Can. J. Math., Vol. XXVII, No. 3, 1975, pp. 618–628

COTORSION THEORIES AND COLOCALIZATION

R. J. McMASTER

Introduction. Let R be an associative ring with unit element. Mod-R and R-Mod will denote the categories of unitary right and left R-modules, respectively, and all modules are assumed to be in Mod-R unless otherwise specified. For all M, $N \in$ Mod-R, $\text{Hom}_R(M, N)$ will usually be abbreviated as $[M, N]$. For the definitions of basic terms, and an exposition on torsion theories in Mod-R, the reader is referred to Lambek [**6**]. Jans [**5**] has called a class of modules which is closed under submodules, direct products, homomorphic images, group extensions, and isomorphic images a *TTF* (torsion-torsionfree) class. Since such a class $\mathcal{T}$ is not closed under injective hulls, while a torsionfree class is closed under injective hulls, we find this terminology misleading and shall instead (following a suggestion by J. Golan) call $\mathcal{T}$ a *Jansian* class from now on. (A torsion class $\mathcal{T}$ which is closed under injective hulls is called *stable*, and hence a stable Jansian class is a true torsion-torsionfree class.)

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory then modules in $\mathcal{T}$ are called *torsion*, and modules in $\mathcal{F}$ are called *torsionfree*. Each $M \in$ Mod-R has a unique maximal torsion submodule, denoted by $\mathcal{T}(M)$. (It is the unique submodule $X \subseteq M$ such that X is torsion and M/X is torsionfree.) A submodule D of M is called *dense* if M/D is torsion. Let $\mathcal{D}_{\mathcal{T}}$ denote the set of all dense right ideals of R. $\mathcal{D}_{\mathcal{T}}$ forms an *idempotent* (or *Gabriel*) *filter*, i.e. it satisfies the following conditions:

(0) $R \in \mathcal{D}_{\mathcal{T}}$,

(1) $D \in \mathcal{D}_{\mathcal{T}}$ and $D \subseteq K \Rightarrow K \in \mathcal{D}_{\mathcal{T}}$,

(2) $D \in \mathcal{D}_{\mathcal{T}}$ and $r \in R \Rightarrow (r : D) \in \mathcal{D}_{\mathcal{T}}$, where $(r : D) = \{x \in R | rx \in D\}$,

(3) $D \in \mathcal{D}_{\mathcal{T}}$ and $(d : K) \in \mathcal{D}_{\mathcal{T}}$ for all $d \in D \Rightarrow D \cap K \in \mathcal{D}_{\mathcal{T}}$.

Gabriel [**4**] has shown that there is a one-to-one correspondence between torsion classes in Mod-R and idempotent filters of right ideals of R: to a torsion class $\mathcal{T}$ associate the idempotent filter $\mathcal{D}_{\mathcal{T}}$, and to an idempotent filter $\mathcal{D}$ associate the torsion class $\mathcal{T}_{\mathcal{D}} = \{M \in \text{Mod-}R | (m : 0) \in \mathcal{D} \text{ for all } m \in M\}$.

Jans [**5**] showed that a torsion class $\mathcal{T}$ is a Jansian class if and only if $\mathcal{D}_{\mathcal{T}}$ contains a unique minimal right ideal T, in which case T is an idempotent two-sided ideal, and $T = \mathcal{C}(R)$ where $(\mathcal{C}, \mathcal{T})$ is the pre-torsion theory with $\mathcal{T}$ as the pre-torsionfree class. Thus there is a one-to-one correspondence between Jansian classes and idempotent ideals of R, with the inverse correspondence given by $T \rightarrow \{M \in \text{Mod-}R | MT = 0\}$.

Given an injective module I_R, one can form the largest torsion theory for which I is torsionfree (where $(\mathcal{T}, \mathcal{F}) \subseteq (\mathcal{T}', \mathcal{F}')$ if $\mathcal{T} \subseteq \mathcal{T}'$), and in fact

Received December 4, 1973.

every torsion theory is of this form for some injective I. For a given torsion theory $(\mathscr{T}, \mathscr{F})$, a module M is called *divisible* (or $\mathscr{T}$*-injective*) if $I(M)/M \in \mathscr{F}$, where $I(M)$ denotes the injective hull of M. Every module M has a *divisible hull* $D(M)$ defined by $D(M)/M = \mathscr{T}(I(M)/M)$. One also defines the *quotient module* $Q(M)$ of M by $Q(M) = D(M/\mathscr{T}(M))$. $Q(M)$ is also called the *localization* of M at I, where I is an injective module such that $(\mathscr{T}, \mathscr{F})$ is the largest torsion theory for which I is torsionfree.

1. Cotorsion theories. Let P_R be a projective module, let $E = [P, P]$, and let $P^* = [P, R]$. As mentioned above, every torsion theory can be thought of as the largest torsion theory for which some injective module I_R is torsionfree, where a module M is torsion if and only if $[M, I] = 0$. We dualize this in the following definitions:

Definition 1.1. (a) A module M is *cotorsion* if $[P, M] = 0$.
(b) A module M is *cotorsionfree* if $[M, X] = 0$ for all X cotorsion.
(c) If $\mathscr{T}^*$ denotes the class of cotorsion modules, and $\mathscr{F}^*$ the class of cotorsionfree modules, then $(\mathscr{F}^*, \mathscr{T}^*)$ is a *cotorsion theory*.
(d) $\epsilon(M)$ is the evaluation mapping $[P, M] \otimes_E P \rightarrow M$, i.e., $\epsilon(M)(\sum g_i \otimes p_i) = \sum g_i(p_i)$.
(e) $T = \epsilon(R)(P^* \otimes_E P)$, the trace ideal of P.

The following lemma appears in [**12**, Proposition 1.2], and is easily proved.

LEMMA 1.2. *$M \in$ Mod-R is cotorsion if and only if $MT = 0$.*

The equivalence of (2) and (5) in the next proposition also has been noted by Sandomierski [**12**, Proposition 1.2].

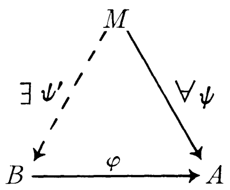
PROPOSITION 1.3. *For all $M \in$ Mod-R, the following conditions are equivalent:*
(1) *M is cotorsionfree.*
(2) $MT = M$.
(3) $M \otimes_R R/T = 0$.
(4) *$\epsilon(M)$ is an epimorphism.*
(5) *M is an epimorphic image of a direct sum of copies of P.*

Proof. (1) $\Leftrightarrow$ (2) M/MT is cotorsion since $(M/MT)T = 0$, hence the projection mapping $M \rightarrow M/MT = 0$. Conversely, for all X cotorsion, and all $\varphi \in [M, X]$, $\varphi(M) = \varphi(MT) = \varphi(M)T \subseteq XT = 0$.
(2) $\Leftrightarrow$ (3) $M/MT \cong M \otimes_R R/T$.
(2) $\Leftrightarrow$ (4) Im $\epsilon(M) = MT$.
(4) $\Leftrightarrow$ (5) This is clear.

Since $T^2 = T$ and $PT = P$, $MT^2 = MT$ and $([P, M] \otimes_E P)T = [P, M] \otimes_E P$ for any M in Mod-R, and thus MT and $[P, M] \otimes_E P$ are cotorsionfree. The class $\mathscr{T}^*$ of cotorsion modules is closed under submodules, direct products, homomorphic images, group extensions, and isomorphic images, i.e. it is a

Jansian class. The class $\mathscr{F}^*$ of cotorsionfree modules is closed under homomorphic images, direct sums, group extensions, isomorphic images, and by [**11**, Proposition 1] minimal epimorphisms (and hence projective covers if they exist).

Definition 1.4. A module M is *codivisible* if for any epimorphism $\varphi: B \to A$ such that Ker φ is cotorsion, any homomorphism $M \to A$ can be extended to a homomorphism $M \to B$, i.e.,



PROPOSITION 1.5. *For all* $M \in$ Mod-R, $[P, M] \otimes_E P$ *is codivisible.*

Proof. We prove that for any $H \in$ Mod-E, $H \otimes_E P$ is codivisible. Let $\varphi: B \to A$ be any epimorphism such that Ker φ is cotorsion. Let ψ be any homomorphism: $H \otimes_E P \to A$. Define $\psi_h: P \to A$ by $\psi_h(p) = \psi(h \otimes p)$ for all $h \in H$, $p \in P$. Then since P is projective there exists $\psi_h': P \to B$ such that $\varphi\psi_h' = \psi_h$. Define $\alpha: H \times_E P \to B$ by $\alpha((h, p)) = \psi_h'(p)$. Since P is projective and $[P, \text{Ker } \varphi] = 0$, $[P, B] \cong [P, A]$, and it is now easily shown that α is bilinear. Therefore there exists $\psi': H \otimes_E P \to B$ such that

$$\varphi\psi'(\sum h_i \otimes p_i) = \varphi(\sum \psi_{h_i}'(p_i)) = \sum \psi_{h_i}(p_i) = \sum \psi(h_i \otimes p_i) = \psi(\sum h_i \otimes p_i)$$

for all $\sum h_i \otimes p_i \in H \otimes_E P$. Thus $\varphi\psi' = \psi$, and hence $H \otimes_E P$ is codivisible.

PROPOSITION 1.6. *For all* $M \in$ Mod-R, Ker $\epsilon(M)$ *is cotorsion.*

Proof. Let $\sum f_i \otimes p_i \in [P, M] \otimes_E P$ such that $\epsilon(M)(\sum f_i \otimes p_i) = \sum f_i(p_i) = 0$. Then for all $f \in P^*$ and $p \in P$, $(\sum f_i \otimes p_i)f(p) = \sum f_i \otimes p_i f(p) = \sum f_i p_i f \otimes p = 0$, since for all $x \in P$, $(\sum f_i p_i f)(x) = \sum f_i(p_i f(x)) = \sum (f_i(p_i))$-$f(x) = (\sum f_i(p_i))f(x) = 0$. Therefore $(\sum f_i \otimes p_i)T = 0$, and Ker $\epsilon(M)$ is cotorsion.

COROLLARY 1.7. *P is a generator* $\Leftrightarrow$ *$\epsilon(M)$ is an isomorphism for all* $M \in$ Mod-R.

Proof. P is a generator $\Leftrightarrow T = R$, i.e. $\epsilon(R)$ is an epimorphism, $\Leftrightarrow$ Ker $\epsilon(M) = 0$ and $MT = M$ for all $M \in$ Mod-R $\Leftrightarrow$ $\epsilon(M)$ is an isomorphism for all $M \in$ Mod-R.

The next theorem is due mainly to Miller [**10**, Theorem 2.1], in particular the equivalence of statements (2) to (7). (2) $\Leftrightarrow$ (5) was also proved by Azumaya [**1**, Theorem 6], along with several more equivalent statements. First we need a lemma, which also appeared in [**10**], but without proof. Since the proof is not completely trivial, we include it here.

LEMMA 1.8. *Let* $\mathscr{H} = \{X \in \text{Mod-}R \mid X'T = X'$ *for all* $X' \subseteq X\}$. *Then* $X \in \mathscr{H}$ *if and only if* $x \in xT$ *for all* $x \in X$. *Also,* $\mathscr{H}$ *is a torsion class* (i.e. it is closed under submodules, direct sums, homomorphic images, group extensions, and isomorphic images).

Proof. Let $X \in \mathscr{H}$, then for all $x \in X$, $xR = xRT = xT$, and therefore $x \in xT$. Conversely, let $X' \subseteq X$. Then for all $x \in X'$, $x \in xT$ and hence $X' = X'T$. Thus $X \in \mathscr{H}$. The non-trivial step in proving that $\mathscr{H}$ is a torsion class is to show that it is closed under direct sums, and this is done by an argument given by Chase [**2**, Proposition 2.2]. Let $X = \oplus_{i \in I} X_i$, where $X_i \in \mathscr{H}$, $i \in I$. Let $X' \subseteq X$, and let $x_{i_1} + \ldots + x_{i_n} \in X'$. We will show by induction on n that there exists $t \in T$ such that $x_{i_j} = x_{i_j}t$ for all $j = 1, \ldots, n$. It is true for $n = 1$ since each $X_i \in \mathscr{H}$. Assume it is true for $n = k - 1$, and let $t_k \in T$ such that $x_{i_k} = x_{i_k}t_k$. Then there exists $t' \in T$ such that $x_{i_j} - x_{i_j}t_k = (x_{i_j} - x_{i_j}t_k)t'$ for $j = 1, \ldots, k - 1$. Let $t = t' - t_kt' + t_k$, then $x_{i_j}t = x_{i_j}t' - x_{i_j}t_kt' + x_{i_j}t_k = x_{i_j}$ for $j = 1, \ldots, k - 1$, and $x_{i_k}t = x_{i_k}t' - x_{i_k}t_kt' + x_{i_k}t_k = x_{i_k}$. Hence it is true for all n, and therefore $X' \in \mathscr{H}$, since $x_{i_1} + \ldots + x_{i_n} \in (x_{i_1} + \ldots + x_{i_n})T$.

THEOREM 1.9. *The following statements are equivalent:*

(1) $\mathscr{T}^*$, *the class of cotorsion modules, is closed under injective hulls.*

(2) $\mathscr{F}^*$, *the class of cotorsionfree modules, is closed under submodules. i.e.,* $\mathscr{F}^* = \mathscr{H}$.

(3) $P \in \mathscr{H}$.

(4) $T \in \mathscr{H}$.

(5) R/T *is flat as a left* R*-module.*

(6) $(p{:}0) + T = R$ *for all* $p \in P$.

(7) $(t{:}0) + T = R$ *for all* $t \in T$.

(8) *Every cotorsionfree module is codivisible.*

(9) $F\colon M \to M/MT$ *for all* $M \in$ Mod-R *is an exact functor.*

Proof. (1) $\Leftrightarrow$ (2) This is well known.

(2) $\Rightarrow$ (7) Since $\mathscr{F}^* = \mathscr{H}$, $\mathscr{F}^*$ is a torsion class by Lemma 1.8, and thus has a corresponding idempotent filter $\mathscr{D}_{\mathscr{F}^*}$. Since $T \in \mathscr{F}^*$, $(t{:}0) \in \mathscr{D}_{\mathscr{F}^*}$ for $t \in T$, i.e., $R/(t{:}0) \in \mathscr{F}^*$ and hence $(t{:}0) + T = R$.

(7) $\Rightarrow$ (5) $R = (t{:}0) + T$ for $t \in T$, and therefore $1 = x + t'$ for some $x \in (t{:}0)$ and $t' \in T$, for any $t \in T$. Hence $t = tx + tt' = tt' \in tT$, for $t \in T$, and ${}_R(R/T)$ is flat by [**2**, Proposition 2.2].

(5) $\Rightarrow$ (2) Let $X \in \mathscr{F}^*$, then for all $X' \subseteq X$,

$$0 \to X' \otimes_R R/T \to X \otimes_R R/T$$

is exact since ${}_R(R/T)$ is flat. But then $X' \otimes_R R/T = 0$ since $X \otimes_R R/T = 0$ by Proposition 1.3, and $X' \in \mathscr{F}^*$. Therefore $\mathscr{F}^* = \mathscr{H}$.

(3) $\Leftrightarrow$ (6) By Lemma 1.8, for all $p \in P$ there exists $t \in T$ such that $p = pt$. Therefore $p(1 - t) = 0$, i.e., $(1 - t) \in (p:0)$, and $R = (p:0) + T$ for all $p \in P$. Conversely, if $(p:0) + T = R$ for $p \in P$, then $1 = x + t$ for some $x \in (p:0)$ and $t \in T$, for all $p \in P$. Hence $p = px + pt = pt \in pT$ for $p \in P$, and $P \in \mathscr{H}$ by Lemma 1.8.

(4) $\Leftrightarrow$ (7) This is proved in the same way as (3) $\Leftrightarrow$ (6).

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (2) Let $X \in \mathscr{F}^*$. Then by Proposition 1.3, X is an epimorphic image of a direct sum of copies of P. But $P \in \mathscr{H}$ and $\mathscr{H}$ is a torsion class, hence $X \in \mathscr{H}$ and $\mathscr{F}^* = \mathscr{H}$.

(2) $\Rightarrow$ (9) Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence in Mod-R. Then

$$A/AT \xrightarrow{f'} B/BT \xrightarrow{g'} C/CT \to 0$$

is always exact. Suppose $f'(a + AT) = 0$, i.e., $f(a) \in BT$, for some $a \in A$. Then since $\mathscr{F}^*$ is closed under submodules, $f(a)R = f(a)RT = f(a)T$, and therefore there exists $t \in T$ such that $f(a) = f(a)t = f(at)$. But f is a monomorphism, and hence $a = at$, i.e., $a + AT = 0$, and f' is a monomorphism.

(9) $\Rightarrow$ (8)

$$0 \longrightarrow \text{Ker } \epsilon(M) \longrightarrow [P, M] \otimes_E P \xrightarrow{\epsilon(M)} MT \longrightarrow 0$$

is an exact sequence for all $M \in$ Mod-R, and therefore, in particular,

$$0 \to \text{Ker } \epsilon(M)/(\text{Ker } \epsilon(M))T \to [P, M] \otimes_E P/([P, M] \otimes_E P)T$$

is exact. But $[P, M] \otimes_E P$ is cotorsionfree, and hence so is Ker $\epsilon(M)$. By Proposition 1.6, Ker $\epsilon(M)$ is also cotorsion, and thus it is zero. Therefore $MT \cong [P, M] \otimes_E P$, and hence is codivisible by Proposition 1.5.

(8) $\Rightarrow$ (1) Let M be a cotorsion module, i.e., $MT = 0$. Let $I(M)$ denote the injective hull of M. We show that $I(M)T = 0$ also. Let π be the projection map: $I(M)T \to I(M)T/I(M)T \cap M$. $I(M)T/I(M)T \cap M$ is cotorsionfree and hence codivisible, and $I(M)T \cap M = \text{Ker } \pi$ is cotorsion since M is cotorsion. Therefore there exists f: $I(M)T/I(M)T \cap M \to I(M)T$ such that $\pi f = 1_{I(M)T/I(M)T \cap M}$, and hence $I(M)T \cap M$ is a direct summand of $I(M)T$. But M essential in $I(M)$ then implies $I(M)T = I(M)T \cap M$. Thus $I(M)T = I(M)T^2 \subseteq MT = 0$.

2. Colocalization. The next result is the dual of a well-known characterization of the localization of M at I. (See, e.g., [**8**, Proposition 1.1].)

PROPOSITION 2.1. *For all $M \in$ Mod-R, let $\varphi: X \to M$ and $\psi: Y \to M$ be homomorphisms with cotorsion kernels and cokernels, where X and Y are cotorsionfree and codivisible modules. Then $X \cong Y$.*

Proof. Since X is cotorsionfree, $X = XT$ and therefore $\varphi(X) \subseteq MT$. But Cok $\varphi = M/\varphi(X)$ is cotorsion, and therefore $MT \subseteq \varphi(X)$. Hence $\varphi(X) = MT$, and similarly $\psi(Y) = MT$. We may regard φ as an epimorphism from X to MT, and ψ as an epimorphism from Y to MT. Since Ker φ is cotorsion and Y is codivisible, there exists $f: Y \to X$ such that $\varphi f = \psi$. Similarly there exists $g: X \to Y$ such that $\psi g = \varphi$. Then $\varphi(1_X - fg) = \varphi - \varphi fg = \varphi - \psi g = \varphi - \varphi = 0$, and therefore $(1_X - fg): X \to$ Ker φ. Hence $1_X = fg$ since X is cotorsionfree and Ker φ is cotorsion. Similarly $1_Y = gf$, and $X \cong Y$.

We are now able to make the following definition.

Definition 2.2. For all $M \in$ Mod-R, $\varphi: X \to M$ is (up to isomorphism) the *colocalization* of M at P if X is cotorsionfree and codivisible, and Ker φ and Cok φ are cotorsion.

Given a projective module P, Lambek and Rattray [**9**] have constructed a cotriple (S', ϵ', δ') on Mod-R, and formed a colocalization of a module M at P by taking the coequalizer of the pair of mappings

$$S'(S'(M)) \underset{S'\epsilon'(M)}{\overset{\epsilon'S'(M)}{\rightrightarrows}} S'(M).$$

For P a finitely generated projective module, they showed that this colocalization of M at P is $[P, M] \otimes_E P$. The next theorem states that this is our colocalization of M at P for any projective P. We will later verify that the two colocalizations are the same for any projective P.

THEOREM 2.3. *For all $M \in$ Mod-R, $[P, M] \otimes_E P$ is the colocalization of M at P.*

Proof. Since clearly $[P, M] \otimes_E P$ is cotorsionfree and Cok $\epsilon(M) = M/MT$ is cotorsion, the result follows from Propositions 1.5 and 1.6.

If we let $F = \_\otimes_E P$: Mod-$E \to$ Mod-R and $U = [P, \_]$: Mod-$R \to$ Mod-E, then F is the left adjoint of U, i.e. there exist natural transformations $\eta: 1_{\text{Mod-}E} \to UF$, given by $\eta(B)(b)(p) = b \otimes p$ for all $B \in$ Mod-E, $b \in B$, $p \in P$, and $\epsilon: FU \to 1_{\text{Mod-}R}$, given by $\epsilon(A)(\sum g_i \otimes p_i) = \sum g_i(p_i)$ for all $A \in$ Mod-R, $\sum g_i \otimes p_i \in [P, A] \otimes_E P$, such that $U\epsilon \circ \eta U = 1_U$ and $\epsilon F \circ F\eta = 1_F$.

We can then form the cotriple $(S^* = FU, \epsilon, \delta)$ on Mod-R. $S^*(M)$ is by Theorem 2.3 the colocalization of M at P for all $M \in$ Mod-R. The coequalizer of the mappings $\epsilon S^*(M)$, $S^*\epsilon(M)$: $S^{*2}(M) \to S^*(M)$ is just the identity on $S^*(M)$, since $\epsilon S^*(M)$ is an isomorphism and therefore $\epsilon S^*(M) = S^*\epsilon(M)$ (since $\epsilon S^*(M)\delta = 1_{S^*(M)} = S^*\epsilon(M)\delta$).

The dual situation (see [**8**, Section 3]) is more complicated. If I is an injective module and $H = [I, I]$, then $[\_, I]$: Mod-$R \to (H\text{-Mod})^{op}$ has a right adjoint

$\mathrm{Hom}_H(\_, {}_HI)$. If we form the triple $(S = \mathrm{Hom}_H([\_, I], {}_HI), \eta, \mu)$ arising from this pair of adjoint functors, then $Q(M)$, the localization of M at I, for all $M \in$ Mod-R, is given by the equalizer of the pair of mappings $\eta S(M)$, $S\eta(M)$: $S(M) \to S^2(M)$. $S(M)$ is torsionfree and divisible, and Ker $\eta(M)$ is torsion, but in general $S(M) \neq Q(M)$. (They are equal if $[M, I]$ is a finitely generated left H-module.) In general, then, Cok $\eta(M)$ is not torsion.

For example, let $R = \mathbf{Z}$. We take the largest torsion theory in Mod-$\mathbf{Z}$ for which $\mathbf{Z}/p\mathbf{Z}$ is torsionfree, where p is a prime number. A $\mathbf{Z}$-module M is torsion if and only if for all $m \in M$, $(m:0) \not\subseteq p\mathbf{Z}$, and $Q(\mathbf{Z})$ is the usual localization of the commutative ring $\mathbf{Z}$ at the prime ideal $p\mathbf{Z}$, i.e., $Q(\mathbf{Z})$ consists of all rational numbers whose denominators are prime to p. Every torsionfree factor module of $Q(\mathbf{Z})$ is divisible (in fact, if D is any dense ideal $DQ(\mathbf{Z}) = Q(\mathbf{Z})$ and hence the localization functor Q preserves all colimits), and therefore $S(\mathbf{Z})$ is the $I(\mathbf{Z}/p\mathbf{Z})$-adic completion of $Q(\mathbf{Z})$ [**8**, Theorem 4.2]. But the $I(\mathbf{Z}/p\mathbf{Z})$-adic topology on $Q(\mathbf{Z})$ coincides with the p-adic topology [**7**, Proposition 4], and thus $S(\mathbf{Z})$ is the ring of p-adic integers. But $S(\mathbf{Z})/\mathbf{Z} =$ Cok $(\eta(\mathbf{Z}): \mathbf{Z} \to S(\mathbf{Z}))$ is not torsion, since for all $z + z_1p + z_2p^2 + \ldots \in S(\mathbf{Z})$, if there exists n, $m \in \mathbf{Z}$ such that $n \notin p\mathbf{Z}$ and $n(z + z_1p + z_2p^2 + \ldots) = m$, then $z + z_1p + z_2p^2 + \ldots = m/n \in Q(\mathbf{Z})$.

Definition 2.4. $\varphi: X \to M$ is a *codivisible cover* of $M \in$ Mod-R if
(1) φ is a minimal epimorphism;
(2) Ker φ is cotorsion;
(3) X is codivisible.

PROPOSITION 2.5. *If a module M has a codivisible cover, then it is unique up to isomorphism.*

Proof. Let $\varphi: X \to M$ and $\psi: Y \to M$ be codivisible covers of M. Then there exists $f: X \to Y$ such that $\psi f = \varphi$ since X is codivisible and Ker ψ is cotorsion. φ an epimorphism and Ker ψ small in Y implies that f is an epimorphism, and Ker f is cotorsion and small in X since Ker $f \subseteq$ Ker φ. Therefore there exists $g: Y \to X$ such that $fg = 1_Y$, hence $X = g(Y) \oplus$ Ker f. But then Ker $f = 0$ since Ker f is small in X, and hence f is an isomorphism.

We will show that if $M \in$ Mod-R has a projective cover, then it has a codivisible cover.

LEMMA 2.6. *If $M \in$ Mod-R is codivisible and $M' \subseteq M$ is a cotorsionfree submodule of M, then M/M' is codivisible.*

Proof. Let $\pi: M \to M/M'$ be the projection map, and let $\varphi: B \to A$ be any epimorphism with Ker φ cotorsion, and $\psi: M/M' \to A$. Since M is codivisible there exists $\psi': M \to B$ such that $\varphi\psi' = \psi\pi$. $\varphi\psi'(M') = \psi\pi(M') = 0$, and therefore $0 = \psi'|_{M'}: M' \to$ Ker φ since M' is cotorsionfree and Ker φ is cotorsion. Therefore ψ' induces a homomorphism $\psi'': M/M' \to B$ such that $\varphi\psi'' = \psi$, and hence M/M' is codivisible.

https://doi.org/10.4153/CJM-1975-072-6 Published online by Cambridge University Press

PROPOSITION 2.7. *If* $\varphi: P(M) \to M$ *is the projective cover of* $M \in$ Mod-R, *then* $\bar{\varphi}: P(M)/(\mathrm{Ker}\ \varphi)T \to M$ *is the codivisible cover of* M, *where* $\bar{\varphi}$ *is the homomorphism induced by* φ.

Proof. Clearly $\bar{\varphi}: P(M)/(\mathrm{Ker}\ \varphi)T$ is a minimal epimorphism, and $\mathrm{Ker}\ \bar{\varphi} = \mathrm{Ker}\ \varphi/(\mathrm{Ker}\ \varphi)T$ is cotorsion. It remains to show that $P(M)/(\mathrm{Ker}\ \varphi)T$ is codivisible, but this follows from the preceding lemma.

COROLLARY 2.8. *If* $\varphi: P(M) \to M$ *is the projective cover of* $M \in$ Mod-R, *then the codivisible cover of* M *in the cotorsion theory determined by* $P(M)$ *is the maximal co-rational extension over* M.

Proof. Courter [**3**, Theorem 2.12] showed that $P(M)/X$ is the maximal co-rational extension over M, where

$$X = \sum_{f \in [P(M), \mathrm{Ker}\,\varphi]} f(P(M)).$$

But if $T_{P(M)}$ denotes the trace ideal of $P(M)$, then it is clear from the proof of Proposition 1.3 that $X = (\mathrm{Ker}\ \varphi)T_{P(M)}$.

COROLLARY 2.9. *If* $\varphi: P(M) \to M$ *is the projective cover of* $M \in$ Mod-R, *then* M *is codivisible if and only if* Ker φ *is cotorsionfree.*

Proof. Ker φ cotorsionfree implies that $\mathrm{Ker}\ \bar{\varphi} = 0$, and hence $M \cong P(M)/(\mathrm{Ker}\ \varphi)T$ which is codivisible. Conversely, if M is codivisible then by Proposition 2.5 $P(M)/(\mathrm{Ker}\ \varphi)T \cong M$, and therefore $\mathrm{Ker}\ \varphi = (\mathrm{Ker}\ \varphi)T$.

THEOREM 2.10. $[P, M] \otimes_E P = [P, MT] \otimes_E P$ *is the codivisible cover of* MT.

Proof. We have already shown that $\epsilon(M): [P, M] \otimes_E P \to MT$ is an epimorphism (Proposition 1.3) with cotorsion kernel (Proposition 1.6), and that $[P, M] \otimes_E P$ is codivisible (Proposition 1.5). Ker $\epsilon(M)$ is small in $[P, M] \otimes_E P$, since if $\mathrm{Ker}\ \epsilon(M) + U = [P, M] \otimes_E P$ for some submodule $U \subseteq [P, M] \otimes_E P$, then $U \supseteq UT = (\mathrm{Ker}\ \epsilon(M))T + UT = ([P, M] \otimes_E P)T = [P, M] \otimes_E P$. Hence $[P, M] \otimes_E P$ is the codivisible cover of MT.

The torsion submodule $\mathscr{T}(M)$ of a module M with respect to a torsion theory $(\mathscr{T}, \mathscr{F})$ is the unique submodule $X \subseteq M$ such that X is torsion and M/X is torsionfree. Dually, M/MT is the unique factor module M/X of M such that M/X is cotorsion and X is cotorsionfree. We call M/MT the *cotorsion factor module* of M. And, we can colocalize in two steps, namely

$[P, M] \otimes_E P \to MT \to M$
codivisible
cover of MT

dualizing $M \to M/\mathscr{T}(M) \to Q(M)$.
divisible
hull of $M/\mathscr{T}(M)$

https://doi.org/10.4153/CJM-1975-072-6 Published online by Cambridge University Press

3. Colocalization as coequalizer. We now return to the colocalization at P obtained by Lambek and Rattray [**9**], and we will show that it is the same as our colocalization at P. They started with a cotriple (S', ϵ', δ') on Mod-R, where S': Mod-$R \to$ Mod-R is defined by

$$S'(M) = \sum_{f:P\to M} P \quad \text{for all } M \in \text{Mod-}R,$$

and an element of $S'(M)$ is written as $\sum_f(f, p_f)$. $S'(M)$ is a right R-module in view of the definitions $\sum_f(f, p_f) + \sum_f(f, q_f) = \sum_f(f, p_f + q_f)$, and $(\sum_f(f, p_f))r = \sum_f(f, p_f r)$ for all $r \in R$. $\epsilon'(M)$: $S'(M) \to M$ is given by $\epsilon'(M)(\sum_f(f, p_f)) = \sum_f f(p_f)$. If k_f: $P \to \sum_f P$ is the canonical injection then $\epsilon'(M)k_f = f$. For any g: $M \to N$ in Mod-R, $S'(g)$: $S'(M) \to S'(N)$ is given by $S'(g)(\sum_f(f, p_f)) = \sum_f(gf, p_f)$, i.e. for the canonical injection k_f, $S'(g)k_f = k_{gf}$. Their colocalization $Q'(M)$ of M at P is given by the coequalizer $\kappa(M)$: $S'(M) \to Q'(M)$ of the pair of mappings $\epsilon' S'(M)$, $S'\epsilon'(M)$: $S'(S'(M)) \to S'(M)$. The following lemma is the dual of [**9**, Lemma 1].

LEMMA 3.1. *For all* $M \in$ Mod-R, $\kappa(M)$ *is the joint coequalizer of all pairs of mappings* u, v: $P \to S'(M)$ *which equalize* $\epsilon'(M)$: $S'(M) \to M$.

Proof. Let u: $P \to S'(M)$, then $\epsilon' S'(M)k_u = u$ and $S'\epsilon'(M)k_u = k_{\epsilon'(M)u}$. Therefore $\kappa(M)$ coequalizes all mappings $(u, k_{\epsilon'(M)u})$. Now let v: $P \to S'(M)$ be such that $\epsilon'(M)u = \epsilon'(M)v$. Then $\kappa(M)$ coequalizes (u, v) since

$$\kappa(M)u = \kappa(M)k_{\epsilon'(M)u} = \kappa(M)k_{\epsilon'(M)v} = \kappa(M)v.$$

Conversely, any mapping which coequalizes all (u, v) such that $\epsilon'(M)u = \epsilon'(M)v$ coequalizes $(u, k_{\epsilon'(M)u})$ in particular, since $\epsilon'(M)k_{\epsilon'(M)u} = \epsilon'(M)u$ by definition of $\epsilon'(M)$, and hence coequalizes $(\epsilon' S'(M), S'\epsilon'(M))$. It follows that $\kappa(M)$ is the joint coequalizer.

LEMMA 3.2. *Let* f: $B \to A$ *be an epimorphism where* B *is a cotorsionfree module and* A *is a codivisible module. Then* Ker f *is cotorsionfree.*

Proof. Let $\bar{f}$: $B/(\text{Ker } f)T \to A$ be the homomorphism induced by f. Then since A is codivisible and $\bar{f}$ is an epimorphism with cotorsion kernel, there exists g: $A \to B/(\text{Ker } f)T$ such that $\bar{f}g = 1_A$. Therefore $(B/(\text{Ker } f)T)T = B/(\text{Ker } f)T = \text{Im } g \oplus \text{Ker } \bar{f} = (\text{Im } g)T \oplus (\text{Ker } \bar{f})T = (\text{Im } g)T$ and hence $\text{Ker } \bar{f} = 0$, i.e., $\text{Ker } f = (\text{Ker } f)T$.

LEMMA 3.3. *For all* $M \in$ Mod-R, MT *is the smallest submodule* $M' \subseteq M$ *such that for all* f: $P \to M$,

$$0 = (P \xrightarrow{f} M \to M/M').$$

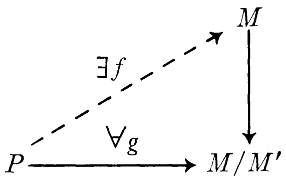
Proof. For all f: $P \to M$,

$$(P \xrightarrow{f} M \to M/MT) = 0$$

since $f(P) \subseteq MT$. Suppose $M' \subseteq M$ is such that for all f: $P \to M$,

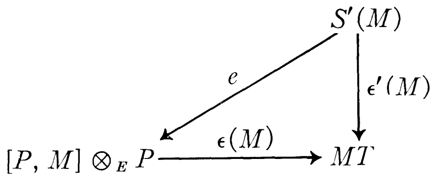
$$(P \xrightarrow{f} M \to M/M') = 0,$$

then for all $g \in [P, M/M']$ since P is projective there exists f: $P \to M$ such that the diagram below commutes, and hence $g = 0$. M/M' is therefore cotorsion, and $MT \subseteq M'$.



THEOREM 3.4. *For all* $M \in$ Mod-R, $[P, M] \otimes_E P$ *is the coequalizer of the pair of mappings* $\epsilon' S'(M)$, $S'\epsilon'(M)$: $S'(S'(M)) \to S'(M)$.

Proof.



$\epsilon(M)$ and $\epsilon'(M)$ both have the same image, namely MT, and we consider them as mappings from $[P, M] \otimes_E P$ to MT and from $S'(M)$ to MT, respectively. Then since $S'(M)$ is projective (since it is a coproduct of copies of P) and Ker $\epsilon(M)$ is small in $[P, M] \otimes_E P$, there exists an epimorphism e: $S'(M) \to [P, M] \otimes_E P$, such that $\epsilon(M)e = \epsilon'(M)$. By Lemma 3.2, Ker e is cotorsionfree since $S'(M)$ is cotorsionfree and $[P, M] \otimes_E P$ is codivisible. But since Ker e is cotorsionfree and Ker $\epsilon(M)$ is cotorsion, Ker $\epsilon(M) =$ Ker $\epsilon'(M)/$Ker e is the cotorsion factor module of Ker $\epsilon'(M)$, i.e. Ker $e =$ (Ker $\epsilon'(M))T$. Hence by Lemma 3.3 Ker e is the smallest submodule X of Ker $\epsilon'(M)$ such that for all f: $P \to$ Ker $\epsilon'(M)$, $0 = (P \xrightarrow{f}$ Ker $\epsilon'(M) \to$ Ker $\epsilon'(M)/X)$. Therefore Ker e is the smallest submodule X of $S'(M)$ such that for all f: $P \to S'(M)$ such that $\epsilon'(M)f = 0$,

$$0 = (P \xrightarrow{f} S'(M) \to S'(M)/X).$$

Hence Ker e is the smallest submodule X of $S'(M)$ such that for all f, f': $P \to S'(M)$ such that $\epsilon'(M)f = \epsilon'(M)f'$,

$$(P \xrightarrow{f} S'(M) \to S'(M)/X) = (P \xrightarrow{f'} S'(M) \to S'(M)/X),$$

i.e.,

$$S'(M) \xrightarrow{e} S'(M)/\mathrm{Ker}\, e \cong [P, M] \otimes_E P$$

is the joint coequalizer of all pairs of mappings $f, f': P \to S'(M)$ which equalize $\epsilon'(M)$. Thus by Lemma 3.1 $Q'(M) \cong [P, M] \otimes_E P$.

References

1. G. Azumaya, *Some properties of TTF-classes*, Proc. Conf. on Orders, Group Rings, and Related Topics, Lecture Notes in Math. *353* (Springer-Verlag, Berlin, 1973).
2. S. U. Chase, *Direct products of modules*, Trans. Amer. Math. Soc. *97* (1960), 457–473.
3. R. C. Courter, *The maximal co-rational extension by a module*, Can. J. Math. *18* (1966), 953–962.
4. P. Gabriel, *Des categories abeliennes*, Bull. Soc. Math. France *90* (1962), 323–448.
5. J. P. Jans, *Some aspects of torsion*, Pacific J. Math. *15* (1965), 1249–1259.
6. J. Lambek, *Torsion theories, additive semantics, and rings of quotients*, Lecture Notes in Math. *177* (Springer-Verlag, Berlin, 1971).
7. ——— *Bicommutators of nice injectives*, J. Algebra *21* (1972), 60–73.
8. ——— *Localization and completion*, J. Pure Appl. Algebra *2* (1972), 343–370.
9. J. Lambek and B. Rattray, *Localization at injectives in complete categories*, Proc. Amer. Math. Soc. *41* (1973), 1–9.
10. R. W. Miller, *TTF classes and quasi-generators*, Pacific J. Math. *51* (1974), 499–507.
11. E. A. Rutter, Jr., *Torsion theories over semi-perfect rings*, Proc. Amer. Math. Soc. *34* (1972), 389–395.
12. F. L. Sandomierski, *Modules over the endomorphism ring of a finitely generated projective module*, Proc. Amer. Math. Soc. *31* (1972), 27–31.

McGill University,
Montreal, Quebec

https://doi.org/10.4153/CJM-1975-072-6 Published online by Cambridge University Press