{ and } [q_{ij}^{(1)}] = (0, \ldots, 0, 1, 0, \ldots, 0).$$
Moreover, $\sum_{ij} q_{ij}^{(1)} = p_1$. Let $v_{ij}^{(1)}$ be partial isometries in $A$ such that $(v_{ij}^{(1)})^*(v_{ij}^{(1)}) = q_{ij}^{(1)}$ and $(v_{ij}^{(1)})^*(v_{ij}^{(1)}) = q_{ij}^{(1)}$, $2 \leq j \leq k(i)$, $i = 1, 2, \ldots, r_1$.

It is routine to check that the $C^*$-subalgebra $B_1$ generated by $\{v_{ij}^{(1)}, 2 \leq j \leq k(i), i = 1, 2, \ldots, r_1\}$ is isomorphic to

$$\mathcal{M}_{k(1)} \oplus \mathcal{M}_{k(2)} \oplus \cdots \oplus \mathcal{M}_{k(r_1)},$$

and $K_0(B_1) \cong \mathbb{Z}^{(r_1)}$. We may assume that $[e_2] = [p_1 + p_2] = [p_1] + [p_2] \in \mathbb{Z}^{(r_2)}$ and

$$[p_1 + p_2] = (m(1), m(2), \ldots, m(r_2)).$$

Suppose that, in $\mathbb{Z}^{(r_2)}$,

$$[q_{1,1}^{(1)}] = (k_{11}, k_{21}, \ldots, k_{m(1),1}, 0, \ldots, 0).$$

Repeating the above argument, we can construct projections $q_{ij}^{(2)} \leq q_{1,1}^{(1)}$, $1 \leq j \leq k(i)$, $i = 1, 2, \ldots, s(1)$, such that

$$q_{ij}^{(2)} \perp q_{ij}^{(2)} \text{ if } i \neq i' \text{ or } j \neq j',
q_{ij}^{(2)} \sim q_{ij}^{(2)} \text{ and } [q_{ij}^{(2)}] = (0, \ldots, 0, 1, 0, \ldots, 0).$$

Let us do the same for each $q_{ij}^{(1)}$. If $[p_2 - p_1] = (s(1), s(2), \ldots, s(r_2))$ (some of $s(i)$ may be zero) then we add $s(i)$ orthogonal but equivalent projections $q_{ij}^{(2)}$ in $(p_2 - p_1)A(p_2 - p_1)$ for each $i$. Suppose that $v_{ij}^{(2)}$ are partial isometries in $A$ such that $(v_{ij}^{(2)})^*(v_{ij}^{(2)}) = q_{ij}^{(2)}$ and $(v_{ij}^{(2)})^*(v_{ij}^{(2)}) = q_{ij}^{(2)}$, $2 \leq j \leq m(i)$, $i = 1, 2, \ldots, r_2$.

The $C^*$-subalgebra $B_2$ generated by $\{v_{ij}^{(2)}, 2 \leq j \leq m(i), i = 1, 2, \ldots, r_2\}$ is isomorphic to

$$\mathcal{M}_{m(1)} \oplus \mathcal{M}_{m(2)} \oplus \cdots \oplus \mathcal{M}_{m(r_1)},$$

$B_1 \subset B_2$,

and $K_0(B_2) \cong \mathbb{Z}^{(r_2)}$.

Continuing this way, we get a sequence of $C^*$-subalgebras $B_1 \subset B_2 \subset \cdots \subset B_n \subset B_{n+1} \subset \cdots$ such that

$$B_n \cong \mathcal{M}_{m^n(1)} \oplus \mathcal{M}_{m^n(2)} \oplus \cdots \oplus \mathcal{M}_{m^n(r_n)},$$

for some integers $m^n(i), i = 1, 2, \ldots, r_n, K_0(B_n) = \mathbb{Z}^{(r_n)}$, and the embedding $B_n \rightarrow B_{n+1}$ gives a homomorphism:

$$\mathbb{Z}^{(r_n)} \xrightarrow{\psi_n} \mathbb{Z}^{(r_{n+1})}.$$

Let $B$ be the $C^*$-subalgebra generated by $\bigcup_{n=1}^{\infty} B_n$. Then $B$ is an AF $C^*$-algebra and $K_0(B) \cong \lim \rightarrow \mathbb{Z}^{(r_n)}$. Since $e_n \in B$, $B$ is a skeleton $C^*$-subalgebra of $A$. If $p$ is a projection in $A$, we may assume that $[p] \in \mathbb{Z}^{(r_n)}$. Therefore there is $q \in B_n \subset B$ such that $p$ is equivalent to $q$ (in the sense of Murray and von Neumann).
Remark 2.10. Separable AF $C^*$-algebras have real rank zero, stable rank one and unperforated $K_0(A)$. Theorem 2.9 shows that separable $C^*$-algebras with real rank zero, stable rank one and unperforated $K_0(A)$ are somewhat similar to separable AF $C^*$-algebras. However, a recent result of M. D. Choi and G. A. Elliott ([7]) provides examples (namely, irrational rotation $C^*$-algebras) of simple $C^*$-algebras with real rank zero, stable rank one and unperforated $K_0(A)$ which are not approximate finite-dimensional. (Note these simple $C^*$-algebras have cancellation [23]. Hence, by [2, 6.5.7], they have stable rank one.) The author would like to raise the following question:

Are separable nuclear (simple) $C^*$-algebras with real rank zero, stable rank one, unperforated $K_0$-groups and trivial $K_1$-flows (see [26]) approximate finite dimensional?

3. Applications.

3.1. Let $A$ be a $C^*$-algebra and denote by $A^{**}$ its enveloping von Neumann algebra. An element $x$ in $A^{**}$ is a multiplier if $xa$ and $ax$ are in $A$ for all $a$ in $A$, $x$ is a left multiplier if $xa$ is in $A$ for all $a$ in $A$, $x$ is a right multiplier if $ax$ is in $A$ for all $a$ in $A$, and $x$ is a quasi-multiplier if $axb$ is in $A$ for all $a$ and $b$ in $A$. We denote the collections of multipliers, left multipliers, right multipliers and quasi-multipliers by $M(A)$, $LM(A)$, $RM(A)$ and $QM(A)$ respectively. If $B$ is a skeleton $C^*$-subalgebra of $A$, then $M(B) \subset M(A)$, $LM(B) \subset LM(A)$, $RM(B) \subset RM(A)$ and $QM(B) \subset QM(A)$ (see [19, 3.7]). (It should be noted that the above inclusions do not hold if $B$ is merely a $C^*$-subalgebra of $A$.) Therefore the results in §2 may help us to determine the structure of $M(A)$, $LM(A)$, $RM(A)$ and $QM(A)$.

It is easy to see that $LM(A) + RM(A) \subset QM(A)$. The question whether $LM(A) + RM(A) = QM(A)$ was raised in [1]. The problem has been studied in [4], [19], [20], [21], among other articles. In this section we will give applications of the results in §2 to this problem.

Recall that a $C^*$-algebra is scattered if it is type I and has scattered spectrum $\Lambda$ (see [16]). Let $X$ be a scattered topological space. Define $X_{[0]} = X, X_{[1]} = X \setminus \{\text{isolated points of } X\}$. If $X_{[\alpha]}$ is defined for some ordinal number $\alpha$, define $X_{[\alpha+1]} = X \setminus \{\text{isolated points in } X_{[\alpha]}\}$; if $\beta$ is a limit ordinal, define $X_{[\beta]} = \cap_{\alpha<\beta} X_{[\alpha]}$. We set $\lambda(X) = \alpha$, where $\alpha$ is the least ordinal such that $X_{[\alpha]}$ is discrete.

The following is a generalization of [19, Theorem 6.3] (see [20, Theorem 3] also).

Theorem 3.2. Let $A$ be a $C^*$-algebra with a scaling approximate identity and $B$ a unital $C^*$-algebra. Then $QM(B \otimes A) = LM(B \otimes A) + RM(B \otimes A)$ implies that $B$ is scattered and $\lambda(B) < \infty$.

Proof. It follows from 2.4 that there is a skeleton $C^*$-subalgebra $A_0$ of $A$ such that there is a $*$-homomorphism from $A_0$ onto $\mathcal{K}$. Thus $B \otimes A_0$ is a skeleton $C^*$-subalgebra of $B \otimes A$ and there is a $*$-homomorphism $\varphi$ such that $\varphi(B \otimes A_0) = B \otimes \mathcal{K}$. By [19, 3.1], if $QM(B \otimes A) = LM(B \otimes A) + RM(B \otimes A)$, then $QM(B \otimes A_0) = LM(B \otimes A_0) + RM(B \otimes A_0)$. It follows from [19, 4.13] that if $QM(B \otimes K) = LM(B \otimes K) + RM(B \otimes K)$, then, by [19, 6.3] (note that the “only if” part of [19, 6.3] works for $\sigma$-unital $C^*$-algebras), $B$ is scattered and $\lambda(B) < \infty$. \hfill \blacksquare
**THEOREM 3.3.** Let $A$ be a $\sigma$-unital simple $C^*$-algebra. Then $QM(A) = LM(A) + RM(A)$ if and only if $A$ is elementary or $A$ is unital.

**PROOF.** Only the “only if” part needs a proof. Assume that $A$ is non-unital and non-elementary. Take a non-elementary stable matroid $C^*$-algebra $M$. By Theorem 2.6, there is a skeleton $C^*$-subalgebra $B$ of $A$ such that $B$ has a quotient which is isomorphic to $M$. If $QM(A) = LM(A) + RM(A)$, then, by [10, 3.1], $QM(B) = LM(B) + RM(B)$. Therefore, by [19, 4.3], $QM(M) = LM(M) + RM(M)$. This contradicts Theorem 6.3 in [19], since $M$ is a stable matroid $C^*$-algebra.

**THEOREM 3.4.** Let $A$ be a $\sigma$-unital $C^*$-algebra and $B$ a $\sigma$-unital, non-unital and non-elementary simple $C^*$-algebra. Then

$$QM(A \otimes B) \neq LM(A \otimes B) + RM(A \otimes B).$$

**PROOF.** Suppose that $M$ is a non-elementary matroid $C^*$-algebra. It follows from Theorem 2.6 that there is a skeleton $C^*$-subalgebra $B_0$ of $B$ such that $B_0$ has a quotient which is isomorphic to $M \otimes K$. Therefore $A \otimes B$ has a skeleton $C^*$-algebra $A \otimes B_0$ with a quotient isomorphic to $A \otimes M \otimes K$. The conclusion then follows from the proof of 3.3.

3.2. L. G. Brown in [4] showed the connection between the problem of whether $QM(A) = LM(A) + RM(A)$ and the problem of perturbations of $C^*$-algebras. Perturbations of $C^*$-algebras have been studied in several different ways (see [6], [8], [9], [17] and [18]). One of them is to ask whether an almost isometric ($\|\varphi\| - 1$ and $\|\varphi\| - 1$ are small) complete order automorphism $\varphi$ of a $C^*$-algebra is close to an isometry.

**THEOREM 3.6.** If $A$ is a $\sigma$-unital, non-elementary simple $C^*$-algebra without identity, then there exists a sequence $\{\varphi_n\}$ of complete order automorphisms of $A$ such that

$$\lim_{n \to \infty} \|\varphi_n\| = 1,$$

$$\lim_{n \to \infty} \|\varphi_n^{-1}\| = 1,$$

but

$$\inf\{\|\theta - \varphi_n\| : n = 1, 2, \ldots, 3\text{ automorphisms of } A\} > 0.$$
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