LOCAL SPACES WITH THREE CELLS AS H-SPACES

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1. Introduction. The question of which finite CW-complexes are H-spaces has been studied for many years. Since a finite CW-complex is an H-space if and only if its localization at each prime \( p \) is an H-space [21], an examination of finite local cell complexes as H-spaces yields results concerning CW-complexes. On the other hand, if it is known that a particular CW-complex is not an H-space, one would like to know for which primes \( p \) its localization at \( p \) fails to be an H-space. The main result of this paper gives a condition equivalent to a three cell local CW-complex’s being an H-space for a prime \( p > 3 \).

An H-space of rank one has the homotopy type of an odd-dimensional sphere \( S^r \). An odd-dimensional sphere \( S^r \) is an H-space if and only if \( r = 1, 3 \) or \( 7 \). Its localization \( S^r_p \) at a prime \( p \) fails to be an H-space only for the prime \( p = 2 \) [1].

The 2-torsion free rank two H-spaces have been classified up to homotopy. The only types \( (q, n) \) which occur are those such that \( \{q, n\} \subset \{1, 3, 7\} \) or \( (q, n) = (1, 2) \) or \( (3, 5) \). There are exactly sixteen homotopy types of torsion-free 1-connected H-spaces. Again the results depend on the prime 2 behaving differently from the other primes [2], [9], [5], [14].

A 1-connected torsion-free CW-complex \( X \) which is an H-space of rank two and type \( (q, n) \) has the same homotopy type as the total space of an \( S^r \)-fibration over the sphere \( S^n \) [16]. Such a total space is homotopically equivalent to a CW-complex \( S^n \cup \epsilon^r \cup \epsilon^{n+r} \) [4]. Localization at a prime \( p \) yields another fibration \( S^n_p \to X_p \to S^n_p \) [19]. These are the fibrations which will be studied here. Always we assume that \( q, n \) and \( p \) are odd and that \( n > q > 2 \).

The main purpose of this paper is to carry through the results of I. M. James and J. H. C. Whitehead [12] for local spherical fibrations over spheres without assuming the existence of a cross-section. James and Whitehead considered fiber bundles \( S^r \to B \to S^n \) and showed that \( B \) is a cell-complex of the form \( S^r \cup_{\alpha} \epsilon^r \cup \epsilon^{n+r} \). For bundles \( S^r \to B \to S^n \) with cross-section (i.e. with \( \alpha = 0 \)), there are elements \( \lambda(B_i) \) in \( \pi_{n+r-1}(S^r) \) such that \( \lambda(B_1) = \pm \lambda(B_2) \) if and only if \( (B_1, S^r) \) and \( (B_2, S^r) \) have the same homotopy type. Also, for a bundle with cross-section, \( \lambda(B) = 0 \) if and only if \( B \) and \( S^r \times S^n \) have the same homotopy type. Furthermore, \( B \) is an H-space if and only if \( \lambda(B) = 0 \) and the spheres \( S^r \) and \( S^n \) are H-spaces.

In Section 2, it will be shown that the total space of a local spherical fibra-
tion \( S^q_p \to E \to S^n_p \) is homotopically equivalent to a local cell complex \( S^q_p \cup_a e^q_p \cup e^{q-1}_p \).

Section 3 is devoted to fibrations with fixed \( \alpha \) and to the construction of an element \( \lambda_a(E) \) in \( \pi_{n+q-1}(S^q_p \cup_a e^q_p) \) for each of these fibrations. If a cross-section exists, then the injection \( i: S^q_p \to S^n_p \cup_a e^q_p \) induces a monomorphism in homotopy, and the element \( i_*^{-1} \lambda_a(E) \) in \( \pi_{n+q-1}(S^q_p) \) is uniquely defined; this element corresponds to James and Whitehead's \( \lambda(B) \). Certain subsets of \( \text{Im} \, i_* \) in \( \pi_{n+q-1}(S^q_p \cup_a e^q_p) \) will be defined in such a way that they correspond to James and Whitehead's set \( \{ \pm \lambda(B) \} \) for fixed \( \alpha = 0 \).

In Section 4, again \( \alpha \) in \( \pi_{n-1}(S^q_p) \) is a fixed element. The main result is:

**Theorem 4.4.** Suppose that \( q \) and \( n \) are odd integers and that \( p \) is an odd prime. Let \( S^q_p \to E \to S^n_p \) be a fibration such that \( E \) has first attaching map \( \alpha \). If \( p > 3 \), then \( \lambda_a(E) = 0 \) if and only if \( E \) is an \( H \)-space. For \( p = 3 \), if \( E \) is an \( H \)-space, then \( \lambda_a(E) = 0 \).

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2. The total space as a local \( CW \)-complex. In this section it will be shown that the total space \( S^q_p \to E \to S^n_p \) is homotopically equivalent to a local \( CW \)-complex. First, some notation and definitions are needed.

The local sphere \( S^r_p \) can be considered as the suspension of \( S^{r-1}_p \) for \( r > 2 \) because the localization of the suspension of a simply-connected space \( X \) has the homotopy type of the suspension of \( X \) localized, i.e., \( (\Sigma X)_p \simeq (\Sigma X)_p \) \([19]\).

Let

\[
S^r_p = \{ [x, t] \mid x \in S^{r-1}_p, -1 \leq t \leq 1; [x_1, 1] = [x_2, 1] \quad \text{and} \quad [x_1, -1] = [x_2, -1] \}
\]

and let the base point \( a_r \) of \( S^r_p \) be \([x, 1]\), where \( x \in S^{r-1}_p \). The local \( r \)-cell \( e^r_p \) is defined to be the cone on \( S^{r-1}_p \) with vertex \( b_r = [x, 0] \), where \( x \in S^{r-1}_p \). As defined by Sullivan \([19]\), a local \( CW \)-complex is a space constructed inductively from a point or local sphere \( S^m_p \) by attaching local cells \( e^r_p \) by maps of local spheres \( S^{r-1}_p \) into the cells of lower dimension.

Define a map \( u_r: e^r_p \to S^r_p \) by \( u_r([x, t]) = [x, 2t - 1] \) for \( x \in S^{r-1}_p \) and \( 0 \leq t \leq 1 \). Then, on the boundary \( S^{r-1}_p \), we have that \( u_r([x, 1]) = a_r \).

The following is a special case of the local form of Proposition 1 in \([17]\) with a modification of the proof \([11]\).

**Proposition 2.1.** Let \( F \to E \to S^n_p \) be a fibration, and suppose that the fiber \( F \) is a local \( CW \)-complex. Then the total space \( E \) has the homotopy type of a local \( CW \)-complex \( K = F \cup (e^q_p \times F) \).
Proof. Let $\chi$ be the map $\iota_*: e^n_p \to S^n_p$, and let $\pi: E \to S^n_p$ be the fiber map. Consider the induced fibration $\pi:X \to e^n_p$. Since the cone $e^n_p$ is contractible, the induced fiber space $E_x$ is fiber homotopy equivalent to the product $e^n_p \times F$. Let $\phi: e^n_p \times F \to E_x$ and $\psi: E_x \to e^n_p \times F$ be fiber homotopy inverses of each other such that the map

$$\phi|\iota_* \times F: \iota_* \times F \to \pi^{-1}(\iota_*)$$

is homotopic to the identity mapping of the fiber $F$. Also, let $\xi: E_x \to E$ and $\rho: e^n_p \times F \to e^n_p$ be the natural projections. Let $F = \pi^{-1}(a_n)$. Then, for $x \in S^{n-1}_p$ and $y \in F$, we have that $\xi(x,y) \in F$. Let $\nu = \xi_\phi|\iota_* \times F$, and use the map $\nu$ to construct the complex $K = F \cup_s (e^n_p \times F)$. The following lemma completes the proof.

**Lemma 2.2.** The spaces $E$ and $K$ are homotopically equivalent.

**Proof.** Let $\theta: K \to E$ be the map induced by $\xi\phi$. A map $\beta: E \to K$ will be defined such that $\theta$ and $\beta$ are homotopy inverses of each other.

Let $h_t: E_x \to E_x$ be a homotopy such that $h_1 = 1$ and $h_0 = \phi\psi$. Using the definition of $e^n_p$ as the cone on $S^{n-1}_p$, define a map $s: e^n_p \to e^n_p$ by:

$$s([x, t]) = [x, 2t] \quad \text{if } 0 \leq t \leq 1/2, \quad x \in S^{n-1}_p;$$

$$= [x, 1] \quad \text{if } 1/2 \leq t \leq 1, \quad x \in S^{n-1}_p.$$  

Then the map $s$ is homotopic to the identity on $e^n_p$ under a homotopy $G$ which keeps each point of $S^{n-1}_p$ fixed; assume that $G|e^n_p \times 0 = 1$ and $G|e^n_p \times 1 = s$. Since $e^n_p$ is a metric space, the fibration $\pi_x: E_x \to e^n_p$ is regular [10]. This means that any homotopy into $e^n_p$ that keeps certain points stable can be lifted to a homotopy which keeps the same points stable. Then, since $G(\pi_x \times 1)(x, t) = G(\pi_x \times 1)(x, t')$ for $0 \leq t, t' \leq 1$ and $\pi_x(x) \in S^{n-1}_p$, there is a homotopy $H: E_x \times 1 \to E$ such that $\pi_x H = G(\pi_x \times 1)$, $H|E_x \times 0$ is the identity on $E_x$, and $H(x, t) = H(x, t')$ for $0 \leq t, t' \leq 1$ and $x \in E_x$ such that $\pi_x(x) \in S^{n-1}_p$. Define a map $\nu: E_x \to E_x$ by $\nu = H|E_x \times 1$. Then, by the properties of the map $H$ listed above, we have that $\pi_x \nu = s \pi_x$, the map $\nu$ is homotopic to the identity on $E_x$, and $\nu(x) = x$ for all $x \in E_x$ such that $\pi_x(x) \in S^{n-1}_p$.

Let $e$ be a point of $E \setminus F$. Then $\pi(e) \in S^n_p \setminus a_n$, and there is exactly one point $x \in e^n_p$ such that $\chi(x) = \pi(e)$. Then the set $\xi^{-1}(e)$ consists of the one point $(x, e) \in E_x$. Let $j: F \to K$ be the inclusion, and let $\eta: e^n_p \times F \to K$ be the map induced by $\nu$. Define a map $\beta: E \to K$ extending the identity on $F$ by:

$$\beta(e) = \eta\psi_{\xi^{-1}(e)} \quad \text{if } 0 \leq t \leq 1/2$$

$$= j\xi_\phi\psi_{\xi^{-1}(e)} \quad \text{if } 1/2 \leq t \leq 1.$$  

Then $\beta$ is a continuous map because the two definitions for $t = 1/2$ agree, and $\xi_\phi\psi_{\xi^{-1}(e)}$ lies in $F$ for $t \geq 1/2$ and equals $e$ for $t = 1$.  

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The maps $\beta \delta$ and $\delta \beta$ are homotopic to the appropriate identities. This completes the proof of the lemma.

**Corollary 2.3** Let $S^s_p \to E \to S^s_p$ be a fibration. Then the total space $E$ is homotopically equivalent to a local CW-complex $K$ with decomposition $S^s_p \cup e^s_p \cup e^{s+q}_p$.

**Proof.** The total space $E$ is homotopically equivalent to a complex $K = S^s_p \cup v(e^s_p \times S^s_p)$ by Proposition 2.1. Let $h: e^s_p \times S^s_p \to K$ be the map determined by $v$; let $k = h(1 \times u_s): e^s_p \times e^s_p \to K$. (This notation, which will be used throughout the rest of this paper, is that used by James and Whitehead [12] in discussing the cellular decomposition of the total space of a bundle.)

Then we have that

$$k(S^{s-1}_p \times e^s_p) \subset S^s_p, \quad k(\text{Int } e^s_p \times S^{s-1}_p) \subset e^s_p \times a_q = e^s_p, \quad \text{and} \quad k(a_{s-1} \times S^{s-1}_p) = \text{a point } e^0.$$ 

This yields a decomposition of $K$ as the local CW-complex $e^0 \cup e^s_p \cup_a e^s_p \cup e^{s+q}_p,$ where

$$e^0 = k(a_{s-1}, a_{q-1}), \quad S^s_p = e^0 \cup e^s_p, \quad e^q = k(e^s_p \times a_{q-1}), \quad \alpha = \nu|S^{s-1}_p \times a_q,$$

and $e^{s+q}_p = k(e^s_p \times e^s_p)$.

which is attached by the map $k|(e^s_p \times e^s_p)$.

**3. Homotopy type of** $(E, S^s_p)$. Let $\alpha \in \pi_{s-1}(S^s_p)$ be a fixed homotopy class. We will consider only those fibrations $S^s_p \to E \to S^s_p$ such that $E$ has the homotopy type of a local CW-complex $K$ with first attaching map $\alpha$. Then $K$ has the form:

$$K = S^s_p \cup_a e^s_p \cup e^{s+q}_p.$$ 

Let $L$ denote the subcomplex of $K$ defined by: $L = S^s_p \cup_a e^s_p$. Certain subsets of $i_\beta \pi_{s+q-1}(S^s_p) \subset \pi_{s+q-1}(L)$ will be designated in such a way that each subset corresponds to a homotopy class of pairs $(E, S^s_p)$.

The map $k$ of the preceding section determines particular generators $i_\mu$ of $\pi_s(L, S^s_p)$ and $i_\nu$ of $\pi_q(S^s_p)$. Let $i_\mu = [k|e^s_p \times a_{q-1}]$, and let $i_\nu = [k|a_{s-1} \times e^s_p]$.

In order to study the homotopy class of the boundary of the map $k$, maps $f$ and $g$ of the boundary of $e^s_p \times e^s_p$ into itself will be defined. Composing the boundary of $k$ with these two maps will lead to expressing the homotopy class of the boundary of $k$ as a sum of two elements. One of these elements determines the homotopy type of the pair $(E, S^s_p)$, and the other element is similar to a Whitehead product of $i_\mu$ and $i_\nu$. We first define this product in general.

Suppose that $A$ is an $H$-space and a subspace of a space $X$. Let $\beta$ in $\pi_\xi(A)$
and \( \gamma \) in \( \pi_n(X, A) \) be represented by the maps:

\[
\begin{align*}
b &: (e^{q}, S^{q-1}) \to (A, \ast), \\
c &: (e^{n_1}, e^{n-1}, e^{n-1}_+) \to (X, A, \ast),
\end{align*}
\]

where

\[
\begin{align*}
e_{n_1} &= \{ [x, t] \in e^{n-1} | t \geq 0 \}, \\
ev_{n-1} &= \{ [x, t] \in e^{n-1} | t = 0 \}, \\
ev_{n-1}^+ &= \{ [x, t] \in e^{n-2} | t \geq 0 \}.
\end{align*}
\]

Let \( Y \) be the space \((e^{q}_1 \times S^{q-1}) \cup (e^{n-1}_+ \times e^{q}) \cup (e^{n-1} \times e^{q})\), which is homotopically equivalent to \(e^{n+q-1}_1\). Consider the map \((b, c): Y \to X\) defined by:

\[
(b, c)(u, v) = \begin{cases} 
  c(u) & \text{if } (u, v) \in e^{q}_1 \times S^{q-1}, \\
  b(v) & \text{if } (u, v) \in e^{n-1}_+ \times e^{q}, \\
  c(u) \cdot b(v) & \text{if } (u, v) \in e^{n-1} \times e^{q},
\end{cases}
\]

where the product means multiplication in the \(H\)-space \(A\). The first two parts of this definition give a representative of the relative Whitehead product \([\beta, \gamma]\) in \(\pi_{n+q-1}(X, A)\), and the last part is the usual map for showing that any Whitehead product (and, in this case, \(\partial[\beta, \gamma] = [\beta, \partial \gamma]\)) is trivial for an \(H\)-space. Since any homotopies \(b_t\) and \(c_t\) yield a homotopy \((b_t, c_t)\), we can define the product:

**Definition 3.1.** \([\beta, \gamma]_X\) is the homotopy class of \((b, c)\) in \(\pi_{n+q-1}(X)\).

Alternately, the representative of the product \([\beta, \gamma]_X\) could be defined as follows: use the \(H\)-structure of \(A\) to deform a representative of the relative Whitehead product \([\beta, \gamma]\) to a map which is trivial on the boundary of \(e^{n}_1 \times e^{q}\). The next proposition lists the properties of this product.

**Proposition 3.2.** Suppose that \(A\) is an \(H\)-space and subspace of a space \(X\). Consider homotopy classes \(\beta, \beta_1\) and \(\beta_2\) in \(\pi_q(A)\) and \(\gamma, \gamma_1\) and \(\gamma_2\) in \(\pi_n(X, A)\). Then:

1. \(j_\ast([\beta, \gamma]_X) = [\beta, \gamma]\), where \(j: (X, \ast) \to (X, A)\) is the inclusion.
2. \([\beta_1 + \beta_2, \gamma]_X = [\beta_1, \gamma]_X + [\beta_2, \gamma]_X\).
3. \([\beta, \gamma_1 + \gamma_2]_X = [\beta, \gamma_1]_X + [\beta, \gamma_2]_X\).
4. Suppose that \(B\) is an \(H\)-space and subspace of a space \(Y\) and that \(f: (X, A) \to (Y, B)\) is a map. Then \(f_\ast([\beta, \gamma]_Y) = [f_\ast \beta, f_\ast \gamma]_Y\).

**Proof.** The first three properties follow immediately from the definition of the product. We now prove the last property.

Let the maps \(b\) and \(c\) represent \(\beta\) and \(\gamma\). Then

\[
\begin{align*}
f(b, c)(u, v) &= fc(u) & \text{if } (u, v) \in e^{q}_1 \times S^{q-1}, \\
&= fb(v) & \text{if } (u, v) \in e^{n-1}_+ \times e^{q}, \\
&= f(c(u) \cdot b(v)) & \text{if } (u, v) \in e^{n-1} \times e^{q}.
\end{align*}
\]
The only difficulty lies in the third line; here we know that
\[ f(c(u) \cdot b(v)) = fm(c(u), b(v)) = fm(c \times b)(u, v), \]
where \( m \) is the multiplication in \( A \). We want to show that \( fm(c \times b) \) is homotopic to \( m'(fc \times fb) \), where \( m' \) is the multiplication in \( B \). For \( [a] \) in \( \pi_1(A \times A) \), projections \( \rho : A \times A \to A \), and diagonal map \( \Delta : S^r \to S^r \times S^r \), we have that
\[
[fma] = f_\ast[m(\rho_1a \times \rho_2a)\Delta] = f_\ast([\rho_1a] + [\rho_2a]) = [fp_1a] + [fp_2a] = [m'(fp_1a \times fp_2a)\Delta] = [m'(f \times f)\Delta].
\]
Then, letting \( a = c \times b \), we find that
\[
[fma] = [m'(fc \times fb)(c \times b)] = [m'(fc \times fb)],
\]
and thus \( f(c \times b) \) is homotopic to \( m'(fc \times fb) \). Therefore,
\[
\text{and the proposition is proved.}
\]
Figure 1. Subsets of $e^r_p$; the maps $f$ and $g$. 
Figure 2. The map \( fg \) on \( e^*_p \times y, y \in S^{n-1} \).
identity. If the points \((x_1, y)\) and \((x_2, y)\) of \(e^{n-1}_p \times S^{i-1}_p\) lie on a line orthogonal to \(e^{n-1}_p \times y\), then \(g(x_1, y) = g(x_2, y)\).

Define a map \(F: (e^{n}_p \times e^{q}_p)^* \rightarrow L\) to be the composition \(kfg\). Then \(F\) is homotopic to \(k\) since \(g\) and \(f\) are homotopic to the identities. Let \(\sigma\) be the homotopy class of \(k\) in the group \(\pi_{n+q}(K, L)\). This group is isomorphic to \(Z(p)\), the integers localized at \(p\), and \(\sigma\) is a generator. The map \(F\) represents \(\partial_1 \sigma\) in \(\pi_{n+q-1}(L, S^q_p)\), where \(\partial_1: \pi_{n+q}(K, L) \rightarrow \pi_{n+q-1}(L, S^q_p)\) is the boundary homomorphism. Let \(G\) be the restriction of \(F\) to \((e^{n}_1 \times S^{i-1}_p) \cup (e^{n-1}_q \times e^{q}_p)\). Then the map \(G\) represents the relative Whitehead product \([i_q, i_n]\) in \(\pi_{n+q-1}(L, S^q_p)\).

Let \(H\) be the restriction of \(F\) to \((e^{n}_1 \times S^{i-1}_p) \cup (e^{n-1}_q \times e^{q}_p)\). Then the image of \(H\) lies in \(S^q_p\). The restriction of \(H\) to the boundary of \(e^{n-1}_q \times e^{q}_p\) is a map which represents the Whitehead product \([\alpha, i_n]\) in \(\pi_{n+q-2}(S^q_p)\). Since \(S^q_p\) is an \(H\)-space, the Whitehead product \([\alpha, i_n]\) is trivial. Use the \(H\)-structure to deform the map \(F\) to a new map which is trivial on \((e^{n-1}_q \times e^{q}_p)^*\). Now call this new map \(F\), and use the names \(H\) and \(G\) for the same restrictions of the new \(F\). Then \(H\) maps \((e^{n-1}_q \times e^{q}_p)^*\) to the point \(e^q\), and \([H] \in \pi_{n+q-1}(S^q_p)\).

Let \(\partial: \pi_{n+q}(K, L) \rightarrow \pi_{n+q-1}(L)\) be the boundary homomorphism, and let \(i: S^q_p \rightarrow L\) be the inclusion.

**Definition 3.3.** \(\lambda_n(E) = i_n[H]\) in \(\pi_{n+q-1}(L)\).

The next proposition follows immediately from the definitions of the maps \(G\) and \(H\) as restrictions of the map \(F\) (Figure 3).

**Proposition 3.4.** \(\partial \sigma = \lambda_n(E) + [i_q, i_n]_L\) and \(\partial_1 \sigma = [i_q, i_n]\).

**Definition 3.5.** \(\Psi_n(E) = \{\psi \in \pi_{n+q-1}(L)|\psi = c\lambda_n(E)\text{ for some unit } c\text{ of } Z(p)\}.

**Theorem 3.6.** Let \(q\) and \(n\) be odd integers, and let \(p\) be an odd prime. Assume that \(S^q_p \rightarrow E_i \rightarrow S^n_p\) is a fibration for \(i = 1\) and \(2\) and that the first attaching maps in the local cellular decompositions of the total spaces are the same. Call the common map \(\alpha\). Then \((E_1, S^q_p)\) and \((E_2, S^n_p)\) have the same homotopy type if and only if \(\Psi_n(E_1) = \Psi_n(E_2)\).

**Proof.** Suppose first that \(\Psi_n(E_1) = \Psi_n(E_2)\). Then, since \(\lambda_n(E_1) \in \Psi_n(E_2)\), there exists a unit \(c\) of \(Z(p)\) such that \(\lambda_n(E_1) = c\lambda_n(E_2)\), i.e., \(i_n[H_1] = ci_n[H_2]\).

Let \(\beta = cj: S^n_q \rightarrow S^n_p\), where \(j\) is the identity mapping. We will apply the local form of the right distributive law: \((\mu + \eta)\gamma = \mu \gamma + \eta \gamma\) for \(\gamma \in \pi_i(S^n_p)\) and \(\mu, \eta \in \pi_r(X)\) such that the Whitehead product \([\mu, \eta] = 0\) [7, Lemma 6.5, p. 166]. Since all Whitehead products in the \(H\)-space \(S^n_p\) are trivial, we have that \(\beta \gamma = (cj)\gamma = c(j\gamma) = c\gamma \) for \(\gamma \in \pi_i(S^n_p)\). It follows that \(\beta \alpha \simeq c\alpha\). Also, we have that \(\beta\) induces isomorphisms in the homotopy groups, and thus \(\beta\) is a homotopy equivalence. Then the map \(\beta\) can be extended to a homotopy equivalence \(\nu: L \rightarrow L\) such that \(\nu_n(i_n) = ci_n\).
Figure 3. The Map $F$. Points on the same dotted line have the same image under $F$. Points in the shaded area of $e_p \times y, y \in S^{n-1}_p$, are mapped to $e^n$ by the map $F$.

Now, letting $\gamma = [H_1]$, we find that $\beta_*[H_1] = e[H_1]$. Then by Propositions 3.2 and 3.4 we have that:

\[
\begin{align*}
\nu_*\partial_1 &= \nu_*i_*[H_1] + \nu_*[i_1, i_{n1}]_L \\
&= i_*\beta_*[H_1] + [\beta_*i_1, \nu*i_{n1}]_L \\
&= i_*c[H_1] + [ci_1, ci_{n2}]_L \\
&= c^2i_*[H_2] + c^2[i_2, i_{n2}]_L \\
&= c^2\partial_2.
\end{align*}
\]

This means that the second attaching map $k_1|S^n+x-1_p$ of the total space $E_1$ is homotopic to $c^2(k_2|S^n+x-1_p)$. Then the map $\nu$ can be extended to a homotopy equivalence $\theta: E_1 \to E_2$. Since $\theta|S^n_p = \beta$, this yields a homotopy equivalence $(E_1, S^n_p) \to (E_2, S^n_p)$.

Before considering the converse, we will localize some of James's results [13]. If $f: X_p \to Y_p$ is a map, we will use $f': X \to Y$ to denote a map such that $(f')_p = f$. James shows that the homomorphism

\[
(i'_n)_*: \pi_{n+q-1}(e^n, S^{n-1}) \to \pi_{n+q-1}(S^q \cup e^n, S^q)
\]

is a monomorphism and that $\pi_{n+q-1}(S^q \cup e^n, S^q) = Z \oplus \text{lm } (i'_n)_*$, where the Whitehead product $[i'_n, i_q]$ is a generator of the infinite cyclic group $Z$. Then
we have that
\[ \pi_{n+q-1}(S^q_p \cup \alpha e^p_e, S^q_p) = Z(p) \oplus (\text{Im } (i'_q)_e)_{\alpha}, \]
where \([i'_n, i'_q] = [i_n, i_q] \) is a generator of \( Z(p) \). James proves that
\[ \partial_1 \sigma = m[i'_n, i'_q] + i'_n \rho', \]
where
\[ i'_q \subset i'_n = m \sigma' \]
and \( \rho' \in \pi_{n+q-1}(e^n, S^{n-1}) \).
Then \( \partial_1 \sigma = m[i'_n, i'_q] + i'_n \rho \). But we know that \( \partial_1 \sigma = [i_n, i_q] \) by Proposition 3.4. Then \( m = 1 \) and \( i'_n \rho = 0 \). Thus \( i'_q \subset i'_n = \sigma \), where \( i'_q \) and \( i'_n \) correspond to the homotopy classes of the same name.

Now we assume that \((E_1, S^q_p)\) and \((E_2, S^q_p)\) have the same homotopy type and let \( \theta: (E_1, S^q_p) \to (E_2, S^q_p) \) be a homotopy equivalence. Then, since the cohomology groups \( H^n(L, S^q_p; Z(p)) \) and \( H^s(S^q_p; Z(p)) \) are both isomorphic to \( Z(p) \),
\[ \theta^*(i_{q2}) = bi_{q1} \]
and \( \theta^*(i_{n2}) = ci_{n1} \)
for some units \( b \) and \( c \) of \( Z(p) \). Then
\[ \theta^*(i_{q2} \subset i_{n2}) = bc(i_{q1} \subset i_{n1}) \text{ in } H^{n+q}(E_1). \]
For \( s = 1 \) and \( 2 \), let \( \sigma_s = (k^*)^{-1}(i_{q2} \subset i_{n1}) \), where \( k^*: H^{n+q}(E_1, L) \to H^{n+q}(E_2) \) is the isomorphism induced by the inclusion \( k \). Then \( \theta^*(\sigma_2) = b c \sigma_1 \) in \( H^{n+q}(E_1, L) \). Since \( \sigma_s \) in cohomology corresponds to the original \( \sigma_s \) in homotopy, we have that \( \theta_*^*(\sigma_1) = b c \sigma_2 \). Also, \( \theta_*^*(i_{q1}) = bi_{q2} \) and \( \theta_*^*(i_{n1}) = ci_{n2} \) in homotopy. Thus,
\[ \theta_*^*(i_{q2} \subset i_{n1}) = \theta_*^*(\partial \sigma_1 - [i_{q1}, i_{n1}]_{L}) = bc \partial \sigma_2 - bc[i_{q2}, i_{n2}]_L. \]
Then \( \theta_*^*[H_1] = b c i_*^*[H_2]. \)
Next we will show that \( \theta_*^*[H_1] = c[H_1] \). Since \( \theta_*^*(i_{q1}) = bi_{q2} \) and each \( i_{q2} \) is the homotopy class of the identity on \( S^q_p \), we have that \( \theta|_{S^q_p} = bi_{q2} \). Then
\[ \theta_*^*[H_1] = (bi_{q1})[H_1] = b[H_1] \text{ [7], and} \]
\[ bi_*[H_1] = i_*b[H_1] = i_*\theta_*[H_1] = \theta_*i_*[H_1] = bci_*[H_2]. \]
Since \( b \) is a unit, this gives that \( i_*[H_1] = ci_*[H_2] \), i.e., \( \lambda_a(E_1) = c\lambda_a(E_2) \).
We have shown that \( \lambda_a(E_1) \in \Psi_a(E_2) \). Therefore, we have that \( \Psi_a(E_1) = \Psi_a(E_2). \)
This completes the proof of the theorem.

4. The total space \( E \) as an \( H \)-space. Suppose that \( \alpha \) is a fixed element of \( \pi_{n-1}(S^q_p) \) and that \( S^q_p \to E \to S^q_p \) is a fibration such that the total space \( E \) has local cellular decomposition \( S^q_p \cup \alpha e^p_e \cup e^{n+q} \). The aim of this section is to show that, for \( p > 3 \), \( E \) is an \( H \)-space if and only if \( \lambda_a(E) = 0 \). In the case \( \alpha = 0 \), Curtis [4] shows that \( E \) is an \( H \)-space if and only if \( E \) has the
same homotopy type as $S^{q}_{p} \times S^{q}_{p}$. For $\alpha \neq 0$, there is a space $E_{\alpha}$ which plays the role of $S^{q}_{p} \times S^{q}_{p}$. The space $E_{\alpha}$ is defined to be the local CW-complex $S^{q}_{p} \cup \alpha(e^{q}_{p} \times S^{q}_{p})$, where $\alpha'(x, y) = \alpha(x) \cdot y$, the product $\cdot$ is multiplication in the $H$-space $S^{q}_{p}$, and $x \in S^{q-1}_{p}$, $y \in S^{q}_{p}$. Stasheff [18] proves that $E_{\alpha}$ is an $H$-space if $n < (1/2)(p - 2)(q + 1)$.

**Proposition 4.1.** If $\alpha$ is nontrivial, then $\lambda_{\alpha}(E_{\alpha}) = 0$. Also $\lambda_{0}(S^{q}_{p} \times S^{q}_{p}) = 0$.

**Proof.** It suffices to show that the map $H$ (of Section 3) is homotopic to the trivial map for these spaces.

The space $S^{q}_{p} \times S^{q}_{p}$ can be represented as $S^{q}_{p} \cup \gamma(e^{q}_{p} \times S^{q}_{p})$, where, for $(x, y) \in S^{q-1}_{p} \times S^{q}_{p}$, $\gamma(x, y) = \alpha(x) \cdot y$. Thus, the map $\gamma$ corresponds to $\alpha'$ in the definition of $E_{\alpha}$, and it will be called $\alpha'$.

Both spaces $S^{q}_{p} \times S^{q}_{p}$ and $E_{\alpha}$ can be decomposed as local CW-complexes $S^{q}_{p} \cup e^{q}_{p} \cup e^{q+p}_{p}$. The first attaching map of $E_{\alpha}$ is $\alpha$ since, on $S^{q-1}_{p} \times S^{q}_{p}$,

$$\beta(x, y) = \alpha(x) \cdot u_{q}(y) \quad \text{if} \quad (x, y) \in S^{q-1}_{p} \times e^{q}_{p};$$

$$\beta(x, y) = x \quad \text{if} \quad (x, y) \in e^{q}_{p} \times S^{q+p-1}_{p}.$$ 

The map $H$, as defined in Section 3, is the composition $\beta f g$ restricted to $(e^{q}_{p} \times S^{q+p-1}_{p}) \cup (e^{q+p-1}_{p} \times e^{q}_{p})$. Then, for $([x, t], y) \in e^{q+p-1}_{p} \times e^{q}_{p}$, $x \in (e^{q+p-1}_{p})'$, $-1 \leq t \leq 1$, we have that

$$H([x, t], y) = (\beta f g)([x, t], y) = \alpha([x, 2t + 1]) \cdot u_{q}(y).$$

For $(s, y) \in e^{q}_{p} \times S^{q+p-1}_{p}$ such that $g(s, y) = ([x, t], y)$, we have that $H(s, y) = [x, 2t + 1]$.

The map $H$ can be extended to a map $J: e^{q}_{p} \times e^{q}_{p} \rightarrow S^{q}_{p}$ by defining

$$J(s, y) = \alpha([x, 2t + 1]) \cdot u_{q}(y),$$

where $s \in e^{q}_{p}$, $y \in e^{q}_{p}$ and $g(s, y) = ([x, t], y)$. Since $H$ can be extended to $e^{q}_{p} \times e^{q}_{p}$, $H$ is homotopic to the trivial map, and thus $\lambda_{\alpha}(E) = i_{*}[H] = 0$.

**Proposition 4.2.** Suppose that $p$ is an odd prime and that $q$ and $n$ are odd. Let $S^{q}_{p} \rightarrow E \rightarrow S^{q}_{p}$ be a fibration such that the total space $E$ has first attaching map $\alpha$. If $E$ is an $H$-space, then the spaces $E$ and $E_{\alpha}$ have the same homotopy type.

**Proof.** Let $m: E \times E \rightarrow E$ be the multiplication. We can assume that $m$ restricted to $S^{q}_{p} \times S^{q}_{p}$ provides an $H$-structure for the fiber $S^{q}_{p}$ [3], [6]. Let

$$\mu = m|_{S^{q}_{p} \times S^{q}_{p}}: S^{q}_{p} \times S^{q}_{p} \rightarrow S^{q}_{p}.$$ 

In $E \times E$ define an equivalence relation $\sim$ by: $(u, v) \sim (u', v')$ if and only if $m(u, v) = m(u', v')$ and $u, u' \in S^{q-1}_{p}$ and $v, v' \in S^{q}_{p}$. Define the map $g: E_{\alpha} \rightarrow E \times E/\sim$ to be the one induced by the product of inclusions $e^{q}_{p} \times S^{q}_{p}$ →

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This map is well-defined because $\mu: S^*_p \times S^*_p = \mu$. Now define the map
\[ m': E \times E/\sim \rightarrow E \text{ to be the one induced by } m: E \times E \rightarrow E, \text{ and let } f: E_a \rightarrow E \text{ be the composition of } m' \text{ and } g. \]

We want to show that the map $f$ is a homotopy equivalence. Let $i_q$, $i_n \in H^*(E)$ and $i'_q$, $i'_n \in H^*(E_a)$ be the generators corresponding to those in homotopy constructed from the map $k$ (in Section 3) for the spaces $E$ and $E_a$. Then, since $f|_{\alpha_{n-1}} \times S^*_p$ is the identity onto $S^*_p$ and $f| e^*_p \times 0$ is the identity onto $e^*_p$, it follows that $f^*(i_q) = i'_q$ and $f^*(i_n) = i'_n$. Thus, we have that
\[ f^*(i_q \cup i_n) = f^*i_q \cup f^*i_n = i'_q \cup i'_n. \]

Since these cup products are the generators in dimension $n + q$, $f^*: H^*(E) \rightarrow H^*(E_a)$ is an isomorphism. Then $f$ is a homotopy equivalence. This completes the proof of the theorem.

**Corollary 4.3.** If $p > 3$, then $E_a$ is an $H$-space.

**Proof.** There exists a fibration $S^q \rightarrow X \rightarrow S^q$ such that the total space $X$ localized at $p > 3$ is an $H$-space homotopic to a local $CW$-complex $S^q \cup e^*_p \cup e^{q+1}_p$ [6]. Then, by the preceding proposition, the spaces $X_p$ and $E_a$ have the same homotopy type, and, thus, $E_a$ is an $H$-space.

**Theorem 4.4.** Suppose that $q$ and $n$ are odd integers and that $p$ is an odd prime. Let $S^*_p \rightarrow E \rightarrow S^*_p$ be a fibration such that $E$ has first attaching map $\alpha$. If $p > 3$, then $\lambda_\alpha(E) = 0$ if and only if $E$ is an $H$-space. For $p = 3$, if $E$ is an $H$-space, then $\lambda_\alpha(E) = 0$.

**Proof.** Suppose that $p \geq 3$ and that $E$ is an $H$-space. Then the spaces $E$ and $E_a$ have the same homotopy type (Proposition 4.2), and thus $\lambda_\alpha(E) = \lambda_\alpha(E_a) = 0$ (Theorem 3.6 and Proposition 4.1).

Now let $p > 3$ and suppose that $\lambda_\alpha(E) = 0$. Then $E$ and $E_a$ have the same homotopy type (Theorem 3.6 and Proposition 4.1). Since $E_a$ is an $H$-space (Corollary 4.3), the space $E$ also is an $H$-space.

This concludes the discussion of $H$-spaces with three local cells.

**References**

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