PLETHYSM OF $S$-FUNCTIONS

A. O. USHER

The $S$-function $\{\mu\} \otimes \{\lambda\}$, $\mu \r {\mathfrak{m}}$, $\lambda \r {\mathfrak{l}}$, where $\{\mu\} \otimes \{\lambda\}$ is the ‘new multiplication’ or plethysm of D. E. Littlewood [1], corresponds, in the sense defined below in (1), to the character afforded by a representation of the symmetric group $S_{\mathfrak{m}\mathfrak{l}}$ induced from a representation of the subgroup $S_{\mathfrak{m}} \subset S_{\mathfrak{l}}$ [3 § 6; 4 § 3.5]. The aim of this paper is to define the latter representation and deduce its character using a somewhat different approach from that in [3].

In Section 2, the character ‘$\{\mu\} \otimes \{\lambda\}$’ of the general linear group, $GL_n$, over the field of complex numbers, is introduced and expressed in a form given by H. O. Foulkes [5] which suggests that one might usefully consider a certain irreducible representation of the wreath product $S_{\mathfrak{m}} \wr S_{\mathfrak{l}}$. It is shown in Section 3 that the character of $S_{\mathfrak{m}\mathfrak{l}}$ induced from the character afforded by this representation has corresponding $S$-function $\{\mu\} \otimes \{\lambda\}$. The connection between the plethysm of $S$-functions and wreath products of symmetric groups has been pointed out by several authors (e.g. [9, § 7; 10 p. 135]) but no proofs seem to be available. Finally, in Section 4 there is a brief summary of one of the possible methods of reducing $\{\mu\} \otimes \{\lambda\}$ into its irreducible components.

2. The $S$-function $\{\mu\} \otimes \{\lambda\}$. Let $\phi$ be any class function defined on $S_{\mathfrak{t}}$, then the Schur characteristic function, or $S$-function, corresponding to $\phi$ is, by definition, the symmetric function,

\begin{equation}
\Phi = \frac{1}{\mathfrak{l}!} \sum_{\rho \in \mathfrak{l}} r_{\rho} \phi_{\rho} S_{\rho}
\end{equation}

where $\phi_{\rho}$ is the value of $\phi$ on the conjugacy class $C_{\rho}$ of $S_{\mathfrak{t}}$,

$\rho = |C_{\rho}|$

$S_{\rho} = S_{\mathfrak{t}1} S_{\mathfrak{t}2} \cdots$ for $\rho = (1^{a_1} 2^{a_2} \cdots) \r \mathfrak{l}$

and with $S_{\mathfrak{t}k}$ ($k$ a positive integer) the $k$th power sum, $t_{1}^{k} + t_{2}^{k} + \ldots$, in the variables $t_{1}, t_{2}, \ldots$

Now if $\lambda = (\lambda_{1}, \lambda_{2}, \ldots, \lambda_{\mathfrak{l}}) \r \mathfrak{l}$ then $\{\lambda\}$ may be defined as the bialternant symmetric function

$\{\lambda\} = \sum_{\rho} \pm t_{1}^{\lambda_{1}+i-1} t_{2}^{\lambda_{2}+i-2} \ldots t_{\mathfrak{l}}^{\lambda_{\mathfrak{l}}} = \frac{|t|^{\lambda_{j}+i-j}|}{|t|^{i-j}} = \frac{\Delta_{\mathfrak{l}}}{\Delta}$, 

say, where the $(i, j)$ entry of the $\mathfrak{l}$th order determinant, $\Delta_{\mathfrak{l}}$, is $t_{i}^{\lambda_{j}}$ and where the sums are taken over all permutations of the suffixes of the $t$’s, with $+$ or $-$ sign according as the permutation is even or odd. It follows from the famous
Frobenius formula for the irreducible characters \( \chi^{(\lambda)} \) of \( S_n \), namely,

\[
S_n \Delta = \sum_{\lambda \vdash l} \chi^{(\lambda)}_{\rho} \Delta_{\lambda}
\]

that \( \{\lambda\} \) is the S-function corresponding to \( \chi^{(\lambda)} \) [2, §§ 5.2, 6.3]. Thus,

\[\{\lambda\} = \frac{1}{l!} \sum_{\rho \vdash l} r_{\rho} \chi^{(\lambda)}_{\rho} S_{\rho}\]

Let the irreducible rational homogeneous representations of weight \( l \) of \( GL_n \) be \( \sigma^{(\lambda)} \), \( \lambda \vdash l \) into not more than \( n \) parts, then the character afforded by \( \sigma^{(\lambda)} \) is \( \{\lambda\} \), where the variables are now the eigenvalues \( t_1, \ldots, t_n \) of \( \xi \in GL_n \). Thus, \( \{1\} = S_1 = t_1 + \ldots + t_n = \text{tr} \xi \) and \( S_\delta = t_1^\delta + \ldots + t_n^\delta = \text{tr} (\xi^\delta) \).

If the degree of the \( \sigma^{(\mu)} \), \( \mu \vdash m \), representation of \( GL_n \) is \( N \) then for \( \xi \in GL_n \), the entries of \( \sigma^{(\omega)}(\xi) \in GL_N \) are homogeneous polynomials of degree \( m \) in the entries of \( \xi \) and \( \sigma^{(\omega)}(GL_n) = R \), a subgroup of \( GL_N \). Next, consider the \( \sigma^{(\lambda)} \) representation of \( GL_N \); the entries of \( \sigma^{(\lambda)}(\eta) \), \( \eta \in GL_N \), are homogeneous polynomials of degree \( l \) in those of \( \eta \) and

\[
\{\lambda\} = \frac{1}{l!} \sum_{\rho \vdash l} r_{\rho} \chi^{(\lambda)}_{\rho} Z_{\rho}
\]

where \( Z_{\rho} \) is defined in terms of the eigenvalues \( t_1^*, \ldots, t_N^* \) of \( \eta \in GL_N \) in exactly the same way as \( S_{\rho} \) in terms of \( t_1, \ldots, t_n \). Now the restriction of \( \sigma^{(\lambda)} \) to \( R \), \( \sigma^{(\lambda)}|_R \), is a representation of \( R \) and hence of \( GL_n \). In this representation \( \xi \in GL_n \) is mapped onto the matrix \( \sigma^{(\lambda)}(\sigma^{(\mu)}(\xi)) \), that is, the matrix \( \sigma^{(\lambda)}(\eta) \) with \( \eta \in R \) and of form \( \sigma^{(\omega)}(\xi) \). The entries of \( \sigma^{(\lambda)}(\sigma^{(\omega)}(\xi)) \) are, of course, homogeneous polynomials of degree \( lm \) in the entries of \( \xi \). The character afforded by \( \sigma^{(\lambda)}|_R \) is written \( \{\mu\} \otimes \{\lambda\} \). Thus,

\[\{\mu\} \otimes \{\lambda\} = \frac{1}{l!} \sum_{\rho \vdash l} r_{\rho} \chi^{(\lambda)}_{\rho} Z_{\rho},\]

a symmetric function of weight \( lm \), constructed from the given S-functions \( \{\mu\}, \{\lambda\} \) of weights \( m \) and \( l \) respectively.

We require \( Z_{\rho} \) in terms of the eigenvalues \( t_i \), \( i = 1, \ldots, n \), of \( \xi \in GL_n \), rather than as a function of the \( t_i^* \), \( j = 1, \ldots, N \). Now,

\[
\{\mu\} = \frac{1}{m!} \sum_{\rho \vdash m} r_{\rho} \chi^{(\omega)}_{\rho} S_{\rho}
\]

where, \( S_{\rho} = S_1^{b_1} S_2^{b_2} \ldots \) for \( \rho = (1^{b_1} 2^{b_2} \ldots) \vdash m \) and \( r_{\rho} = |C_{\rho}| \) of \( S_m \). That is,

\[
\text{tr} \sigma^{(\omega)}(\xi) = \frac{1}{m!} \sum_{\rho \vdash m} r_{\rho} \chi^{(\omega)}_{\rho} (\text{tr} \xi)^{b_1} (\text{tr} \xi^2)^{b_2} \ldots \quad \text{for all } \xi \in GL_n.
\]

Replace \( \xi \) with \( \xi^2 \), hence \( S_h \) with \( S_{2h} \) then, since

\[
Z_{\eta} = t_1^{\eta^2} + \ldots + t_N^{\eta^2} = \text{tr} \eta^2 = \text{tr} (\sigma^{(\omega)}(\xi)^2) = \text{tr} (\sigma^{(\omega)}(\xi^2))
\]
we have,

\[ Z_q = \frac{1}{m!} \sum_{\rho} r_\rho \chi_\rho^{(a)} (\text{tr } \xi^a)^{b_1} (\text{tr } \xi^{2a})^{b_2} \ldots \]

Thus, if we write \( \{\mu\}^{(g)} \) for \( Z_q \)

(4) \[ \{\mu\}^{(g)} = \frac{1}{m!} \sum_{\rho} r_\rho \chi_\rho^{(a)} S_\rho \]

where \( g_\rho = (q^{a_1} 2q^{a_2} \ldots) \vdash qm. \)

Finally, since \( Z_\rho = Z_1^{a_1} Z_2^{a_2} \ldots = \{\mu\} (\{\mu\}^{(g)})^2 \ldots = \{\mu\}_\rho \), say, then (3) becomes

(5) \[ \{\mu\} \otimes \{\lambda\} = \frac{1}{m!} \sum_{\rho} r_\rho \chi_\rho^{(k)} \{\mu\}_\rho, \]

a form, used by H. O. Foulkes [5, § 5], which invites comparison with a certain irreducible representation of the wreath product \( S_m \wr S_l \).

3. The character of \( S_{lm} \) corresponding to the \( S \)-function \( \{\mu\} \otimes \{\lambda\} \).

Following the definitions and notation of A. Kerber [8, pp. 24–25], we let \( (y; x) = (y_1, \ldots, y_i; x) \) be a general element of the wreath product \( S_m \wr S_l \), where \( y \) maps the set \( \Omega = \{1, \ldots, l\} \) into \( S_m \) and \( x \in S_l \). The basis group of \( S_m \wr S_l, S_m^* \), with elements of form \( (y; 1_{S_l}) \), \( y : \Omega \to S_m \), is the direct product \( S_m \times \ldots \times S_m \) of \( l \) copies of \( S_m \). The complement \( S'_l \) of \( S_m^* \) is isomorphic to \( S_l \) and its elements are of the form \( (e; x) \), \( x \in S_l \), \( e \) the identity of \( S_m^* \). Thus, the factor group \( (S_m \wr S_l)/S_m^* \) is \( S'_l \) and if \( x \) is a given element of \( S_l \), then the set of elements \( \{y; x\} \) constitute a coset of \( S_m^* \) in \( S_m \wr S_l \).

From the definition of \( S_m \wr S_l \), it is easily seen that the cycle decomposition of elements \( (y; x), x \in C_\rho \) of \( S_l \) and \( \rho = (1^e 2^e \ldots s^e \ldots) \) a partition of \( l \) into \( r \) parts, is of the form

(6) \[ \nu_\rho = \nu_1 \oplus \ldots \oplus \nu_{\nu_1} \oplus 2\nu_{\nu_1+1} \oplus \ldots \oplus 2\nu_{\nu_1+\nu_2} \oplus \ldots \]

a direct sum of \( r \) partitions, where the first \( a_1 \) terms are of form \( \nu_i \), the next \( a_2 \) of form \( 2 \nu_i \), the next \( a_3 \) of form \( 3 \nu_i \), \ldots, with \( \nu_i = (s^e 2s^e \ldots) \vdash sm \) for \( \nu_i = (1^e 2^e \ldots) \vdash \).

Now, Kerber shows [8, §§ 5, 6] that certain irreducible representations of \( S_m \wr S_l \) are of the form \( \{\mu; \lambda\} = (\sigma \otimes \rho^{(x)}) \) where, \( \rho^{(x)} \) is the (irreducible) representation of \( S_m \wr S_l \) derived from the irreducible representation \( \rho^{(x)} \) of the factor group \( S'_l \), \( \sigma \) is the (irreducible) Kronecker product representation \( \rho^{(x)} \otimes \ldots \otimes \rho^{(y)} \) (\( l \) factors) of \( S_m^* \), with \( \rho^{(y)} \) the irreducible representation (of degree \( n_y \)) of \( S_m \), and \( \sigma \) is the (irreducible) representation, derived from \( \sigma \) by permuting the columns of the matrices \( \sigma((y; 1_{S_l})) \). The representation \( \sigma \) is given by \( \sigma((y; x)) \) with \((i_1, \ldots, i_l; j_1, \ldots, j_l)\) entry equal to

\[ \rho^{(x)}_{s_1 s_2 \ldots s_{l-1}}(a)(y_1) \rho^{(y)}_{s_{l-1} s_{l-2} \ldots s_1}(a)(y_2) \ldots \rho^{(y)}_{s_{l-1} s_{l-2} \ldots s_1}(a)(y_l), (1 \leq i_1, j_1 \leq n_x). \]

Therefore the \((i_1, \ldots, i_l; j_1, \ldots, j_l)\) entry of \( \sigma((y; x)) \), if \( x \in C_\rho \) with
\( \rho = (1^a 2^a \ldots) \), is equal to

\[
\rho_{i_1 i_2} (y_1) \cdots \rho_{i_a i_{a+1}} (y_{a+1}) \rho_{i_{a+1} i_{a+2}} (y_{a+2}) \cdots
\]

which includes, corresponding to an \( s \)-cycle (say the first) in the \( k \)th to

\( (k + s - 1) \)th parts of \( \rho \), the product of factors

\[
\rho_{i_k i_{k+1}} (y_k) \cdot \rho_{i_{k+1} i_{k+2}} (y_{k+1}) \cdots \rho_{i_{k+s-1} i_k} (y_{k+s-1}).
\]

Hence,

\[
\text{tr} \left( (y; x) \right) = \text{tr} \rho^{(a)} (y_1) \cdots \text{tr} \rho^{(a)} (y_{a+1}) \text{tr} \rho^{(a)} (y_{a+1} y_{a+2}) \cdots
\]

\[
\text{tr} \rho^{(a)} (y_k y_{k+1} \cdots y_{k+s-1}) = \chi^{(a)} (x^1) \chi^{(a)} (x^2) \cdots \chi^{(a)} (x^r) (r \text{ factors})
\]

where \( y_1 \in C_{r_1}, \ldots, y_{a+1} \in C_{r_{a+1}}, y_{a+1} y_{a+2} \in C_{r_{a+1} r_{a+2}}, \ldots, y_k y_{k+1} \cdots y_{k+s-1} \in C_{r_{k+s-1} \cdots r_{k+s-2} r_{k+s-1}} \) of \( S_m \) and here all the \( y_i \) in \( \rho^{(a)} (y_i) \) are considered as elements of a single \( S_m \), since the factors of \( \sigma \) are all equivalent and so may be made equal.

Thus, the value of the character afforded by the irreducible representation \((\mu; \lambda) \equiv (\sigma \otimes \rho^{(a)})\) of \( S_m \triangleright S_1 \) on \( (y; x) \) with \( x \in C_\rho, \rho = (1^a 2^a \ldots) \mapsto l \) into \( r \) parts is equal to \( \Pi_{i=1}^r \chi^{(a)} (x^i) \).

Finally, we show that the \( S \)-function corresponding to the character \( \phi \), say, afforded by the induced representation, \((\mu; \lambda) \mapsto S_m \triangleright S_1 \) is \( \{\rho\} \otimes \{X\} \). Now, the element \( (y; x) \in S_m \triangleright S_1 \) with \( x \in C_\rho \), from (6), corresponds to a partition of \( m \) of the form \( \nu_\rho \) and therefore belongs to the conjugacy class \( C_{\nu_\rho} \) of \( S_m \).

Thus [6, Theorem 16.7.2], the value of the character \( \phi \) on \( (y; x) \) is

\[
\phi((y; x)) = \phi_{\nu_\rho} = \frac{(lm)!}{l!r_\rho} \sum_{(y; x) \in C_{\nu_\rho} \cap S_m \triangleright S_1} \chi^{(\lambda)} (x) \left( \prod_{i=1}^r r_{x_i} \chi^{(a)} (x) \right)
\]

the sum being over all \( (y; x) \) of \( S_m \triangleright S_1 \) of the form \( \nu_\rho \). But the number of cosets of \( S_m \) in \( S_m \triangleright S_1 \) corresponding to a particular \( \rho \mapsto l \) is \( r_\rho \), the number of ways of building \( \nu_\rho \) in each of these cosets is

\[
\sum_{i \in \nu_\rho} \left( \prod_{i=1}^r r_{x_i} \right)
\]

and every one of these occurs \( (m!)^{l-r} \) times in each such coset. Hence,

\[
\phi_{\nu_\rho} = \frac{(lm)!}{l!r_\rho} \sum_{l \in \nu_\rho} \left( \prod_{i=1}^r r_{x_i} \right) \frac{r_\rho}{(ml)!} \chi^{(\lambda)} \left( \sum_{i \in \nu_\rho} \prod_{i=1}^r r_{x_i} \chi^{(a)} (x) \right),
\]

for given \( \nu_\rho \). Now, the corresponding \( S \)-function,

\[
\Phi = \frac{1}{(lm)!} \sum_{l \in \nu_\rho} r_{\cdot} \phi_{\cdot \cdot} S_{\cdot \cdot} = \frac{1}{(lm)!} \sum_{\rho \neq \cdot} \sum_{l \in \nu_\rho} r_{\rho} \phi_{\rho \cdot} S_{\rho \cdot}
\]

since \( \phi_{\cdot} = 0 \) unless \( \cdot = \nu_\rho \) for some \( \rho \mapsto 1 \). Thus,

\[
\Phi = \frac{1}{l!} \sum_{\rho \neq \cdot} r_{\rho} \chi^{(\lambda)} \left( \prod_{l \in \nu_\rho} r_{\rho \cdot} \chi^{(a)} \right) S_{\rho \cdot},
\]

now summed over all \( \nu_\rho \).
But \( S_{r_0} = S_{r_1} \ldots S_{r_{a_1}} S_{2r_{a_1} + 1} \ldots S_{2r_{a_1} + 2} \ldots (r \text{ factors}) \) for \( \rho = (1^{a_1} 2^{a_2} \ldots) \vdash l \) into \( r \) parts. Therefore,

\[
\Phi = \frac{1}{l!} \sum \rho \chi_\rho^{(\lambda)} \left( \sum r_{r_1} x_{r_1}^{(\mu)} S_{r_1} \right) \ldots \left( \sum r_{a_1} x_{a_1}^{(\mu)} S_{a_1} \right)
\times \left( \sum r_{a_1 + 1} x_{a_1 + 1}^{(\mu)} S_{2a_1 + 1} \right) \ldots = \frac{1}{l!} \sum \rho \chi_\rho^{(\lambda)} \{\mu\}_\rho,
\]

as required.

4. The reduction of \( \{\mu\} \otimes \{\lambda\} \). We conclude with a brief reference to the problem of reducing the \( S \)-function \( \{\mu\} \otimes \{\lambda\} \) to a sum of \( S \)-functions, that is, to the decomposition of the character \( \phi \) of \( S_{lm} \) with corresponding \( S \)-function \( \{\mu\} \otimes \{\lambda\} \) to a sum of irreducible characters of \( S_{lm} \). Many methods (e.g. [1], also [4, p. 166] for more references) have been devised for this reduction; we consider \( \{\mu\} \otimes \{\lambda\} \) in the form (5).

The \( \chi_\rho^{(\lambda)} \) may be found from the character table of \( S_l \), or by applying the Littlewood-Richardson recurrence rule [2, § 5.3, Theorem II] and the order of \( C_\rho \) is

\[
r_\rho = \frac{l!}{1^{a_1} a_1! 2^{a_2} a_2! \ldots}
\]

for \( \rho = (1^{a_1} 2^{a_2} \ldots) \vdash l \). The differential operator method of H. O. Foulkes [5] gives a simple determinantal procedure for the coefficient of \( \{\nu\} \), \( \nu \vdash lm \), in \( \{\mu\} \); it is also very useful in conjunction with other methods which may determine the coefficients of \( S \)-functions \( \{\nu\} \) in \( \{\mu\} \otimes \{\lambda\} = \sum_{\sigma \vdash lm} C_{\mu\lambda\nu} \{\sigma\} \) corresponding to certain - but not all - forms of the partition \( \nu \) of \( lm \).

If, however, each \( \{\mu\}^{(\nu)} \) in \( \{\mu\}_\rho \) were expressed as a sum of \( S \)-functions, the problem would then reduce to that of the ordinary multiplication of \( S \)-functions [2, § 6.3, Theorem V]. We have,

\[
\{\mu\}^{(\nu)} = \frac{1}{m!} \sum_{\rho \vdash m} r_\rho x_\rho^{(\mu)} S_{\rho \nu}
\]

from (4). But

\[
S_\rho = \sum_{\mu \vdash m} \chi_\rho^{(\mu)} \{\mu\}
\]

for each \( \rho \vdash m \). In particular for \( q\rho \vdash qm \),

\[
S_{q\rho} = \sum_{\sigma \vdash qm} \chi_{q\rho}^{(\sigma)} \{\sigma\}
\]

Thus,

\[
\{\mu\}^{(\nu)} = \frac{1}{m!} \sum_{\rho \vdash m} r_\rho x_\rho^{(\mu)} \chi_{q\rho}^{(\nu)} \{\sigma\}
\]
where, $\chi^{(r)}$, $\chi^{(e)}$ are irreducible characters of $S_m$ and $S_{qm}$ respectively. Hence $[\mu] \otimes [\lambda]$ becomes a sum of products of $S$-functions, the coefficients in which are integral multiples of products of characters of $S_i$, $S_m$ and $S_{qm}$ ($q$ a divisor of $lm$). Now we require the values of $\chi^{(r)}$, $\sigma \vdash qm$, on classes of form $C_{\phi\rho}$, $\rho \vdash m$, only. But D. E. Littlewood [2, § 8.1] has shown that $\chi_{\phi\rho}^{(e)}$ may be expressed in terms of the irreducible characters $\chi^{(a)}$ of $S_m$; we therefore require the irreducible characters of only $S_i$ and $S_m$.

References

2. ——— The theory of group characters (Oxford, 1940).
10. D. Knutson, Lecture notes on $\lambda$-rings and the theory of the symmetric group, Lecture notes in Math. vol. 308 (Springer-Verlag, 1973)

Royal Holloway College (University of London), Englefield Green, Surrey